# FUNDAMENTA NOVA

## THEORIAE

# FUNCTIONUM ELLIPTICARUM

AUCTORE

D. CAROLO GUSTAVO JACOBO JACOBI

PROF. ORD. IN UNIV. REGIOM.

---

REGIOMONTI

SUMTIBUS FRATRUM BORNTRAEGER

1829.

# DE

# TRANSFORMATIONE FUNCTIONUM ELLIPTICARUM.

$$z = \frac{u(u+v^5)y - v^3(u^2+v^2)(u+v^5)y^3 + v^{10}(1+u^3v)y^5}{u^2(1+u^3v) - u^2v(u^2+v^2)(u+v^5)y^2 + u^3v^6(u+v^5)y^4},$$

erui:

$$\frac{dz}{\sqrt{(1-z^2)(1-u^8z^2)}} = \frac{u+v^5}{u(1+u^3v)} \cdot \frac{dy}{\sqrt{(1-y^2)(1-v^8y^2)}}.$$

Iam cum ex aequatione:

$$u^6 - v^6 + 5u^2v^2(u^2-v^2) + 4uv(1-u^4v^4) = 0$$

sequatur:

$$\frac{(u+v^5)(v-u^5)}{uv(1+u^3v)(1-uv^3)} = \frac{uv(1-u^4v^4)-(u^6-v^6)}{uv(1+u^3v)(1-uv^3)} = 5,$$

fieri videmus:

$$\frac{dz}{\sqrt{(1-z^2)(1-u^8z^2)}} = \frac{5dx}{\sqrt{(1-x^2)(1-u^8x^2)}}.$$

Ita transformatione bis adhibita pervenitur ad multiplicationem.

Haec duo exempla, videlicet transformationes tertii et quinti ordinis, iam prius in litteris exhibui, quas mense Iunio a. 1827 ad Cl$^{m}$. Schumacher dedi. Vide *Nova Astronomica* Nr. 123. Nec non ibidem methodi, qua eruta sunt, generalitatem praedicabam. Alterum biennio ante iam a Cl$^{o}$. Legendre inventum erat.

## DE NOTATIONE NOVA FUNCTIONUM ELLIPTICARUM.

### 17.

Missis factis quaestionibus algebraicis, accuratius inquiramus in naturam analyticam functionum nostrarum. Antea autem notationis modum, cuius in sequentibus usus erit, indicemus necesse est.

Posito $\int_0^\varphi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = u$, angulum $\varphi$ *amplitudinem* functionis $u$ vocare geometrae consueverunt. Hunc igitur angulum in sequentibus denotabimus per ampl $u$ seu brevius per:

$$\varphi = \text{am}\, u.$$

Ita, ubi

$$u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

erit:

$$x = \sin \text{am}\, u.$$

Insuper posito:

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = K,$$

vocabimus $K-u$ complementum functionis $u$; complementi amplitudinem designabimus per *coam*, ita ut sit:

$$\operatorname{am}(K-u) = \operatorname{coam} u.$$

Expressionem $\sqrt{1-k^2\sin^2\operatorname{am} u} = \frac{d\operatorname{am} u}{du}$, duce Cl^o. Legendre, denotabimus per:

$$\Delta\operatorname{am} u = \sqrt{1-k^2\sin^2\operatorname{am} u}.$$

Complementum, quod vocatur a Cl^o. Legendre, moduli $k$ designabo per $k'$, ita ut sit:

$$kk+k'k' = 1.$$

Porro e notatione nostra erit:

$$K' = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k'k'\sin^2\varphi}}.$$

Modulus, qui subintelligi debet, ubi opus erit, sive uncis inclusus addetur sive in margine adiicietur. Modulo non addito, in sequentibus eundem ubique modulum $k$ subintelligas.

Ipsas expressiones $\sin\operatorname{am} u$, $\sin\operatorname{coam} u$, $\cos\operatorname{am} u$, $\cos\operatorname{coam} u$, $\Delta\operatorname{am} u$, $\Delta\operatorname{coam} u$ etc. ac generaliter *functiones trigonometricas amplitudinis* in sequentibus *functionum ellipticarum* nomine insignire convenit, ita ut ei nomini aliam quandam tribuamus notionem atque hactenus factum est ab analystis. Ipsam $u$ dicemus *argumentum functionis ellipticae*, ita ut, posito $x = \sin\operatorname{am} u$, sit $u = \arg\sin\operatorname{am} x$. E notatione proposita erit:

$$\sin\operatorname{coam} u = \frac{\cos\operatorname{am} u}{\Delta\operatorname{am} u}$$

$$\cos\operatorname{coam} u = \frac{k'\sin\operatorname{am} u}{\Delta\operatorname{am} u}$$

$$\Delta\operatorname{coam} u = \frac{k'}{\Delta\operatorname{am} u}$$

$$\operatorname{tg}\operatorname{coam} u = \frac{1}{k'\operatorname{tg}\operatorname{am} u}$$

$$\operatorname{cotg}\operatorname{coam} u = \frac{k'}{\operatorname{cotg}\operatorname{am} u}.$$

# FORMULAE IN ANALYSI FUNCTIONUM ELLIPTICARUM FUNDAMENTALES.

18.

Ponamus $\operatorname{am} u = a$, $\operatorname{am} v = b$, $\operatorname{am}(u+v) = \sigma$, $\operatorname{am}(u-v) = \vartheta$; notae sunt formulae additionis et subtractionis functionum ellipticarum fundamentales:

$$\sin\sigma = \frac{\sin a \cos b \,\Delta b + \sin b \cos a \,\Delta a}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\cos\sigma = \frac{\cos a \cos b - \sin a \sin b \,\Delta a \Delta b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\Delta\sigma = \frac{\Delta a \Delta b - k^2 \sin a \sin b \cos a \cos b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\sin\vartheta = \frac{\sin a \cos b \,\Delta b - \sin b \cos a \,\Delta a}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\cos\vartheta = \frac{\cos a \cos b + \sin a \sin b \,\Delta a \Delta b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\Delta\vartheta = \frac{\Delta a \Delta b + k^2 \sin a \sin b \cos a \cos b}{1 - k^2 \sin^2 a \sin^2 b}.$$

Ut in promptu sint omnia, quorum in posterum usus erit, adnotemus adhuc formulas sequentes, quae facile demonstrantur, et quarum facile augetur numerus:

$$(1.)\qquad \sin\sigma + \sin\vartheta = \frac{2 \sin a \cos b \,\Delta b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$(2.)\qquad \cos\sigma + \cos\vartheta = \frac{2 \cos a \cos b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$(3.)\qquad \Delta\sigma + \Delta\vartheta = \frac{2 \Delta a \Delta b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$(4.)\qquad \sin\sigma - \sin\vartheta = \frac{2 \sin b \cos a \,\Delta a}{1 - k^2 \sin^2 a \sin^2 b}$$

$$(5.)\qquad \cos\vartheta - \cos\sigma = \frac{2 \sin a \sin b \,\Delta a \Delta b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$(6.)\qquad \Delta\vartheta - \Delta\sigma = \frac{2 k^2 \sin a \sin b \cos a \cos b}{1 - k^2 \sin^2 a \sin^2 b}$$

$$\sin\sigma\,\sin\vartheta = \frac{\sin^2 a - \sin^2 b}{1 - k^2\sin^2 a\sin^2 b} \tag{7.}$$

$$1 + k^2\sin\sigma\,\sin\vartheta = \frac{\Delta^2 b + k^2\sin^2 a\cos^2 b}{1 - k^2\sin^2 a\sin^2 b} \tag{8.}$$

$$1 + \sin\sigma\,\sin\vartheta = \frac{\cos^2 b + \sin^2 a\,\Delta^2 b}{1 - k^2\sin^2 a\sin^2 b} \tag{9.}$$

$$1 + \cos\sigma\,\cos\vartheta = \frac{\cos^2 a + \cos^2 b}{1 - k^2\sin^2 a\sin^2 b} \tag{10.}$$

$$1 + \Delta\sigma\,\Delta\vartheta = \frac{\Delta^2 a + \Delta^2 b}{1 - k^2\sin^2 a\sin^2 b} \tag{11.}$$

$$1 - k^2\sin\sigma\,\sin\vartheta = \frac{\Delta^2 a + k^2\sin^2 b\cos^2 a}{1 - k^2\sin^2 a\sin^2 b} \tag{12.}$$

$$1 - \sin\sigma\,\sin\vartheta = \frac{\cos^2 a + \sin^2 b\,\Delta^2 a}{1 - k^2\sin^2 a\sin^2 b} \tag{13.}$$

$$1 - \cos\sigma\,\cos\vartheta = \frac{\sin^2 a\,\Delta^2 b + \sin^2 b\,\Delta^2 a}{1 - k^2\sin^2 a\sin^2 b} \tag{14.}$$

$$1 - \Delta\sigma\,\Delta\vartheta = \frac{k^2(\sin^2 a\cos^2 b + \sin^2 b\cos^2 a)}{1 - k^2\sin^2 a\sin^2 b} \tag{15.}$$

$$(1 \pm \sin\sigma)(1 \pm \sin\vartheta) = \frac{(\cos b \pm \sin a\,\Delta b)^2}{1 - k^2\sin^2 a\sin^2 b} \tag{16.}$$

$$(1 \pm \sin\sigma)(1 \mp \sin\vartheta) = \frac{(\cos a \pm \sin b\,\Delta a)^2}{1 - k^2\sin^2 a\sin^2 b} \tag{17.}$$

$$(1 \pm k\sin\sigma)(1 \pm k\sin\vartheta) = \frac{(\Delta b \pm k\sin a\cos b)^2}{1 - k^2\sin^2 a\sin^2 b} \tag{18.}$$

$$(1 \pm k\sin\sigma)(1 \mp k\sin\vartheta) = \frac{(\Delta a \pm k\sin b\cos a)^2}{1 - k^2\sin^2 a\sin^2 b} \tag{19.}$$

$$(1 \pm \cos\sigma)(1 \pm \cos\vartheta) = \frac{(\cos a \pm \cos b)^2}{1 - k^2\sin^2 a\sin^2 b} \tag{20.}$$

$$(1 \pm \cos\sigma)(1 \mp \cos\vartheta) = \frac{(\sin a\,\Delta b \mp \sin b\,\Delta a)^2}{1 - k^2\sin^2 a\sin^2 b} \tag{21.}$$

$$(1 \pm \Delta\sigma)(1 \pm \Delta\vartheta) = \frac{(\Delta a \pm \Delta b)^2}{1 - k^2\sin^2 a\sin^2 b} \tag{22.}$$

$$(1 \pm \Delta\sigma)(1 \mp \Delta\vartheta) = \frac{k^2\sin^2(a \mp b)}{1 - k^2\sin^2 a\sin^2 b} \tag{23.}$$

(24.) $$\sin\sigma\cos\vartheta = \frac{\sin a\cos a\,\Delta b + \sin b\cos b\,\Delta a}{1-k^2\sin^2 a\sin^2 b}$$

(25.) $$\sin\vartheta\cos\sigma = \frac{\sin a\cos a\,\Delta b - \sin b\cos b\,\Delta a}{1-k^2\sin^2 a\sin^2 b}$$

(26.) $$\sin\sigma\,\Delta\vartheta = \frac{\cos b\sin a\,\Delta a + \cos a\sin b\,\Delta b}{1-k^2\sin^2 a\sin^2 b}$$

(27.) $$\sin\vartheta\,\Delta\sigma = \frac{\cos b\sin a\,\Delta a - \cos a\sin b\,\Delta b}{1-k^2\sin^2 a\sin^2 b}$$

(28.) $$\cos\sigma\,\Delta\vartheta = \frac{\cos a\cos b\,\Delta a\,\Delta b - k'k'\sin a\sin b}{1-k^2\sin^2 a\sin^2 b}$$

(29.) $$\cos\vartheta\,\Delta\sigma = \frac{\cos a\cos b\,\Delta a\,\Delta b + k'k'\sin a\sin b}{1-k^2\sin^2 a\sin^2 b}$$

(30.) $$\sin(\sigma+\vartheta) = \frac{2\sin a\cos a\,\Delta b}{1-k^2\sin^2 a\sin^2 b}$$

(31.) $$\sin(\sigma-\vartheta) = \frac{2\sin b\cos b\,\Delta a}{1-k^2\sin^2 a\sin^2 b}$$

(32.) $$\cos(\sigma+\vartheta) = \frac{\cos^2 a - \sin^2 a\,\Delta^2 b}{1-k^2\sin^2 a\sin^2 b}$$

(33.) $$\cos(\sigma-\vartheta) = \frac{\cos^2 b - \sin^2 b\,\Delta^2 a}{1-k^2\sin^2 a\sin^2 b}.$$

## DE IMAGINARIIS FUNCTIONUM ELLIPTICARUM VALORIBUS. PRINCIPIUM DUPLICIS PERIODI.

### 19.

Ponamus $\sin\varphi = i\,\mathrm{tg}\,\psi$, ubi $i$ loco $\sqrt{-1}$ positum est more plerisque geometris usitato, erit $\cos\varphi = \sec\psi = \frac{1}{\cos\psi}$, unde $d\varphi = \frac{i\,d\psi}{\cos\psi}$. Hinc fit:

$$\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{i\,d\psi}{\sqrt{\cos^2\psi + k^2\sin^2\psi}} = \frac{i\,d\psi}{\sqrt{1-k'k'\sin^2\psi}}.$$

Quam e notatione nostra in hanc abire videmus aequationem:

(1.) $$\sin\mathrm{am}(iu,k) = i\,\mathrm{tg}\,\mathrm{am}(u,k').$$

Hinc sequitur:

(2.) $$\cos\mathrm{am}(iu,k) = \sec\mathrm{am}(u,k')$$

(3.) $$\mathrm{tg}\,\mathrm{am}(iu,k) = i\sin\mathrm{am}(u,k')$$

$$(4.)\qquad \Delta\,\mathrm{am}(iu,k) = \frac{\Delta\,\mathrm{am}(u,k')}{\cos\mathrm{am}(u,k')} = \frac{1}{\sin\mathrm{coam}(u,k')}$$

$$(5.)\qquad \sin\mathrm{coam}(iu,k) = \frac{1}{\Delta\,\mathrm{am}(u,k')}$$

$$(6.)\qquad \cos\mathrm{coam}(iu,k) = \frac{ik'}{k}\cos\mathrm{coam}(u,k')$$

$$(7.)\qquad \mathrm{tg}\,\mathrm{coam}(iu,k) = \frac{-i}{k'\sin\mathrm{am}(u,k')}$$

$$(8.)\qquad \Delta\,\mathrm{coam}(iu,k) = k'\sin\mathrm{coam}(u,k').$$

Aliud, quod hinc fluit, formularum systema hoc est:

$$(9.)\qquad \sin\mathrm{am}\,2iK' = 0$$

$$(10.)\qquad \sin\mathrm{am}\,iK' = \infty,\ \text{vel si placet}\ \pm i\infty$$

$$(11.)\qquad \sin\mathrm{am}(u+2iK') = \sin\mathrm{am}\,u$$

$$(12.)\qquad \cos\mathrm{am}(u+2iK') = -\cos\mathrm{am}\,u$$

$$(13.)\qquad \Delta\,\mathrm{am}(u+2iK') = -\Delta\,\mathrm{am}\,u$$

$$(14.)\qquad \sin\mathrm{am}(u+iK') = \frac{1}{k\sin\mathrm{am}\,u}$$

$$(15.)\qquad \cos\mathrm{am}(u+iK') = \frac{-i\,\Delta\,\mathrm{am}\,u}{k\sin\mathrm{am}\,u} = \frac{-ik'}{k\cos\mathrm{coam}\,u}$$

$$(16.)\qquad \mathrm{tg}\,\mathrm{am}(u+iK') = \frac{i}{\Delta\,\mathrm{am}\,u}$$

$$(17.)\qquad \Delta\,\mathrm{am}(u+iK') = -i\,\mathrm{cotg}\,\mathrm{am}\,u$$

$$(18.)\qquad \sin\mathrm{coam}(u+iK') = \frac{\Delta\,\mathrm{am}\,u}{k\cos\mathrm{am}\,u} = \frac{1}{k\sin\mathrm{coam}\,u}$$

$$(19.)\qquad \cos\mathrm{coam}(u+iK') = \frac{ik'}{k\cos\mathrm{am}\,u}$$

$$(20.)\qquad \mathrm{tg}\,\mathrm{coam}(u+iK') = \frac{-i}{k'}\Delta\,\mathrm{am}\,u$$

$$(21.)\qquad \Delta\,\mathrm{coam}(u+iK') = ik'\,\mathrm{tg}\,\mathrm{am}\,u.$$

E formulis praecedentibus, quae et ipsae tamquam fundamentales in analysi functionum ellipticarum considerari debent, elucet:

*a)* functiones ellipticas argumenti imaginarii $iv$, moduli $k$, transformari posse in alias argumenti realis $v$, moduli $k' = \sqrt{1-k^2}$. Unde generaliter functiones ellipticas argumenti imaginarii $u+iv$, moduli $k$, componere licet e functionibus ellipticis argumenti $u$, moduli $k$, et aliis argumenti $v$, moduli $k'$.

b) functiones ellipticas duplici gaudere periodo, altera reali, altera imaginaria, siquidem modulus $k$ est realis. Utraque fit imaginaria, ubi modulus et ipse est imaginarius. Quod *principium duplicis periodi* nuncupabimus. E quo, cum universam, quae fingi potest, amplectatur periodicitatem analyticam, elucet functiones ellipticas non aliis adnumerari debere transcendentibus, quae quibusdam gaudent elegantiis, fortasse pluribus illas aut maioribus, sed speciem quandam iis inesse perfecti et absoluti.

## THEORIA ANALYTICA TRANSFORMATIONIS FUNCTIONUM ELLIPTICARUM.

20.

Vidimus in antecedentibus, quoties functiones elementi $x$ rationales integrae $A, B, C, D, U, V$ ita determinentur, ut sit:

$$\begin{aligned} V+U &= (1+x)AA \\ V-U &= (1-x)BB \\ V+\lambda U &= (1+kx)CC \\ V-\lambda U &= (1-kx)DD, \end{aligned}$$

posito $y = \frac{U}{V}$, fore:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2 y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}},$$

designante $M$ quantitatem constantem. Iam expressiones illarum functionum analyticas generales proponamus.

Sit $n$ numerus impar quilibet, sint $m, m'$ numeri integri quilibet positivi seu negativi, qui tamen factorem communem non habeant, qui et ipse numerum $n$ metitur, ponamus:

$$\omega = \frac{mK+m'iK'}{n};$$

fit:

$$U = \frac{x}{M}\left(1-\frac{x^2}{\sin^2 \mathrm{am}\, 4\omega}\right)\left(1-\frac{x^2}{\sin^2 \mathrm{am}\, 8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2 \mathrm{am}\, 2(n-1)\omega}\right)$$

$$V = \left(1-k^2x^2\sin^2 \mathrm{am}\, 4\omega\right)\left(1-k^2x^2\sin^2 \mathrm{am}\, 8\omega\right)\cdots\left(1-k^2x^2\sin^2 \mathrm{am}\, 2(n-1)\omega\right)$$

$$A = \left(1+\frac{x}{\sin \mathrm{coam}\, 4\omega}\right)\left(1+\frac{x}{\sin \mathrm{coam}\, 8\omega}\right)\cdots\left(1+\frac{x}{\sin \mathrm{coam}\, 2(n-1)\omega}\right)$$

$$B = \left(1-\frac{x}{\sin \mathrm{coam}\, 4\omega}\right)\left(1-\frac{x}{\sin \mathrm{coam}\, 8\omega}\right)\cdots\left(1-\frac{x}{\sin \mathrm{coam}\, 2(n-1)\omega}\right)$$

$$C = \left(1+kx\sin \mathrm{coam}\, 4\omega\right)\left(1+kx\sin \mathrm{coam}\, 8\omega\right)\cdots\left(1+kx\sin \mathrm{coam}\, 2(n-1)\omega\right)$$

$$D = \left(1-kx\sin \mathrm{coam}\, 4\omega\right)\left(1-kx\sin \mathrm{coam}\, 8\omega\right)\cdots\left(1-kx\sin \mathrm{coam}\, 2(n-1)\omega\right)$$

$$\lambda = k^n\left\{\sin \mathrm{coam}\, 4\omega \sin \mathrm{coam}\, 8\omega \ldots \sin \mathrm{coam}\, 2(n-1)\omega\right\}^4$$

$$M = (-1)^{\frac{n-1}{2}}\left\{\frac{\sin \mathrm{coam}\, 4\omega \sin \mathrm{coam}\, 8\omega \ldots \sin \mathrm{coam}\, 2(n-1)\omega}{\sin \mathrm{am}\, 4\omega \;\sin \mathrm{am}\, 8\omega \ldots \sin \mathrm{am}\, 2(n-1)\omega}\right\}^2.$$

Quibus positis, ubi $x = \sin \mathrm{am}\, u$, fit $y = \frac{U}{V} = \sin \mathrm{am}\left(\frac{u}{M}, \lambda\right)$.

Antequam ipsam aggrediamur formularum demonstrationem, earum transformationem quandam indicabimus. Quem in finem sequentes adnotamus formulas, quae statim e formulis §. 18. decurrunt:

$$\text{(1.)} \quad \sin \mathrm{am}(u+\alpha)\sin \mathrm{am}(u-\alpha) = \frac{\sin^2 \mathrm{am}\, u - \sin^2 \mathrm{am}\, \alpha}{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, \alpha}$$

$$\text{(2.)} \quad \frac{[1+\sin \mathrm{am}(u+\alpha)][1+\sin \mathrm{am}(u-\alpha)]}{\cos^2 \mathrm{am}\, \alpha} = \frac{\left(1+\frac{\sin \mathrm{am}\, u}{\sin \mathrm{coam}\, \alpha}\right)^2}{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, \alpha}$$

$$\text{(3.)} \quad \frac{[1-\sin \mathrm{am}(u+\alpha)][1-\sin \mathrm{am}(u-\alpha)]}{\cos^2 \mathrm{am}\, \alpha} = \frac{\left(1-\frac{\sin \mathrm{am}\, u}{\sin \mathrm{coam}\, \alpha}\right)^2}{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, \alpha}$$

$$\text{(4.)} \quad \frac{[1+k\sin \mathrm{am}(u+\alpha)][1+k\sin \mathrm{am}(u-\alpha)]}{\Delta^2 \mathrm{am}\, \alpha} = \frac{(1+k\sin \mathrm{am}\, u \sin \mathrm{coam}\, \alpha)^2}{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, \alpha}$$

$$\text{(5.)} \quad \frac{[1-k\sin \mathrm{am}(u+\alpha)][1-k\sin \mathrm{am}(u-\alpha)]}{\Delta^2 \mathrm{am}\, \alpha} = \frac{(1-k\sin \mathrm{am}\, u \sin \mathrm{coam}\, \alpha)^2}{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, \alpha}.$$

E quibus formulis etiam sequitur:

$$\text{(6.)} \quad \frac{\cos \mathrm{am}(u+\alpha)\cos \mathrm{am}(u-\alpha)}{\cos^2 \mathrm{am}\, \alpha} = \frac{1-\frac{\sin^2 \mathrm{am}\, u}{\sin^2 \mathrm{coam}\, \alpha}}{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, \alpha}$$

$$\text{(7.)} \quad \frac{\Delta \mathrm{am}(u+\alpha)\Delta \mathrm{am}(u-\alpha)}{\Delta^2 \mathrm{am}\, \alpha} = \frac{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{coam}\, \alpha}{1-k^2\sin^2 \mathrm{am}\, u \sin^2 \mathrm{am}\, \alpha}.$$

Posito $x = \sin\operatorname{am} u$, nanciscimur e formula (1.):

$$\frac{1-\dfrac{x^2}{\sin^2\operatorname{am}\alpha}}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{-\sin\operatorname{am}(u+\alpha)\sin\operatorname{am}(u-\alpha)}{\sin^2\operatorname{am}\alpha},$$

e formulis (2.), (3.):

$$\frac{\left(1\pm\dfrac{x}{\sin\operatorname{coam}\alpha}\right)^2}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{[1\pm\sin\operatorname{am}(u+\alpha)][1\pm\sin\operatorname{am}(u-\alpha)]}{\cos^2\operatorname{am}\alpha},$$

e formulis (4.), (5.):

$$\frac{(1\pm kx\sin\operatorname{coam}\alpha)^2}{1-k^2x^2\sin^2\operatorname{am}\alpha} = \frac{[1\pm k\sin\operatorname{am}(u+\alpha)][1\pm k\sin\operatorname{am}(u-\alpha)]}{\Delta^2\operatorname{am}\alpha}.$$

Hinc ubi loco $\alpha$ successive ponitur $4\omega$, $8\omega$, $\dots 2(n-1)\omega$, loco $-\alpha$ autem $4n\omega-\alpha$, obtinemus:

$$(8.)\quad \frac{U}{V} = \frac{\dfrac{x}{M}\left(1-\dfrac{x^2}{\sin^2\operatorname{am}4\omega}\right)\left(1-\dfrac{x^2}{\sin^2\operatorname{am}8\omega}\right)\cdots\left(1-\dfrac{x^2}{\sin^2\operatorname{am}2(n-1)\omega}\right)}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{\sin\operatorname{am}u\sin\operatorname{am}(u+4\omega)\sin\operatorname{am}(u+8\omega)\cdots\sin\operatorname{am}(u+4(n-1)\omega)}{[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^2}$$

$$(9.)\quad \frac{(1+x)AA}{V} = \frac{(1+x)\left\{\left(1+\dfrac{x}{\sin\operatorname{coam}4\omega}\right)\left(1+\dfrac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1+\dfrac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{[1+\sin\operatorname{am}u][1+\sin\operatorname{am}(u+4\omega)][1+\sin\operatorname{am}(u+8\omega)]\cdots[1+\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(10.)\quad \frac{(1-x)BB}{V} = \frac{(1-x)\left\{\left(1-\dfrac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\dfrac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\dfrac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{[1-\sin\operatorname{am}u][1-\sin\operatorname{am}(u+4\omega)][1-\sin\operatorname{am}(u+8\omega)]\cdots[1-\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(11.)\quad \frac{(1+kx)CC}{V} = \frac{(1+kx)\{[1+kx\sin\operatorname{coam}4\omega][1+kx\sin\operatorname{coam}8\omega]\cdots[1+kx\sin\operatorname{coam}2(n-1)\omega]\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{[1+k\sin\operatorname{am}u][1+k\sin\operatorname{am}(u+4\omega)][1+k\sin\operatorname{am}(u+8\omega)]\cdots[1+k\sin\operatorname{am}(u+4(n-1)\omega)]}{[\Delta\operatorname{am}4\omega\,\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}$$

$$(12.)\quad \frac{(1-kx)DD}{V} = \frac{(1-kx)\{[1-kx\sin\operatorname{coam}4\omega][1-kx\sin\operatorname{coam}8\omega]\cdots[1-kx\sin\operatorname{coam}2(n-1)\omega]\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{[1-k\sin\operatorname{am}u][1-k\sin\operatorname{am}(u+4\omega)][1-k\sin\operatorname{am}(u+8\omega)]\cdots[1-k\sin\operatorname{am}(u+4(n-1)\omega)]}{[\Delta\operatorname{am}4\omega\,\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}$$

Hinc etiam sequuntur formulae:

$$(13.)\quad \frac{\sqrt{1-x^2}AB}{V} = \sqrt{1-x^2}\,\frac{\left(1-\frac{x^2}{\sin^2\operatorname{coam}4\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{coam}2(n-1)\omega}\right)}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{\cos\operatorname{am}u\cos\operatorname{am}(u+4\omega)\cos\operatorname{am}(u+8\omega)\cdots\cos\operatorname{am}(u+4(n-1)\omega)}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2}$$

$$(14.)\quad \frac{\sqrt{1-k^2x^2}CD}{V} = \sqrt{1-k^2x^2}\,\frac{[1-k^2x^2\sin^2\operatorname{coam}4\omega][1-k^2x^2\sin^2\operatorname{coam}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{coam}2(n-1)\omega]}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{\Delta\operatorname{am}u\,\Delta\operatorname{am}(u+4\omega)\,\Delta\operatorname{am}(u+8\omega)\cdots\Delta\operatorname{am}(u+4(n-1)\omega)}{[\Delta\operatorname{am}4\omega\,\Delta\operatorname{am}8\omega\cdots\Delta\operatorname{am}2(n-1)\omega]^2}.$$

## DEMONSTRATIO FORMULARUM ANALYTICARUM PRO TRANSFORMATIONE.

21.

Iam demonstremus, posito:

$$1-y = (1-x)\frac{\left\{\left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]}$$

$$= \frac{[1-\sin\operatorname{am}u][1-\sin\operatorname{am}(u+4\omega)][1-\sin\operatorname{am}(u+8\omega)]\cdots[1-\sin\operatorname{am}(u+4(n-1)\omega)]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2},$$

et reliquas erui formulas et hanc:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}},$$

siquidem:

$$\lambda = k^n[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^4$$

$$M = (-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}4\omega\sin\operatorname{coam}8\omega\cdots\sin\operatorname{coam}2(n-1)\omega]^2}{[\sin\operatorname{am}4\omega\sin\operatorname{am}8\omega\cdots\sin\operatorname{am}2(n-1)\omega]^2}.$$

E formula proposita apparet minime mutari $y$, quoties $u$ abit in $u+4\omega$. Tum enim quivis factor in subsequentem abit, ultimus vero in primum. Unde generaliter $y$ non mutatur, siquidem loco $u$ ponatur $u+4p\omega$, designante $p$ numerum integrum positivum seu negativum. Ubi vero $u=0$, fit:

$$1-y = \frac{[1-\sin\operatorname{am}4\omega][1-\sin\operatorname{am}8\omega]\cdots[1-\sin\operatorname{am}4(n-1)\omega]}{[\cos\operatorname{am}4\omega\cos\operatorname{am}8\omega\cdots\cos\operatorname{am}2(n-1)\omega]^2} = 1,$$

sive $y = 0$. Facile enim patet fore:

$$-\sin\operatorname{am}4(n-1)\omega = \sin\operatorname{am}4\omega$$
$$-\sin\operatorname{am}4(n-2)\omega = \sin\operatorname{am}8\omega,$$
$$\cdots\cdots\cdots\cdots$$

unde:

$$[1-\sin\operatorname{am}4\omega][1-\sin\operatorname{am}4(n-1)\omega] = \cos^2\operatorname{am}4\omega$$
$$[1-\sin\operatorname{am}8\omega][1-\sin\operatorname{am}4(n-2)\omega] = \cos^2\operatorname{am}8\omega$$
$$\cdots\cdots\cdots\cdots$$
$$[1-\sin\operatorname{am}2(n-1)\omega][1-\sin\operatorname{am}2(n+1)\omega] = \cos^2\operatorname{am}2(n-1)\omega.$$

Iam quia $y = 0$, quoties $u = 0$, neque mutatur $y$, ubi loco $u$ ponitur $u+4p\omega$, generaliter evanescit $y$, quoties $u$ valores induit:

$$0,\ 4\omega,\ 8\omega,\ \ldots\ldots,\ 4(n-2)\omega,\ 4(n-1)\omega,$$

quibus respondent valores quantitatis $x = \sin\operatorname{am}u$:

$$0,\ \sin\operatorname{am}4\omega,\ \sin\operatorname{am}8\omega,\ \ldots,\ \sin\operatorname{am}4(n-2)\omega,\ \sin\operatorname{am}4(n-1)\omega,$$

quos ita etiam exhibere licet:

$$0,\ \pm\sin\operatorname{am}4\omega,\ \pm\sin\operatorname{am}8\omega,\ \ldots,\ \pm\sin\operatorname{am}2(n-1)\omega,$$

sive etiam hunc in modum:

$$0,\ \pm\sin\operatorname{am}2\omega,\ \pm\sin\operatorname{am}4\omega,\ \ldots,\ \pm\sin\operatorname{am}(n-1)\omega.$$

Qui valores elementi $x$, quos evanescente $y$ induere potest, omnes inter se diversi erunt, eorumque numerus erit $n$. Iam ex aequatione inter $x$ et $y$ supposita, e qua profecti sumus, elucet, positis:

$$V = [1-k^2x^2\sin^2\operatorname{am}4\omega][1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]$$
$$= [1-k^2x^2\sin^2\operatorname{am}2\omega][1-k^2x^2\sin^2\operatorname{am}4\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}(n-1)\omega],$$

$y = \frac{U}{V}$, fieri $U$ functionem elementi $x$ rationalem integram $n^{\text{ti}}$ ordinis. Quae cum simul cum $y$ evanescat pro valoribus quantitatis $x$ numero $n$ et inter se diversis sequentibus:

$$0,\ \pm\sin\operatorname{am}2\omega,\ \pm\sin\operatorname{am}4\omega,\ \ldots,\ \pm\sin\operatorname{am}(n-1)\omega,$$

necessario formam induit:

$$U = \frac{x}{M}\left(1-\frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1-\frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right)$$
$$= \frac{x}{M}\left(1-\frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\left(1-\frac{x^2}{\sin^2 \operatorname{am} 8\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2 \operatorname{am} 2(n-1)\omega}\right),$$

designante $M$ constantem. Cum, posito $x = 1$, fiat $1-y = 0$, $y = 1$, obtinemus ex aequatione $y = \frac{U}{V}$:

$$1 = \frac{\left(1-\frac{1}{\sin^2 \operatorname{am} 2\omega}\right)\left(1-\frac{1}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1-\frac{1}{\sin^2 \operatorname{am}(n-1)\omega}\right)}{M[1-k^2\sin^2 \operatorname{am} 2\omega][1-k^2\sin^2 \operatorname{am} 4\omega]\cdots[1-k^2\sin^2 \operatorname{am}(n-1)\omega]}$$
$$= \frac{(-1)^{\frac{n-1}{2}}[\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^2}{M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2},$$

unde:

$$M = \frac{(-1)^{\frac{n-1}{2}}[\sin \operatorname{coam} 2\omega \sin \operatorname{coam} 4\omega \cdots \sin \operatorname{coam}(n-1)\omega]^2}{[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2}.$$

Inter functiones $U$, $V$ memorabilis intercedit correlatio, illam dico supra memoratam, cuius beneficio fit, ut, posito $\frac{1}{kx}$ loco $x$, simul $y$ in $\frac{1}{\lambda y}$ abeat, designante $\lambda$ constantem.

Posito enim $\frac{1}{kx}$ loco $x$, abit:

$$U = \frac{x}{M}\left(1-\frac{x^2}{\sin^2 \operatorname{am} 2\omega}\right)\left(1-\frac{x^2}{\sin^2 \operatorname{am} 4\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2 \operatorname{am}(n-1)\omega}\right)$$

in hanc expressionem:

$$(-1)^{\frac{n-1}{2}}\frac{V}{Mx^n}\cdot\frac{1}{k^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2}.$$

Contra vero, eadem substitutione facta,

$$V = [1-k^2x^2\sin^2 \operatorname{am} 2\omega][1-k^2x^2\sin^2 \operatorname{am} 4\omega]\cdots[1-k^2x^2\sin^2 \operatorname{am}(n-1)\omega]$$

in hanc expressionem abit:

$$(-1)^{\frac{n-1}{2}}\frac{U}{x^n}\cdot M[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^2.$$

Unde, loco $x$ posito $\frac{1}{kx}$, $y = \frac{U}{V}$ abit in:

$$\frac{V}{U}\cdot\frac{1}{MM.k^n[\sin \operatorname{am} 2\omega \sin \operatorname{am} 4\omega \cdots \sin \operatorname{am}(n-1)\omega]^4},$$

sive $y$ in $\frac{1}{\lambda y}$, siquidem ponitur:

$$\begin{aligned}\lambda &= MMk^n[\sin\operatorname{am}2\omega \sin\operatorname{am}4\omega \cdots \sin\operatorname{am}(n-1)\omega]^4\\ &= k^n[\sin\operatorname{coam}2\omega \sin\operatorname{coam}4\omega \cdots \sin\operatorname{coam}(n-1)\omega]^4.\end{aligned}$$

Id quod demonstrandum erat.

Ex aequatione proposita:

$$1-y = (1-x)\frac{\left\{\left(1-\frac{x}{\sin\operatorname{coam}4\omega}\right)\left(1-\frac{x}{\sin\operatorname{coam}8\omega}\right)\cdots\left(1-\frac{x}{\sin\operatorname{coam}2(n-1)\omega}\right)\right\}^2}{[1-k^2x^2\sin^2\operatorname{am}4\omega]\,[1-k^2x^2\sin^2\operatorname{am}8\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}2(n-1)\omega]},$$

posito $\frac{1}{kx}$ loco $x$, $\frac{1}{\lambda y}$ loco $y$, quod ex antecedentibus licet, eruimus:

$$\frac{1}{\lambda y}-1 = \frac{1-kx}{\lambda U}\{[1-kx\sin\operatorname{coam}4\omega][1-kx\sin\operatorname{coam}8\omega]\cdots[1-kx\sin\operatorname{coam}2(n-1)\omega]\}^2,$$

quod ductum in $\lambda y = \frac{\lambda U}{V}$ praebet:

$$1-\lambda y = (1-kx)\frac{\{[1-kx\sin\operatorname{coam}4\omega][1-kx\sin\operatorname{coam}8\omega]\cdots[1-kx\sin\operatorname{coam}2(n-1)\omega]\}^2}{V}.$$

Ceterum patet $y = \frac{U}{V}$ abire in $-y$, ubi $x$ in $-x$ mutatur, quo facto igitur statim etiam $1+y$, $1+\lambda y$ ex $1-y$, $1-\lambda y$ obtinemus.

Iam igitur eiusmodi invenimus functiones elementi $x$ rationales integras $U$, $V$, ut sit:

$$\begin{aligned}V+U &= V(1+y) = (1+x)AA\\ V-U &= V(1-y) = (1-x)BB\\ V+\lambda U &= V(1+\lambda y) = (1+kx)CC\\ V-\lambda U &= V(1-\lambda y) = (1-kx)DD,\end{aligned}$$

designantibus $A$, $B$, $C$, $D$ et ipsis functiones elementi $x$ rationales integras. Hinc autem secundum principia transformationis initio stabilita statim sequitur:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Multiplicatorem $M$, quem vocabimus, ex observatione §. 14. facta obtinemus. Unde iam omnes formulae analyticae generales, quae theoriam transformationis functionum ellipticarum concernunt, demonstratae sunt.

22.

Demonstratio proposita ex ea, quam dedimus in *Novis Astronomicis* a Cl°. Schumacher editis Nr. 127, eruitur, ubi ponitur $\omega$ loco $\frac{K}{n}$, $(-1)^{\frac{n-1}{2}}M$ loco M, aliis omnibus immutatis manentibus. Ipsum theorema analyticum generale de transformatione sub forma paulo alia iam prius ibidem Nr. 123 cum analystis communicaveram. Demonstrationem Cl. Legendre, summus in hac doctrina arbiter, ibidem Nr. 130 benigne et praeclare recensere voluit. Observat ibi vir multis nominibus venerandus aequationem:

$$V\frac{dU}{dx}-U\frac{dV}{dx} = \frac{ABCD}{M} = \frac{T}{M},$$

cuius beneficio demonstratio conficitur, et quae nobis e principiis transformationis mere algebraicis sequebatur, etiam sine illis analytice probari posse. Quod cum ex ipsa viri clarissimi sententia egregiam theorematі nostro lucem affundat, praeeunte illo, paucis hunc in modum demonstremus.

Aequationem propositam:

$$V\frac{dU}{dx}-U\frac{dV}{dx} = \frac{ABCD}{M} = \frac{T}{M}$$

ita quoque exhibere licet:

$$\frac{dU}{Udx}-\frac{dV}{Vdx} = \frac{d\log U}{dx}-\frac{d\log V}{dx} = \frac{ABCD}{MUV} = \frac{T}{MUV}.$$

Invenimus autem:

$$U = \frac{x}{M}\left(1-\frac{x^2}{\sin^2\operatorname{am}2\omega}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}4\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}(n-1)\omega}\right)$$

$$V = [1-k^2x^2\sin^2\operatorname{am}2\omega][1-k^2x^2\sin^2\operatorname{am}4\omega]\cdots[1-k^2x^2\sin^2\operatorname{am}(n-1)\omega],$$

unde:

$$\frac{d\log U}{dx}-\frac{d\log V}{dx} = \frac{1}{x}+\Sigma\left\{\frac{-2x}{\sin^2\operatorname{am}2q\omega-x^2}+\frac{2k^2x\sin^2\operatorname{am}2q\omega}{1-k^2x^2\sin^2\operatorname{am}2q\omega}\right\},$$

numero $q$ in summa designata tributis valoribus $1, 2, 3, \ldots, \frac{n-1}{2}$. Porro invenimus:

$$AB = \left(1-\frac{x^2}{\sin^2 \operatorname{coam} 2\omega}\right)\left(1-\frac{x^2}{\sin^2 \operatorname{coam} 4\omega}\right)\cdots\left(1-\frac{x^2}{\sin^2 \operatorname{coam}(n-1)\omega}\right).$$

$$CD = [1-k^2x^2\sin^2\operatorname{coam} 2\omega][1-k^2x^2\sin^2\operatorname{coam} 4\omega]\cdots[1-k^2x^2\sin^2\operatorname{coam}(n-1)\omega],$$

unde:

$$\frac{T}{MUV} = \frac{ABCD}{MUV} = \frac{x\prod\left(1-\frac{x^2}{\sin^2\operatorname{coam} 2p\omega}\right)(1-k^2x^2\sin^2\operatorname{coam} 2p\omega)}{x^2\prod\left(1-\frac{x^2}{\sin^2\operatorname{am} 2p\omega}\right)(1-k^2x^2\sin^2\operatorname{am} 2p\omega)},$$

siquidem in productis, brevitatis causa praefixo signo $\Pi$ denotatis, elemento $p$ valores tribuuntur $1, 2, 3, \ldots, \frac{n-1}{2}$. Hanc expressionem in fractiones simplices discerpere licet, ita ut formam induat:

$$\frac{1}{x}+\sum\left(\frac{A^{(q)}x}{\sin^2\operatorname{am} 2q\omega - x^2}+\frac{B^{(q)}x}{1-k^2x^2\sin^2\operatorname{am} 2q\omega}\right),$$

quo facto ut evictum habeamus, quod propositum est, demonstrari debet fore:

$$A^{(q)} = -2, \quad B^{(q)} = 2k^2\sin^2\operatorname{am} 2q\omega.$$

Denotabimus in sequentibus praefixo signo $\Pi^{(q)}$ productum ita formatum, ut elemento $p$ valores tribuantur $1, 2, 3, \ldots, \frac{n-1}{2}$, omisso tamen valore $p = q$. Hinc e praeceptis fractionum simplicium theoriae abunde notis sequitur:

$$A^{(q)} = (1-k^2\sin^2\operatorname{am} 2q\omega\sin^2\operatorname{coam} 2q\omega)\frac{\prod\left(\dfrac{1-\dfrac{\sin^2\operatorname{am} 2q\omega}{\sin^2\operatorname{coam} 2p\omega}}{1-k^2\sin^2\operatorname{am} 2q\omega\sin^2\operatorname{am} 2p\omega}\right)}{\prod^{(q)}\left(\dfrac{1-\dfrac{\sin^2\operatorname{am} 2q\omega}{\sin^2\operatorname{am} 2p\omega}}{1-k^2\sin^2\operatorname{am} 2q\omega\sin^2\operatorname{coam} 2p\omega}\right)}.$$

Iam e formulis supra a nobis exhibitis fit:

$$\frac{1-\dfrac{\sin^2\operatorname{am} 2q\omega}{\sin^2\operatorname{coam} 2p\omega}}{1-k^2\sin^2\operatorname{am} 2q\omega\sin^2\operatorname{am} 2p\omega} = \frac{\cos\operatorname{am}(2q+2p)\omega\cos\operatorname{am}(2q-2p)\omega}{\cos^2\operatorname{am} 2p\omega}$$

$$\frac{1-\dfrac{\sin^2\operatorname{am} 2q\omega}{\sin^2\operatorname{am} 2p\omega}}{1-k^2\sin^2\operatorname{am} 2q\omega\sin^2\operatorname{coam} 2p\omega} = \frac{\cos\operatorname{coam}(2p+2q)\omega\cos\operatorname{coam}(2p-2q)\omega}{\cos^2\operatorname{coam} 2p\omega}.$$

Facile autem patet, sublatis qui in denominatore et numeratore iidem inveniuntur factoribus, fieri:

$$\Pi \frac{\cos \operatorname{am}(2q+2p)\omega \cos \operatorname{am}(2q-2p)\omega}{\cos^2 \operatorname{am} 2p\omega} = \frac{\pm 1}{\cos \operatorname{am} 2q\omega}$$

$$\Pi^{(q)} \frac{\cos \operatorname{coam}(2p+2q)\omega \cos \operatorname{coam}(2p-2q)\omega}{\cos^2 \operatorname{coam} 2p\omega} = \frac{\mp 1}{\cos \operatorname{coam} 2q\omega} \cdot \frac{\cos^2 \operatorname{coam} 2q\omega}{\cos \operatorname{coam} 4q\omega} = \frac{\mp \cos \operatorname{coam} 2q\omega}{\cos \operatorname{coam} 4q\omega},$$

unde:

$$A^{(q)} = \frac{-(1-k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2q\omega) \cos \operatorname{coam} 4q\omega}{\cos \operatorname{am} 2q\omega \cos \operatorname{coam} 2q\omega}.$$

At e nota de duplicatione formula fit:

$$\begin{aligned} \cos \operatorname{coam} 4q\omega &= \frac{2k' \sin \operatorname{am} 2q\omega \cos \operatorname{am} 2q\omega \,\Delta \operatorname{am} 2q\omega}{1-2k^2 \sin^2 \operatorname{am} 2q\omega + k^2 \sin^4 \operatorname{am} 2q\omega} \\ &= \frac{2k' \sin \operatorname{am} 2q\omega \cos \operatorname{am} 2q\omega \,\Delta \operatorname{am} 2q\omega}{\Delta^2 \operatorname{am} 2q\omega - k^2 \sin^2 \operatorname{am} 2q\omega \cos^2 \operatorname{am} 2q\omega} \\ &= \frac{2 \cos \operatorname{am} 2q\omega \cos \operatorname{coam} 2q\omega}{1-k^2 \sin^2 \operatorname{am} 2q\omega \sin^2 \operatorname{coam} 2q\omega}, \end{aligned}$$

unde tandem, quod demonstrandum erat, $A^{(q)} = -2$. Prorsus simili modo alteram aequationem: $B^{(q)} = 2k^2 \sin^2 \operatorname{am} 2q\omega$ probare licet; quod tamen, iam invento $A^{(q)} = -2$, facilius ita fit.

Facile patet, loco $x$ posito $\frac{1}{kx}$, non mutari expressionem:

$$\Pi \frac{\left(1-\frac{x^2}{\sin^2 \operatorname{coam} 2p\omega}\right)(1-k^2x^2 \sin^2 \operatorname{coam} 2p\omega)}{(1-k^2x^2 \sin^2 \operatorname{am} 2p\omega)\left(1-\frac{x^2}{\sin^2 \operatorname{am} 2p\omega}\right)},$$

quam vidimus aequalem poni posse expressioni:

$$1 + \Sigma \frac{-2x^2}{\sin^2 \operatorname{am} 2q\omega - x^2} + \Sigma \frac{B^{(q)} x^2}{1-k^2x^2 \sin^2 \operatorname{am} 2q\omega}.$$

Haec autem expressio, posito $\frac{1}{kx}$ loco $x$, abit in hanc:

$$1 + \Sigma \frac{2}{1-k^2x^2 \sin^2 \operatorname{am} 2q\omega} + \Sigma \frac{-B^{(q)}}{k^2(\sin^2 \operatorname{am} 2q\omega - x^2)}$$

$$= 1 + \Sigma\left(2 - \frac{B^{(q)}}{k^2 \sin^2 \operatorname{am} 2q\omega}\right) + \Sigma \frac{2k^2x^2 \sin^2 \operatorname{am} 2q\omega}{1-k^2x^2 \sin^2 \operatorname{am} 2q\omega} + \Sigma \frac{-B^{(q)}}{k^2 \sin^2 \operatorname{am} 2q\omega} \cdot \frac{x^2}{\sin^2 \operatorname{am} 2q\omega - x^2},$$

unde ut immutata illa maneat, quod debet, fieri oportet:

$$B^{(q)} = 2k^2 \sin^2 \operatorname{am} 2q\omega.$$

Q. D. E.

23.

E formula (14.) §. 20. sequitur:

$$\sqrt{1-\lambda^2y^2} = \sqrt{1-k^2x^2}\,\frac{CD}{V}$$
$$= \sqrt{1-k^2x^2}\,\frac{[1-k^2x^2\sin^2\mathrm{coam}\,2\omega][1-k^2x^2\sin^2\mathrm{coam}\,4\omega]\cdots[1-k^2x^2\sin^2\mathrm{coam}(n-1)\omega]}{[1-k^2x^2\sin^2\mathrm{am}\,2\omega]\,[1-k^2x^2\sin^2\mathrm{am}\,4\omega]\cdots[1-k^2x^2\sin^2\mathrm{am}\,(n-1)\omega]}.$$

Posito $x = 1$, unde etiam $y = 1$, ac $\sqrt{1-\lambda^2} = \lambda'$, fit:

$$\lambda' = k'\left\{\frac{\Delta\,\mathrm{coam}\,2\omega\,\Delta\,\mathrm{coam}\,4\omega\cdots\Delta\,\mathrm{coam}\,(n-1)\omega}{\Delta\,\mathrm{am}\,2\omega\,\Delta\,\mathrm{am}\,4\omega\cdots\Delta\,\mathrm{am}\,(n-1)\omega}\right\}^2.$$

Iam vero est:

$$\Delta\,\mathrm{coam}\,u = \frac{k'}{\Delta\,\mathrm{am}\,u},$$

unde:

$$(1.)\qquad \lambda' = \frac{k'^n}{[\Delta\,\mathrm{am}\,2\omega\,\Delta\,\mathrm{am}\,4\omega\cdots\Delta\,\mathrm{am}\,(n-1)\omega]^4}.$$

Porro in usum vocatis formulis:

$$(2.)\qquad \lambda = k^n[\sin\mathrm{coam}\,2\omega\,\sin\mathrm{coam}\,4\omega\cdots\sin\mathrm{coam}\,(n-1)\omega]^4$$

$$(3.)\qquad M = (-1)^{\frac{n-1}{2}}\frac{[\sin\mathrm{coam}\,2\omega\,\sin\mathrm{coam}\,4\omega\cdots\sin\mathrm{coam}\,(n-1)\omega]^2}{[\sin\mathrm{am}\,2\omega\,\sin\mathrm{am}\,4\omega\cdots\sin\mathrm{am}\,(n-1)\omega]^2},$$

nanciscimur:

$$(4.)\qquad \frac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\frac{\lambda}{k^n}} = [\sin\mathrm{am}\,2\omega\,\sin\mathrm{am}\,4\omega\cdots\sin\mathrm{am}\,(n-1)\omega]^2$$

$$(5.)\qquad \sqrt{\frac{\lambda k'^n}{\lambda' k^n}} = [\cos\mathrm{am}\,2\omega\,\cos\mathrm{am}\,4\omega\cdots\cos\mathrm{am}\,(n-1)\omega]^2$$

$$(6.)\qquad \sqrt{\frac{k'^n}{\lambda'}} = [\Delta\,\mathrm{am}\,2\omega\,\Delta\,\mathrm{am}\,4\omega\cdots\Delta\,\mathrm{am}\,(n-1)\omega]^2$$

$$(7.)\qquad \frac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\frac{\lambda'}{k'^n}} = [\mathrm{tg}\,\mathrm{am}\,2\omega\,\mathrm{tg}\,\mathrm{am}\,4\omega\cdots\mathrm{tg}\,\mathrm{am}\,(n-1)\omega]^2$$

$$(8.)\qquad \sqrt{\frac{\lambda}{k^n}} = [\sin\mathrm{coam}\,2\omega\,\sin\mathrm{coam}\,4\omega\cdots\sin\mathrm{coam}\,(n-1)\omega]^2$$

(9.) $$\frac{(-1)^{\frac{n-1}{2}}}{M}\sqrt{\frac{\lambda\lambda' k'^{n-2}}{k^n}} = [\cos\operatorname{coam} 2\omega \cos\operatorname{coam} 4\omega \cdots \cos\operatorname{coam}(n-1)\omega]^2$$

(10.) $$\sqrt{\lambda' k'^{n-2}} = [\Delta\operatorname{coam} 2\omega\, \Delta\operatorname{coam} 4\omega \cdots \Delta\operatorname{coam}(n-1)\omega]^2$$

(11.) $$(-1)^{\frac{n-1}{2}} M\sqrt{\frac{1}{\lambda' k'^{n-2}}} = [\operatorname{tg}\operatorname{coam} 2\omega\, \operatorname{tg}\operatorname{coam} 4\omega \cdots \operatorname{tg}\operatorname{coam}(n-1)\omega]^2.$$

Harum formularum ope formulae (8.), (13.), (14.), §. 20. in sequentes abeunt:

(12.) $$\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{k^n}{\lambda}} \sin\operatorname{am} u \sin\operatorname{am}(u+4\omega)\sin\operatorname{am}(u+8\omega)\cdots\sin\operatorname{am}(u+4(n-1)\omega)$$

(13.) $$\cos\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{\lambda' k^n}{\lambda k'^n}} \cos\operatorname{am} u \cos\operatorname{am}(u+4\omega)\cos\operatorname{am}(u+8\omega)\cdots\cos\operatorname{am}(u+4(n-1)\omega)$$

(14.) $$\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{\lambda'}{k'^n}}\, \Delta\operatorname{am} u\, \Delta\operatorname{am}(u+4\omega)\, \Delta\operatorname{am}(u+8\omega)\cdots\Delta\operatorname{am}(u+4(n-1)\omega),$$

unde etiam:

(15.) $$\operatorname{tg}\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{k'^n}{\lambda'}}\, \operatorname{tg}\operatorname{am} u\, \operatorname{tg}\operatorname{am}(u+4\omega)\, \operatorname{tg}\operatorname{am}(u+8\omega)\cdots\operatorname{tg}\operatorname{am}(u+4(n-1)\omega).$$

Aliud ita invenitur formularum systema. Ex aequatione (4.) sequitur:

$$\frac{\lambda}{M^2 k^n} = [\sin\operatorname{am} 2\omega \sin\operatorname{am} 4\omega \cdots \sin\operatorname{am}(n-1)\omega]^4,$$

unde:

$$y = \sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = \frac{x}{M}\prod \frac{1-\dfrac{x^2}{\sin^2\operatorname{am} 2p\omega}}{1-k^2x^2\sin^2\operatorname{am} 2p\omega} = \frac{kM}{\lambda}\, x\prod \frac{x^2-\sin^2\operatorname{am} 2p\omega}{x^2-\dfrac{1}{k^2\sin^2\operatorname{am} 2p\omega}},$$

sive:

$$0 = x\prod(x^2-\sin^2\operatorname{am} 2p\omega) - \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)\prod\left(x^2-\frac{1}{k^2\sin^2\operatorname{am} 2p\omega}\right).$$

Radices huius aequationis $n^{\text{ti}}$ ordinis sunt:

$$x = \sin\operatorname{am} u,\quad \sin\operatorname{am}(u+4\omega),\quad \sin\operatorname{am}(u+8\omega),\ldots,\quad \sin\operatorname{am}(u+4(n-1)\omega),$$

unde aequationem nanciscimur identicam:

$$x\prod(x^2-\sin^2\operatorname{am} 2p\omega) - \frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)\prod\left(x^2-\frac{1}{k^2\sin^2\operatorname{am} 2p\omega}\right)$$
$$= [x-\sin\operatorname{am} u][x-\sin\operatorname{am}(u+4\omega)][x-\sin\operatorname{am}(u+8\omega)]\cdots[x-\sin\operatorname{am}(u+4(n-1)\omega)].$$

Hinc prodit summa radicum:

(16.) $$\sum \sin \operatorname{am}(u+4q\omega) = \frac{\lambda}{kM} \sin \operatorname{am}\left(\frac{u}{M}, \lambda\right).$$

Eodem modo invenitur:

(17.) $$\sum \cos \operatorname{am}(u+4q\omega) = \frac{(-1)^{\frac{n-1}{2}}\lambda}{kM} \cos \operatorname{am}\left(\frac{u}{M}, \lambda\right)$$

(18.) $$\sum \Delta \operatorname{am}(u+4q\omega) = \frac{(-1)^{\frac{n-1}{2}}}{M} \Delta \operatorname{am}\left(\frac{u}{M}, \lambda\right)$$

(19.) $$\sum \operatorname{tg} \operatorname{am}(u+4q\omega) = \frac{\lambda'}{k'M} \operatorname{tg} \operatorname{am}\left(\frac{u}{M}, \lambda\right),$$

in quibus formulis numero $q$ tribuuntur valores $0, 1, 2, 3, \ldots, n-1$. Quas formulas etiam hunc in modum repraesentare convenit:

$$\frac{\lambda}{kM} \sin \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \sin \operatorname{am} u + \sum [\sin \operatorname{am}(u+4q\omega) + \sin \operatorname{am}(u-4q\omega)]$$

$$\frac{(-1)^{\frac{n-1}{2}}\lambda}{kM} \cos \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \cos \operatorname{am} u + \sum [\cos \operatorname{am}(u+4q\omega) + \cos \operatorname{am}(u-4q\omega)]$$

$$\frac{(-1)^{\frac{n-1}{2}}}{M} \Delta \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \Delta \operatorname{am} u + \sum [\Delta \operatorname{am}(u+4q\omega) + \Delta \operatorname{am}(u-4q\omega)]$$

$$\frac{\lambda'}{k'M} \operatorname{tg} \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \operatorname{tg} \operatorname{am} u + \sum [\operatorname{tg} \operatorname{am}(u+4q\omega) + \operatorname{tg} \operatorname{am}(u-4q\omega)],$$

ubi numero $q$ tribuuntur valores $1, 2, 3, \ldots, \frac{n-1}{2}$. Iam adnotentur formulae:

$$\sin \operatorname{am}(u+4q\omega) + \sin \operatorname{am}(u-4q\omega) = \frac{2 \cos \operatorname{am} 4q\omega \, \Delta \operatorname{am} 4q\omega \sin \operatorname{am} u}{1-k^2 \sin^2 \operatorname{am} 4q\omega \sin^2 \operatorname{am} u}$$

$$\cos \operatorname{am}(u+4q\omega) + \cos \operatorname{am}(u-4q\omega) = \frac{2 \cos \operatorname{am} 4q\omega \cos \operatorname{am} u}{1-k^2 \sin^2 \operatorname{am} 4q\omega \sin^2 \operatorname{am} u}$$

$$\Delta \operatorname{am}(u+4q\omega) + \Delta \operatorname{am}(u-4q\omega) = \frac{2 \Delta \operatorname{am} 4q\omega \, \Delta \operatorname{am} u}{1-k^2 \sin^2 \operatorname{am} 4q\omega \sin^2 \operatorname{am} u}$$

$$\operatorname{tg} \operatorname{am}(u+4q\omega) + \operatorname{tg} \operatorname{am}(u-4q\omega) = \frac{2 \Delta \operatorname{am} 4q\omega \sin \operatorname{am} u \cos \operatorname{am} u}{\cos^2 \operatorname{am} 4q\omega - \Delta^2 \operatorname{am} 4q\omega \sin^2 \operatorname{am} u} \text{*)},$$

quarum ope formulae (16.) — (19.) in has abeunt:

*) cf. §. 18. formulas (1.), (2.), (3.); formula postrema e formulis (10.), (30.) fluit, ubi reputas esse $\operatorname{tg}\sigma + \operatorname{tg}\vartheta = \frac{\sin(\sigma+\vartheta)}{\cos\sigma\cos\vartheta}$.

$$(20.)\qquad \frac{\lambda}{kM}\sin\mathrm{am}\left(\frac{u}{M},\lambda\right) = \sin\mathrm{am}\,u + \sum\frac{2\cos\mathrm{am}\,4q\omega\,\Delta\,\mathrm{am}\,4q\omega\,\sin\mathrm{am}\,u}{1-k^2\sin^2\mathrm{am}\,4q\omega\,\sin^2\mathrm{am}\,u}$$

$$(21.)\qquad \frac{(-1)^{\frac{n-1}{2}}\lambda}{kM}\cos\mathrm{am}\left(\frac{u}{M},\lambda\right) = \cos\mathrm{am}\,u + \sum\frac{2\cos\mathrm{am}\,4q\omega\,\cos\mathrm{am}\,u}{1-k^2\sin^2\mathrm{am}\,4q\omega\,\sin^2\mathrm{am}\,u}$$

$$(22.)\qquad \frac{(-1)^{\frac{n-1}{2}}}{M}\Delta\,\mathrm{am}\left(\frac{u}{M},\lambda\right) = \Delta\,\mathrm{am}\,u + \sum\frac{2\Delta\,\mathrm{am}\,4q\omega\,\Delta\,\mathrm{am}\,u}{1-k^2\sin^2\mathrm{am}\,4q\omega\,\sin^2\mathrm{am}\,u}$$

$$(23.)\qquad \frac{\lambda'}{k'M}\mathrm{tg}\,\mathrm{am}\left(\frac{u}{M},\lambda\right) = \mathrm{tg}\,\mathrm{am}\,u + \sum\frac{2\Delta\,\mathrm{am}\,4q\omega\,\sin\mathrm{am}\,u\cos\mathrm{am}\,u}{\cos^2\mathrm{am}\,4q\omega - \Delta^2\mathrm{am}\,4q\omega\,\sin^2\mathrm{am}\,u},$$

quae etiam obtinentur, ubi formulae supra propositae e methodis notis in fractiones simplices resolvuntur.

## DE VARIIS EIUSDEM ORDINIS TRANSFORMATIONIBUS. TRANSFORMATIONES DUAE REALES, MAIORIS MODULI IN MINOREM ET MINORIS IN MAIOREM.

24.

Elemento $\omega$ vidimus tribui posse valorem quemlibet schematis $\frac{mK+m'iK'}{n}$, designantibus $m, m'$ numeros integros positivos seu negativos, qui tamen, quoties $n$ est numerus compositus, nullum ipsius $n$ factorem communem habent. Facile autem patet, ubi $q$ sit primus ad $n$, valores $\omega = \frac{qmK+qm'iK'}{n}$ substitutiones diversas non exhibituros esse. Hinc ubi ipse $n$ est numerus primus, valores elementi $\omega$, qui transformationes diversas suppeditant, erunt omnes:

$$\frac{K}{n},\quad \frac{iK'}{n},\quad \frac{K+iK'}{n},\quad \frac{K+2iK'}{n},\quad \frac{K+3iK'}{n},\quad \dots,\quad \frac{K+(n-1)iK'}{n},$$

sive etiam:

$$\frac{K}{n},\quad \frac{iK'}{n},\quad \frac{K+iK'}{n},\quad \frac{2K+iK'}{n},\quad \frac{3K+iK'}{n},\quad \dots,\quad \frac{(n-1)K+iK'}{n},$$

aut, si placet:

$$\frac{K}{n},\quad \frac{iK'}{n},\quad \frac{K\pm iK'}{n},\quad \frac{K\pm 2iK'}{n},\quad \frac{K\pm 3iK'}{n},\quad \dots,\quad \frac{K\pm\frac{n-1}{2}iK'}{n},$$

sive etiam:

$$\frac{K}{n},\quad \frac{iK'}{n},\quad \frac{K \pm iK'}{n},\quad \frac{2K \pm iK'}{n},\quad \frac{3K \pm iK'}{n}, \ldots, \frac{\frac{n-1}{2} K \pm iK'}{n},$$

quorum est numerus $n+1$. Ac reapse vidimus in transformationibus tertii et quinti ordinis, supra tamquam exemplis propositis, aequationes inter $u = \sqrt[4]{k}$ et $v = \sqrt[4]{\lambda}$, quas *aequationes modulares* nuncupabimus, resp. ad quartum et sextum gradum ascendisse. Quoties vero $n$ est numerus compositus, iste valde augetur numerus; accedunt enim casus, quibus sive $m$, sive $m'$ sive etiam uterque factorem habet cum $n$ communem, modo ne utrisque $m, m'$ idem communis sit cum $n$. Generaliter autem valet theorema:

»*numerum substitutionum $n^{ti}$ ordinis inter se diversarum, quarum ope transformare liceat functiones ellipticas, aequare summam factorum ipsius $n$, qui tamen numerus, quoties $n$ per quadratum dividitur, et substitutiones amplectitur ex transformatione et multiplicatione mixtas, adeoque, quoties $n$ ipsum est quadratum, ipsam multiplicationem.*«

Ista igitur factorum summa designabit gradum, ad quem pro dato numero $n$ aequatio modularis ascendet, ubi adnotandum est, quoties $n$ sit numerus quadratus, unam e radicum numero praebituram esse $k = \lambda$, ac generaliter, quoties $n = m^2 v$, designante $m^2$ quadratum maximum, per quod numerum $n$ dividere licet, e numero radicum fore etiam omnes radices aequationis modularis, quae ad ipsum $v$ pertinet.

Inter valores elementi $\omega$ supra propositos, qui casu, quo $n$ est primus, quem, cum in eum reliqui redeant, sive unice sive prae ceteris considerare convenit, universam transformationum copiam suggerunt, duo tantum generaliter loquendo *) inveniuntur, qui transformationes reales suppeditant, hos dico $\omega = \frac{K}{n}$, $\omega' = \frac{iK'}{n}$. Illam in sequentibus vocabimus transformationem *primam*, hanc *secundam;* modulosque, qui his respondent, designabimus resp. per $\lambda$, $\lambda_1$ eorumque complementa per $\lambda'$, $\lambda'_1$. Argumenta amplitudinis $\frac{\pi}{2}$, quae his modulis respondent, (functiones integras vocat Cl. Legendre), designabimus per $\Lambda$, $\Lambda_1$, $\Lambda'$, $\Lambda'_1$. Formulae nostrae generales pro his casibus evadunt sequentes.

---

*) Nam infinitis casibus pro modulis specialibus fit, ut par radicum imaginariarum aequationum modularium sibi aequale evadat ideoque reale fiat.

## I.

### FORMULAE PRO TRANSFORMATIONE REALI PRIMA MODULI $k$ IN MODULUM $\lambda$.

$$\lambda = k^n \left\{ \sin\operatorname{coam}\frac{2K}{n} \sin\operatorname{coam}\frac{4K}{n} \cdots\cdots \sin\operatorname{coam}\frac{(n-1)K}{n} \right\}^4$$

$$\lambda' = \frac{k'^n}{\left\{ \Delta\operatorname{am}\frac{2K}{n}\, \Delta\operatorname{am}\frac{4K}{n} \cdots\cdots \Delta\operatorname{am}\frac{(n-1)K}{n} \right\}^4}$$

$$\mathrm{M} = \left\{ \frac{\sin\operatorname{coam}\frac{2K}{n} \sin\operatorname{coam}\frac{4K}{n} \cdots\cdots \sin\operatorname{coam}\frac{(n-1)K}{n}}{\sin\operatorname{am}\frac{2K}{n}\; \sin\operatorname{am}\frac{4K}{n} \cdots\cdots \sin\operatorname{am}\frac{(n-1)K}{n}} \right\}^2$$

$$\sin\operatorname{am}\left(\frac{u}{\mathrm{M}}, \lambda\right) = \frac{\frac{\sin\operatorname{am} u}{\mathrm{M}} \left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{2K}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{4K}{n}}\right) \cdots \left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1 - k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am} u\right)\left(1 - k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am} u\right) \cdots \left(1 - k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am} u\right)}$$

$$= (-1)^{\frac{n-1}{2}} \sqrt{\frac{k^n}{\lambda}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4K}{n}\right) \sin\operatorname{am}\left(u + \frac{8K}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)K}{n}\right)$$

$$\cos\operatorname{am}\left(\frac{u}{\mathrm{M}}, \lambda\right) = \frac{\cos\operatorname{am} u \left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{2K}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{4K}{n}}\right) \cdots \left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{(n-1)K}{n}}\right)}{\left(1 - k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am} u\right)\left(1 - k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am} u\right) \cdots \left(1 - k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am} u\right)}$$

$$= \sqrt{\frac{\lambda' k^n}{\lambda k'^n}} \cos\operatorname{am} u \cos\operatorname{am}\left(u + \frac{4K}{n}\right) \cos\operatorname{am}\left(u + \frac{8K}{n}\right) \cdots \cos\operatorname{am}\left(u + \frac{4(n-1)K}{n}\right)$$

$$\Delta\operatorname{am}\left(\frac{u}{\mathrm{M}}, \lambda\right) = \frac{\Delta\operatorname{am} u \left(1 - k^2\sin^2\operatorname{coam}\frac{2K}{n}\sin^2\operatorname{am} u\right)\left(1 - k^2\sin^2\operatorname{coam}\frac{4K}{n}\sin^2\operatorname{am} u\right) \cdots \left(1 - k^2\sin^2\operatorname{coam}\frac{(n-1)K}{n}\sin^2\operatorname{am} u\right)}{\left(1 - k^2\sin^2\operatorname{am}\frac{2K}{n}\sin^2\operatorname{am} u\right)\left(1 - k^2\sin^2\operatorname{am}\frac{4K}{n}\sin^2\operatorname{am} u\right) \cdots \left(1 - k^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\sin^2\operatorname{am} u\right)}$$

$$= \sqrt{\frac{\lambda'}{k'^n}} \Delta\operatorname{am} u\, \Delta\operatorname{am}\left(u + \frac{4K}{n}\right) \Delta\operatorname{am}\left(u + \frac{8K}{n}\right) \cdots \Delta\operatorname{am}\left(u + \frac{4(n-1)K}{n}\right)$$

$$\sqrt{\frac{1\mp\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}{1\pm\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}}$$

$$=\sqrt{\frac{1-\sin\operatorname{am}u}{1+\sin\operatorname{am}u}}\cdot\frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4K}{n}}\right)\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{8K}{n}}\right)\cdots\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2(n-1)K}{n}}\right)}{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4K}{n}}\right)\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{8K}{n}}\right)\cdots\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2(n-1)K}{n}}\right)}$$

$$\sqrt{\frac{1\mp\lambda\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}{1\pm\lambda\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)}}$$

$$=\sqrt{\frac{1-k\sin\operatorname{am}u}{1+k\sin\operatorname{am}u}}\cdot\frac{\left(1-k\sin\operatorname{coam}\frac{4K}{n}\sin\operatorname{am}u\right)\left(1-k\sin\operatorname{coam}\frac{8K}{n}\sin\operatorname{am}u\right)\cdots\left(1-k\sin\operatorname{coam}\frac{2(n-1)K}{n}\sin\operatorname{am}u\right)}{\left(1+k\sin\operatorname{coam}\frac{4K}{n}\sin\operatorname{am}u\right)\left(1+k\sin\operatorname{coam}\frac{8K}{n}\sin\operatorname{am}u\right)\cdots\left(1+k\sin\operatorname{coam}\frac{2(n-1)K}{n}\sin\operatorname{am}u\right)}$$

$$\frac{\lambda}{kM}\sin\operatorname{am}\left(\frac{u}{M},\lambda\right)=\sin\operatorname{am}u+2\sum\frac{(-1)^q\cos\operatorname{am}\frac{2qK}{n}\,\Delta\operatorname{am}\frac{2qK}{n}\sin\operatorname{am}u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}$$

$$\frac{\lambda}{kM}\cos\operatorname{am}\left(\frac{u}{M},\lambda\right)=\cos\operatorname{am}u+2\sum\frac{(-1)^q\cos\operatorname{am}\frac{2qK}{n}\cos\operatorname{am}u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}$$

$$\frac{1}{M}\,\Delta\operatorname{am}\left(\frac{u}{M},\lambda\right)=\Delta\operatorname{am}u+2\sum\frac{\Delta\operatorname{am}\frac{2qK}{n}\,\Delta\operatorname{am}u}{1-k^2\sin^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}$$

$$\frac{\lambda'}{k'M}\operatorname{tg}\operatorname{am}\left(\frac{u}{M},\lambda\right)=\operatorname{tg}\operatorname{am}u+2\sum\frac{\Delta\operatorname{am}\frac{2qK}{n}\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}\frac{2qK}{n}-\Delta^2\operatorname{am}\frac{2qK}{n}\sin^2\operatorname{am}u}.$$

## II.

*A.* FORMULAE PRO TRANSFORMATIONE REALI SECUNDA, MODULI $k$ IN MODULUM $\lambda_1$, SUB FORMA IMAGINARIA.

$$\lambda_1 = k^n \left\{ \sin\operatorname{coam}\frac{2iK'}{n} \sin\operatorname{coam}\frac{4iK'}{n} \cdots\cdots \sin\operatorname{coam}\frac{(n-1)iK'}{n} \right\}^4$$

$$\lambda_1' = \frac{k'^n}{\left\{ \Delta\operatorname{am}\frac{2iK'}{n}\, \Delta\operatorname{am}\frac{4iK'}{n} \cdots\cdots \Delta\operatorname{am}\frac{(n-1)iK'}{n} \right\}^4}$$

$$\mathrm{M}_1 = (-1)^{\frac{n-1}{2}} \left\{ \frac{\sin\operatorname{coam}\frac{2iK'}{n} \sin\operatorname{coam}\frac{4iK'}{n} \cdots \sin\operatorname{coam}\frac{(n-1)iK'}{n}}{\sin\operatorname{am}\frac{2iK'}{n}\ \sin\operatorname{am}\frac{4iK'}{n} \cdots \sin\operatorname{am}\frac{(n-1)iK'}{n}} \right\}^2$$

$$\sin\operatorname{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right) = \frac{\frac{\sin\operatorname{am} u}{\mathrm{M}_1}\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{2iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{4iK'}{n}}\right)\cdots\cdot\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{(n-1)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{3iK'}{n}}\right)\cdots\cdot\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{k^n}{\lambda_1}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4iK'}{n}\right) \sin\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots\cdot \sin\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\cos\operatorname{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right) = \frac{\cos\operatorname{am} u\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{2iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{4iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{(n-1)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{\lambda_1' k^n}{\lambda_1 k'^n}} \cos\operatorname{am} u \cos\operatorname{am}\left(u + \frac{4iK'}{n}\right) \cos\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots\cdot \cos\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\Delta\operatorname{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right) = \frac{\Delta\operatorname{am} u\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{(n-2)iK'}{n}}\right)}{\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{iK'}{n}}\right)\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{3iK'}{n}}\right)\cdots\left(1 - \frac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{(n-2)iK'}{n}}\right)}$$

$$= \sqrt{\frac{\lambda_1'}{k'^n}} \Delta\operatorname{am} u\, \Delta\operatorname{am}\left(u + \frac{4iK'}{n}\right) \Delta\operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots\cdot \Delta\operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right)$$

$$\sqrt{\frac{1-\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}{1+\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}}$$

$$=\sqrt{\frac{1-\sin\operatorname{am}u}{1+\sin\operatorname{am}u}}\cdot\frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2iK'}{n}}\right)\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4iK'}{n}}\right)\cdots\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{(n-1)iK'}{n}}\right)}{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{2iK'}{n}}\right)\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{4iK'}{n}}\right)\cdots\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{(n-1)iK'}{n}}\right)}$$

$$\sqrt{\frac{1-\lambda_1\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}{1+\lambda_1\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)}}$$

$$=\sqrt{\frac{1-k\sin\operatorname{am}u}{1+k\sin\operatorname{am}u}}\cdot\frac{\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{iK'}{n}}\right)\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{3iK'}{n}}\right)\cdots\left(1-\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{(n-2)iK'}{n}}\right)}{\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{iK'}{n}}\right)\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{3iK'}{n}}\right)\cdots\left(1+\frac{\sin\operatorname{am}u}{\sin\operatorname{coam}\frac{(n-2)iK'}{n}}\right)}$$

$$\frac{\lambda_1}{kM_1}\sin\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)=\sin\operatorname{am}u-\frac{2}{k}\sum\frac{\cos\operatorname{am}\frac{(2q-1)iK'}{n}\,\Delta\operatorname{am}\frac{(2q-1)iK'}{n}\sin\operatorname{am}u}{\sin^2\operatorname{am}\frac{(2q-1)iK'}{n}-\sin^2\operatorname{am}u}$$

$$\frac{(-1)^{\frac{n-1}{2}}\lambda_1}{kM_1}\cos\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)=\cos\operatorname{am}u+\frac{2(-1)^{\frac{n-1}{2}}}{ik}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)iK'}{n}\,\Delta\operatorname{am}\frac{(2q-1)iK'}{n}\cos\operatorname{am}u}{\sin^2\operatorname{am}\frac{(2q-1)iK'}{n}-\sin^2\operatorname{am}u}$$

$$\frac{(-1)^{\frac{n-1}{2}}}{M_1}\Delta\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)=\Delta\operatorname{am}u+\frac{2(-1)^{\frac{n-1}{2}}}{i}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)iK'}{n}\cos\operatorname{am}\frac{(2q-1)iK'}{n}\,\Delta\operatorname{am}u}{\sin^2\operatorname{am}\frac{(2q-1)iK'}{n}-\sin^2\operatorname{am}u}$$

$$\frac{\lambda_1'}{k'M_1}\operatorname{tg}\operatorname{am}\left(\frac{u}{M_1},\lambda_1\right)=\operatorname{tg}\operatorname{am}u+2\sum\frac{(-1)^q\,\Delta\operatorname{am}\frac{2qiK'}{n}\sin\operatorname{am}u\cos\operatorname{am}u}{\cos^2\operatorname{am}\frac{2qiK'}{n}-\Delta^2\operatorname{am}\frac{2qiK'}{n}\sin^2\operatorname{am}u}.$$

*B.* FORMULAE PRO TRANSFORMATIONE REALI SECUNDA SUB FORMA REALI.

$$\lambda_1 = \frac{k^n}{\left\{\Delta\,\mathrm{am}\left(\frac{2K'}{n}, k'\right)\Delta\,\mathrm{am}\left(\frac{4K'}{n}, k'\right)\cdots\Delta\,\mathrm{am}\left(\frac{(n-1)K'}{n}, k'\right)\right\}^4}$$

$$\lambda_1' = k'^n\left\{\sin\mathrm{coam}\left(\frac{2K'}{n}, k'\right)\sin\mathrm{coam}\left(\frac{4K'}{n}, k'\right)\cdots\sin\mathrm{coam}\left(\frac{(n-1)K'}{n}, k'\right)\right\}^4$$

$$\mathrm{M}_1 = \left\{\frac{\sin\mathrm{coam}\left(\frac{2K'}{n}, k'\right)\sin\mathrm{coam}\left(\frac{4K'}{n}, k'\right)\cdots\sin\mathrm{coam}\left(\frac{(n-1)K'}{n}, k'\right)}{\sin\mathrm{am}\left(\frac{2K'}{n}, k'\right)\sin\mathrm{am}\left(\frac{4K'}{n}, k'\right)\cdots\sin\mathrm{am}\left(\frac{(n-1)K'}{n}, k'\right)}\right\}^2$$

$$\sin\mathrm{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right) = \frac{\frac{\sin\mathrm{am}\,u}{\mathrm{M}_1}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{2K'}{n}, k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{4K'}{n}, k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{(n-1)K'}{n}, k'\right)}\right\}}{\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{K'}{n}, k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{3K'}{n}, k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{(n-2)K'}{n}, k'\right)}\right\}}$$

$$\cos\mathrm{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right) = \frac{\cos\mathrm{am}u\left\{1-\sin^2\mathrm{am}u\Delta^2\mathrm{am}\left(\frac{2K'}{n}, k'\right)\right\}\left\{1-\sin^2\mathrm{am}u\Delta^2\mathrm{am}\left(\frac{4K'}{n}, k'\right)\right\}\cdot\cdot\left\{1-\sin^2\mathrm{am}u\Delta^2\mathrm{am}\left(\frac{(n-1)K'}{n}, k'\right)\right\}}{\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{K'}{n}, k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{3K'}{n}, k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{(n-2)K'}{n}, k'\right)}\right\}}$$

$$\Delta\,\mathrm{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right) = \frac{\Delta\,\mathrm{am}u\left\{1-\sin^2\mathrm{am}u\,\Delta^2\mathrm{am}\left(\frac{K'}{n}, k'\right)\right\}\left\{1-\sin^2\mathrm{am}u\,\Delta^2\mathrm{am}\left(\frac{3K'}{n}, k'\right)\right\}\cdot\cdot\left\{1-\sin^2\mathrm{am}u\,\Delta^2\mathrm{am}\left(\frac{(n-2)K'}{n}, k'\right)\right\}}{\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{K'}{n}, k'\right)}\right\}\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{3K'}{n}, k'\right)}\right\}\cdots\left\{1+\frac{\sin^2\mathrm{am}\,u}{\mathrm{tg}^2\mathrm{am}\left(\frac{(n-2)K'}{n}, k'\right)}\right\}}$$

$$\sqrt{\frac{1-\sin\mathrm{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right)}{1+\sin\mathrm{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right)}}$$

$$= \sqrt{\frac{1-\sin\mathrm{am}u}{1+\sin\mathrm{am}u}}\cdot\frac{\left\{1-\sin\mathrm{am}u\Delta\mathrm{am}\left(\frac{2K'}{n}, k'\right)\right\}\left\{1-\sin\mathrm{am}u\,\Delta\mathrm{am}\left(\frac{4K'}{n}, k'\right)\right\}\cdot\cdot\left\{1-\sin\mathrm{am}u\,\Delta\mathrm{am}\left(\frac{(n-1)K'}{n}, k'\right)\right\}}{\left\{1+\sin\mathrm{am}u\Delta\mathrm{am}\left(\frac{2K'}{n}, k'\right)\right\}\left\{1+\sin\mathrm{am}u\,\Delta\mathrm{am}\left(\frac{4K'}{n}, k'\right)\right\}\cdot\cdot\left\{1+\sin\mathrm{am}u\,\Delta\mathrm{am}\left(\frac{(n-1)K'}{n}, k'\right)\right\}}$$

$$\sqrt{\frac{1-\lambda_1 \sin\mathrm{am}\left(\frac{u}{M_1}, \lambda_1\right)}{1+\lambda_1 \sin\mathrm{am}\left(\frac{u}{M_1}, \lambda_1\right)}}$$

$$=\sqrt{\frac{1-k\sin\mathrm{am}u}{1+k\sin\mathrm{am}u}}\cdot\frac{\left\{1-\Delta\mathrm{am}\left(\frac{K'}{n},k'\right)\sin\mathrm{am}u\right\}\left\{1-\Delta\mathrm{am}\left(\frac{3K'}{n},k'\right)\sin\mathrm{am}u\right\}\cdots\left\{1-\Delta\mathrm{am}\left(\frac{(n-2)K'}{n},k'\right)\sin\mathrm{am}u\right\}}{\left\{1+\Delta\mathrm{am}\left(\frac{K'}{n},k'\right)\sin\mathrm{am}u\right\}\left\{1+\Delta\mathrm{am}\left(\frac{3K'}{n},k'\right)\sin\mathrm{am}u\right\}\cdots\left\{1+\Delta\mathrm{am}\left(\frac{(n-2)K'}{n},k'\right)\sin\mathrm{am}u\right\}}$$

$$\frac{\lambda_1}{kM_1}\sin\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)=\sin\mathrm{am}u+\frac{2}{k}\sum\frac{\Delta\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin\mathrm{am}u}{\sin^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\mathrm{am}u}$$

$$\frac{(-1)^{\frac{n-1}{2}}\lambda_1}{kM_1}\cos\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)=\cos\mathrm{am}u-\frac{2(-1)^{\frac{n-1}{2}}}{k}\sum\frac{(-1)^q\sin\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\Delta\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\cos\mathrm{am}u}{\sin^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\mathrm{am}u}$$

$$\frac{(-1)^{\frac{n-1}{2}}}{M_1}\Delta\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)=\Delta\mathrm{am}u-2(-1)^{\frac{n-1}{2}}\sum\frac{(-1)^q\sin\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\Delta\mathrm{am}u}{\sin^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)+\cos^2\mathrm{am}\left(\frac{(2q-1)K'}{n},k'\right)\sin^2\mathrm{am}u}$$

$$\frac{\lambda_1'}{k'M_1}\mathrm{tg}\,\mathrm{am}\left(\frac{u}{M_1},\lambda_1\right)=\mathrm{tg}\,\mathrm{am}u+2\sum\frac{(-1)^q\cos\mathrm{am}\left(\frac{2qK'}{n},k'\right)\Delta\mathrm{am}\left(\frac{2qK'}{n},k'\right)\sin\mathrm{am}u\cos\mathrm{am}u}{1-\Delta^2\mathrm{am}\left(\frac{2qK'}{n},k'\right)\sin^2\mathrm{am}u}.$$

---

In formulis pro transformatione prima positum est $(-1)^{\frac{n-1}{2}}M$ loco $M$. Formulas pro transformatione secunda dupliciter exhibere placuit, et sub forma imaginaria et sub forma reali, in quibus praeterea loco $k\sin\mathrm{am}\frac{2miK'}{n}$, $k\sin\mathrm{coam}\frac{2miK'}{n}$, etc. ubique scriptum est $\frac{-1}{\sin\mathrm{am}\frac{(n-2m)iK'}{n}}$, $\frac{1}{\sin\mathrm{coam}\frac{(n-2m)iK'}{n}}$, etc.: id quod, sicuti reductio in formam realem, ope formularum $§^i$. 19. facile transactum est. Ubi signum ambiguum $\pm$ positum est, alterum $+$ eligendum est, ubi $\frac{n-1}{2}$ est numerus par, alterum $-$, ubi $\frac{n-1}{2}$ est numerus impar; de signo $\mp$ contrarium valet. In summis praefixo $\Sigma$ designatis numero $q$ valores $1, 2, 3, \ldots, \frac{n-1}{2}$ tribuendi sunt.

E formulis pro transformatione prima propositis patet, quoties $u$ fiat successive:

$$0, \quad \frac{K}{n}, \quad \frac{2K}{n}, \quad \frac{3K}{n}, \quad \frac{4K}{n}, \quad \dots,$$

fore $\operatorname{am}\left(\frac{u}{M}, \lambda\right)$:

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad 2\pi, \quad \dots,$$

unde obtinemus:

$$\frac{K}{nM} = \Lambda.$$

Contra vero videmus in transformatione secunda, quoties $u$ fiat: $0, K, 2K, 3K, \dots$ sive $\operatorname{am} u$: $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots,$ fieri $\operatorname{am}\left(\frac{u}{M_1}, \lambda_1\right)$ et ipsam $= 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots,$ unde hoc casu:

$$\frac{K}{M_1} = \Lambda_1.$$

Ceterum e formulis pro modulis $\lambda, \lambda', \lambda_1, \lambda_1'$ exhibitis elucet, crescente $n$, modulos $\lambda, \lambda_1'$ rapide ad nihilum convergere, ideoque simul modulos $\lambda', \lambda_1$ proxime accedere ad unitatem. Itaque transformationem moduli primam dicere convenit *maioris in minorem*, secundam *minoris in maiorem*.

## DE TRANSFORMATIONIBUS COMPLEMENTARIIS SEU QUOMODO E TRANSFORMATIONE MODULI IN MODULUM ALIA DERIVATUR COMPLEMENTI IN COMPLEMENTUM.

25.

In formula supra inventa:

$$\operatorname{tg}\operatorname{am}\left(\frac{u}{M}, \lambda\right) = \sqrt{\frac{k'^n}{\lambda'}} \operatorname{tg}\operatorname{am} u \operatorname{tg}\operatorname{am}(u+4\omega) \operatorname{tg}\operatorname{am}(u+8\omega) \cdots \operatorname{tg}\operatorname{am}(u+4(n-1)\omega)$$

ponamus $u = iu'$, $\omega = i\omega'$, ita ut sit $\omega = \frac{mK+m'iK'}{n}$, $\omega' = \frac{m'K'-miK}{n}$. Iam vero est (§. 19.):

$$\operatorname{tg}\operatorname{am}(iu', k) = i \sin\operatorname{am}(u', k')$$
$$\operatorname{tg}\operatorname{am}(iu', \lambda) = i \sin\operatorname{am}(u', \lambda'),$$

unde formulam allegatam in sequentem abire videmus:

$$\sin\operatorname{am}\left(\frac{u'}{M},\lambda'\right)=(-1)^{\frac{n-1}{2}}\sqrt{\frac{k'^n}{\lambda'}}\sin\operatorname{am}u'\sin\operatorname{am}(u'+4\omega')\sin\operatorname{am}(u'+8\omega')\cdot\cdot\sin\operatorname{am}(u'+4(n-1)\omega')\quad(\text{mod. }k')$$

Porro invenimus formulas:

$$\lambda' = \frac{k'^n}{[\Delta\operatorname{am}2\omega\,\Delta\operatorname{am}4\omega\cdots\Delta\operatorname{am}(n-1)\omega]^4}$$

$$M = (-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\,\omega\cdots\sin\operatorname{coam}(n-1)\omega]^2}{[\sin\operatorname{am}2\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}(n-1)\omega]^2},$$

quae e formulis:

$$\Delta\operatorname{am}(iu,k) = \frac{1}{\sin\operatorname{coam}(u,k')}$$

$$\sin\operatorname{coam}(iu,k) = \frac{1}{\Delta\operatorname{am}(u,k')},$$

unde etiam sequitur:

$$\frac{\sin\operatorname{coam}(iu,k)}{\sin\operatorname{am}(iu,k)} = \frac{-i}{\operatorname{tg}\operatorname{am}(u,k')\,\Delta\operatorname{am}(u,k')} = \frac{-i\sin\operatorname{coam}(u,k')}{\sin\operatorname{am}(u,k')},$$

in sequentes abeunt:

$$\lambda' = k'^n[\sin\operatorname{coam}2\omega'\sin\operatorname{coam}4\omega'\cdots\sin\operatorname{coam}(n-1)\omega']^4\qquad(\text{mod. }k')$$

$$M = \frac{[\sin\operatorname{coam}2\omega'\sin\operatorname{coam}4\omega'\cdots\sin\operatorname{coam}(n-1)\omega']^2}{[\sin\operatorname{am}2\omega'\sin\operatorname{am}4\omega'\cdots\sin\operatorname{am}(n-1)\omega']^2}\qquad(\text{mod. }k')$$

His formulis comparatis cum illis, quae transformationi moduli $k$ in modulum $\lambda$ inserviunt:

$$\sin\operatorname{am}\left(\frac{u}{M},\lambda\right) = \sqrt{\frac{k^n}{\lambda}}\sin\operatorname{am}u\sin\operatorname{am}(u+4\omega)\sin\operatorname{am}(u+8\omega)\cdots\sin\operatorname{am}(u+4(n-1)\omega)$$

$$\lambda = k^n[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^4$$

$$M = (-1)^{\frac{n-1}{2}}\frac{[\sin\operatorname{coam}2\omega\sin\operatorname{coam}4\omega\cdots\sin\operatorname{coam}(n-1)\omega]^2}{[\sin\operatorname{am}2\omega\sin\operatorname{am}4\omega\cdots\sin\operatorname{am}(n-1)\omega]^2},$$

elucet theorema, quod maximi momenti censeri debet in theoria transformationis:

*Quaecunque de transformatione moduli $k$ in modulum $\lambda$ proponi possint formulae, easdem valere, mutato $k$ in $k'$, $\lambda$ in $\lambda'$, $\omega$ in $\omega' = \frac{\omega}{i}$, $M$ in $(-1)^{\frac{n-1}{2}}M$.* Transformationem autem complementi in complementum, dicto modo e transformatione proposita derivatam, dicemus ***transformationem complementariam.***

Facile patet, transformationum realium moduli $k$ transformationes reales moduli $k'$ complementarias esse, ita tamen ut primae moduli $k'$, secundae moduli $k$ prima moduli $k'$ complementaria sit. Ubi enim in theoremate modo proposito ponitur $\omega = \frac{\pm K}{n}$, $\omega = \frac{\pm iK'}{n}$, quod transformationibus moduli $k$ primae et secundae respondet, fit $\omega' = \frac{\omega}{i} = \frac{\mp iK}{n}$, $\omega' = \frac{\omega}{i} = \frac{\pm K'}{n}$, quod transformationibus moduli $k'$ respondet resp. secundae et primae. Nec non, cum crescente modulo decrescat complementum ac vice versa, transformatio moduli in modulum ubi est maioris in minorem, transformatio complementi in complementum seu transformatio complementaria minoris in maiorem esse debet ac vice versa. Videmus igitur, mutato $k$ in $k'$, abire $\lambda$ in $\lambda'_1$, $\lambda_1$ in $\lambda'$. Nec non multiplicator M, transformationi primae eiusque complementariae communis*), abibit in $M_1$, qui ad transformationem secundam eiusque complementariam pertinet, ac vice versa $M_1$ in M. Hinc e formulis supra inventis:

$$\Lambda = \frac{K}{n\mathrm{M}}, \quad \Lambda_1 = \frac{K}{\mathrm{M}_1}$$

sequuntur hae:

$$\Lambda'_1 = \frac{K'}{n\mathrm{M}_1}, \quad \Lambda' = \frac{K'}{\mathrm{M}},$$

unde proveniunt formulae summi momenti in hac theoria:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}; \quad \frac{\Lambda'_1}{\Lambda_1} = \frac{1}{n}\cdot\frac{K'}{K}.$$

Hae formulae genuinum transformationis propositae characterem constituunt, unde patet, bono iure singulas nos transformationes ad singulos numeros $n$ retulisse. Adnotabo, quoties $n$ sit numerus compositus $= n'n''$, e singulis radicibus realibus aequationum modularium seu e singulis modulis realibus, in quos datum modulum $k$ per substitutionem $n^{ti}$ ordinis transformare liceat, provenire aequationes huiusmodi:

$$\frac{\Lambda'}{\Lambda} = \frac{n'}{n''}\cdot\frac{K'}{K},$$

*) Hoc generaliter tantum neglecto signo valet; vidimus enim, quod in altera transformatione erat $M$, in complementaria esse $(-1)^{\frac{n-1}{2}} M$; at nostris casibus eo, quod in transformatione prima loco $M$ positum est $(-1)^{\frac{n-1}{2}}$ M (v. supra), signi ambiguitas tollitur, ita ut transformationibus realibus complementariis omnino idem sit multiplicator M.

quae singulis discerptionibus numeri $n$ in duos factores respondent. E quarum igitur numero, quoties $n$ est numerus quadratus, erit etiam haec:

$$\frac{\Lambda'}{\Lambda} = \frac{K'}{K}, \quad \text{unde } \lambda = k,$$

quae docet, casu quo $n$ est quadratum, e numero substitutionum esse unam, quae multiplicationem suppeditet.

## DE TRANSFORMATIONIBUS SUPPLEMENTARIIS AD MULTIPLICATIONEM.

26.

Revocemus formulas:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}, \qquad \frac{\Lambda_1'}{\Lambda_1} = \frac{1}{n}\cdot\frac{K'}{K},$$

quibus hunc in modum scriptis:

$$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}$$
$$\frac{K'}{K} = n\frac{\Lambda_1'}{\Lambda_1},$$

elucet, *eodem modo pendere modulum* $\lambda$ *a modulo* $k$ *atque modulum* $k$ *a modulo* $\lambda_1$, *sive eodem modo pendere modulum* $k$ *a modulo* $\lambda$ *atque modulum* $\lambda_1$ *a modulo* $k$. Itaque per transformationem primam seu maioris in minorem, qua $k$ in $\lambda$, transformabitur $\lambda_1$ in $k$; per transformationem secundam seu minoris in maiorem, qua $k$ in $\lambda_1$, transformabitur $\lambda$ in $k$. Itaque *post transformationem primam adhibita secunda seu post secundam adhibita prima, modulus* $k$ *in se redit, seu transformationes prima et secunda successive adhibitae, utro ordine placet, multiplicationem praebent.*

Vocemus $M'$ multiplicatorem, qui eodem modo a $\lambda$ pendet atque $M_1$ a $k$, $M_1$ multiplicatorem, qui eodem modo a $\lambda_1$ pendet atque $M$ a $k$, ita ut obtineantur aequationes:

$$\frac{dy}{\sqrt{(1-y^2)(1-\lambda^2y^2)}} = \frac{dx}{M\sqrt{(1-x^2)(1-k^2x^2)}}$$
$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{dy}{M'\sqrt{(1-y^2)(1-\lambda^2y^2)}},$$

quarum altera transformationi moduli $k$ in modulum $\lambda$ per transformationem primam, altera transformationi moduli $\lambda$ in modulum $k$ per transformationem secundam respondet. Ex his aequationibus provenit:

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{dx}{MM'\sqrt{(1-x^2)(1-k^2x^2)}}, \quad \text{unde } z = \sin\operatorname{am}\left(\frac{u}{MM'}\right).$$

At ex aequatione $\Lambda_1 = \frac{K}{M_1}$ mutando $k$ in $\lambda$, quo facto $K$ in $\Lambda$, $\lambda_1$ in $k$, $\Lambda_1$ in $K$, $M_1$ in $M'$ abit, obtinetur $K = \frac{\Lambda}{M'}$, qua aequatione comparata cum illa $\Lambda = \frac{K}{nM}$, provenit $\frac{1}{MM'} = n$, unde:

$$\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{n\,dx}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

Eodem modo ex aequatione $\Lambda = \frac{K}{nM}$ mutando $k$ in $\lambda_1$, quo facto $K$ in $\Lambda_1$, $\lambda$ in $k$, $\Lambda$ in $K$, $M_1$ in $M'_1$ abit, provenit $K = \frac{\Lambda_1}{nM'_1}$, qua aequatione comparata cum hac $\Lambda_1 = \frac{K}{M_1}$, provenit $\frac{1}{M_1M'_1} = n$; unde videmus, duobus illis casibus post binas transformationes successive adhibitas multiplicari argumentum per numerum $n$.

Ubi post transformationem moduli $k$ in modulum $\lambda$ modulus $\lambda$ rursus in modulum $k$ transformatur, ita ut multiplicatio proveniat, hanc transformationem illius *supplementariam ad multiplicationem* seu simpliciter *supplementariam* nuncupabimus.

Apponamus cum exempli causa tum in usum sequentium formulas pro transformatione *primae supplementaria* seu moduli $\lambda$ in modulum $k$, quae erit ipsius $\lambda$ secunda, eas tamen sub altera tantum forma imaginaria, cum reductio ad realem in promptu sit. Quas confestim obtinemus formulas, ubi in iis, quae supra de transformatione moduli $k$ secunda propositae sunt (v. tab. II. A. §. 24.), loco $k$ ponimus $\lambda$, $k$ loco $\lambda_1$, $\frac{u}{M}$ loco $u$, $M' = \frac{1}{nM}$ loco $M_1$, unde $\frac{u}{MM'} = nu$ loco $\frac{u}{M_1}$. In his formulis, sed in his tantum, modulus $\lambda$ valebit, nisi diserte adiectus sit modulus $k$; ceterum brevitatis causa positum est $y = \sin\operatorname{am}\left(\frac{u}{M}, \lambda\right)$; numero $q$, ut supra, tribuendi sunt valores:

$$1, 2, 3, \ldots, \tfrac{n-1}{2}.$$

# FORMULAE PRO TRANSFORMATIONE MODULI λ IN MODULUM $k$, SEU PRIMAE SUPPLEMENTARIA *).

27.

$$k = \lambda^n \left\{ \sin\operatorname{coam}\frac{2i\Lambda'}{n} \sin\operatorname{coam}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n} \right\}^4$$

$$k' = \frac{\lambda'^n}{\left\{ \Delta\operatorname{am}\frac{2i\Lambda'}{n} \Delta\operatorname{am}\frac{4i\Lambda'}{n} \cdots \Delta\operatorname{am}\frac{(n-1)i\Lambda'}{n} \right\}^4}$$

$$\frac{1}{nM} = (-1)^{\frac{n-1}{2}} \left\{ \frac{\sin\operatorname{coam}\frac{2i\Lambda'}{n} \sin\operatorname{coam}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}{\sin\operatorname{am}\frac{2i\Lambda'}{n} \sin\operatorname{am}\frac{4i\Lambda'}{n} \cdots \sin\operatorname{am}\frac{(n-1)i\Lambda'}{n}} \right\}^2$$

$$\sin\operatorname{am}(nu, k) = \frac{nMy\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{2i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{4i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)}$$

$$= \sqrt{\frac{\lambda^n}{k}} \sin\operatorname{am}\frac{u}{M} \sin\operatorname{am}\left(\frac{u}{M} + \frac{4i\Lambda'}{n}\right) \sin\operatorname{am}\left(\frac{u}{M} + \frac{8i\Lambda'}{n}\right) \cdots \sin\operatorname{am}\left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n}\right)$$

$$\cos\operatorname{am}(nu, k) = \frac{\sqrt{1-y^2}\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)}$$

$$= \sqrt{\frac{k'\lambda^n}{k\lambda'^n}} \cos\operatorname{am}\frac{u}{M} \cos\operatorname{am}\left(\frac{u}{M} + \frac{4i\Lambda'}{n}\right) \cos\operatorname{am}\left(\frac{u}{M} + \frac{8i\Lambda'}{n}\right) \cdots \cos\operatorname{am}\left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n}\right)$$

$$\Delta\operatorname{am}(nu, k) = \frac{\sqrt{1-\lambda^2 y^2}\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}{\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{i\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{3i\Lambda'}{n}}\right)\cdots\left(1 - \frac{y^2}{\sin^2\operatorname{am}\frac{(n-2)i\Lambda'}{n}}\right)}$$

$$= \sqrt{\frac{k'}{\lambda'^n}} \Delta\operatorname{am}\frac{u}{M} \Delta\operatorname{am}\left(\frac{u}{M} + \frac{4i\Lambda'}{n}\right) \Delta\operatorname{am}\left(\frac{u}{M} + \frac{8i\Lambda'}{n}\right) \cdots \Delta\operatorname{am}\left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n}\right)$$

*) In formulis huius paragraphi omnes functiones ellipticae, quibus modulus non adscriptus est, modulo λ gaudent. B.

$$\sqrt{\frac{1-\sin\operatorname{am}(nu,k)}{1+\sin\operatorname{am}(nu,k)}}=\sqrt{\frac{1-y}{1+y}}\cdot\frac{\left(1-\dfrac{y}{\sin\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1-\dfrac{y}{\sin\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1-\dfrac{y}{\sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}{\left(1+\dfrac{y}{\sin\operatorname{coam}\frac{2i\Lambda'}{n}}\right)\left(1+\dfrac{y}{\sin\operatorname{coam}\frac{4i\Lambda'}{n}}\right)\cdots\left(1+\dfrac{y}{\sin\operatorname{coam}\frac{(n-1)i\Lambda'}{n}}\right)}$$

$$\sqrt{\frac{1-k\sin\operatorname{am}(nu,k)}{1+k\sin\operatorname{am}(nu,k)}}=\sqrt{\frac{1-\lambda y}{1+\lambda y}}\cdot\frac{\left(1-\dfrac{y}{\sin\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1-\dfrac{y}{\sin\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1-\dfrac{y}{\sin\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}{\left(1+\dfrac{y}{\sin\operatorname{coam}\frac{i\Lambda'}{n}}\right)\left(1+\dfrac{y}{\sin\operatorname{coam}\frac{3i\Lambda'}{n}}\right)\cdots\left(1+\dfrac{y}{\sin\operatorname{coam}\frac{(n-2)i\Lambda'}{n}}\right)}$$

$$\sin\operatorname{am}(nu,k)=\frac{\lambda y}{kn\mathrm{M}}-\frac{2y}{kn\mathrm{M}}\sum\frac{\cos\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\,\Delta\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n}-y^2}$$

$$\cos\operatorname{am}(nu,k)=\frac{(-1)^{\frac{n-1}{2}}\lambda\sqrt{1-y^2}}{kn\mathrm{M}}+\frac{2\sqrt{1-y^2}}{ikn\mathrm{M}}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\,\Delta\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n}-y^2}$$

$$\Delta\operatorname{am}(nu,k)=\frac{(-1)^{\frac{n-1}{2}}}{n\mathrm{M}}\sqrt{1-\lambda^2y^2}+\frac{2\sqrt{1-\lambda^2y^2}}{in\mathrm{M}}\sum\frac{(-1)^q\sin\operatorname{am}\frac{(2q-1)i\Lambda'}{n}\cos\operatorname{am}\frac{(2q-1)i\Lambda'}{n}}{\sin^2\operatorname{am}\frac{(2q-1)i\Lambda'}{n}-y^2}$$

$$\operatorname{tg}\operatorname{am}(nu,k)=\frac{\lambda'}{k'n\mathrm{M}}\cdot\frac{y}{\sqrt{1-y^2}}+\frac{2\lambda'y\sqrt{1-y^2}}{k'n\mathrm{M}}\sum\frac{(-1)^q\,\Delta\operatorname{am}\frac{2qi\Lambda'}{n}}{\cos^2\operatorname{am}\frac{2qi\Lambda'}{n}-y^2\Delta^2\operatorname{am}\frac{2qi\Lambda'}{n}}.$$

---

Theorema analyticum generale, transformationem illam primae supplementariam concernens, iam initio mensis Augusti a. 1827 cum Cl°. Legendre communicavi, cuius etiam ille in nota supra citata (*Nova Astronomica* a. 1827. Nr. 130) mentionem iniicere voluit. Simile formularum systema pro transformatione altera secundae supplementaria seu transformatione moduli $\lambda_1$ in modulum $k$ stabiliri potuisset. Quae omnia ut dilucidiora fiant, adiecta tabula formulas fundamentales pro transformationibus prima et secunda earumque complementariis et supplementariis conspectui exponere placuit*).

Nec non e numero transformationum imaginariarum una quaeque suam habet supplementariam ad multiplicationem. Supponamus, quod licet, numeros $m$, $m'$ §. 20. factorem communem non habere: sit porro $m\mu' - \mu m' = 1$, designantibus $\mu$, $\mu'$ numeros integros positivos seu negativos. Iam si in formulis nostris generalibus de transformatione propositis §. 20. sqq. ponitur $\omega = \frac{\mu K + \mu' iK'}{nM}$, ac $k$ et $\lambda$ inter se commutantur, formulas obtines, quae ad supplementariam transformationis pertinent. Posito $m = 1$, $m' = 0$, fit $\mu = 0$, $\mu' = 1$, unde $\frac{\mu K + \mu' iK'}{nM} = \frac{iK'}{nM} = \frac{i\Lambda'}{n}$, quod primae supplementariam praebet, uti vidimus.

*) In quatuor paginis sequentibus inveniuntur:
*Transformationes reales functionum ellipticarum earumque complementariae et supplementariae*, quae primae huius operis editioni in tabula separata adiectae erant.

B.

## A. TRANSFORMATIO PRIMA CUM SUPPLEMENTARIA.

(a) $$\lambda = k^n \sin^4 \operatorname{coam}\frac{2K}{n} \sin^4 \operatorname{coam}\frac{4K}{n} \cdots \sin^4 \operatorname{coam}\frac{(n-1)K}{n} \qquad (\text{mod.}\, k)$$

(aa) $$k = \lambda^n \sin^4 \operatorname{coam}\frac{2i\Lambda'}{n} \sin^4 \operatorname{coam}\frac{4i\Lambda'}{n} \cdots \sin^4 \operatorname{coam}\frac{(n-1)i\Lambda'}{n} \qquad (\text{mod.}\, \lambda)$$

$$= \frac{\lambda^n}{\Delta^4 \operatorname{am}\frac{2\Lambda'}{n}\, \Delta^4 \operatorname{am}\frac{4\Lambda'}{n} \cdots \Delta^4 \operatorname{am}\frac{(n-1)\Lambda'}{n}} \qquad (\text{mod.}\, \lambda')$$

(b) $$\mathrm{M} = \frac{\sin^2 \operatorname{coam}\frac{2K}{n} \sin^2 \operatorname{coam}\frac{4K}{n} \cdots \sin^2 \operatorname{coam}\frac{(n-1)K}{n}}{\sin^2 \operatorname{am}\frac{2K}{n} \sin^2 \operatorname{am}\frac{4K}{n} \cdots \sin^2 \operatorname{am}\frac{(n-1)K}{n}} \qquad (\text{mod.}\, k)$$

(bb) $$\frac{1}{n\mathrm{M}} = \frac{\sin^2 \operatorname{coam}\frac{2\Lambda'}{n} \sin^2 \operatorname{coam}\frac{4\Lambda'}{n} \cdots \sin^2 \operatorname{coam}\frac{(n-1)\Lambda'}{n}}{\sin^2 \operatorname{am}\frac{2\Lambda'}{n} \sin^2 \operatorname{am}\frac{4\Lambda'}{n} \cdots \sin^2 \operatorname{am}\frac{(n-1)\Lambda'}{n}} \qquad (\text{mod.}\, \lambda')$$

$$\sin\operatorname{am}(u, k) = x; \quad \sin\operatorname{am}\left(\frac{u}{\mathrm{M}}, \lambda\right) = y; \quad \sin\operatorname{am}(nu, k) = z$$

(c) $$y = (-1)^{\frac{n-1}{2}} \sqrt{\frac{k^n}{\lambda}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4K}{n}\right) \sin\operatorname{am}\left(u + \frac{8K}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)K}{n}\right) \quad (\text{mod.}\, k)$$

$$= \frac{\frac{x}{\mathrm{M}}\left(1 - \frac{x^2}{\sin^2\operatorname{am}\frac{2K}{n}}\right)\left(1 - \frac{x^2}{\sin^2\operatorname{am}\frac{4K}{n}}\right)\cdots\left(1 - \frac{x^2}{\sin^2\operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1 - k^2x^2\sin^2\operatorname{am}\frac{2K}{n}\right)\left(1 - k^2x^2\sin^2\operatorname{am}\frac{4K}{n}\right)\cdots\left(1 - k^2x^2\sin^2\operatorname{am}\frac{(n-1)K}{n}\right)} \qquad (\text{mod.}\, k)$$

(cc) $$z = \sqrt{\frac{\lambda^n}{k}} \sin\operatorname{am}\frac{u}{\mathrm{M}} \sin\operatorname{am}\left(\frac{u}{\mathrm{M}} + \frac{4i\Lambda'}{n}\right) \sin\operatorname{am}\left(\frac{u}{\mathrm{M}} + \frac{8i\Lambda'}{n}\right) \cdots \sin\operatorname{am}\left(\frac{u}{\mathrm{M}} + \frac{4(n-1)i\Lambda'}{n}\right) \quad (\text{mod.}\, \lambda)$$

$$= \frac{n\mathrm{M}y\left(1 + \frac{y^2}{\operatorname{tg}^2\operatorname{am}\frac{2\Lambda'}{n}}\right)\left(1 + \frac{y^2}{\operatorname{tg}^2\operatorname{am}\frac{4\Lambda'}{n}}\right)\cdots\left(1 + \frac{y^2}{\operatorname{tg}^2\operatorname{am}\frac{(n-1)\Lambda'}{n}}\right)}{\left(1 + \lambda^2y^2\operatorname{tg}^2\operatorname{am}\frac{2\Lambda'}{n}\right)\left(1 + \lambda^2y^2\operatorname{tg}^2\operatorname{am}\frac{4\Lambda'}{n}\right)\cdots\left(1 + \lambda^2y^2\operatorname{tg}^2\operatorname{am}\frac{(n-1)\Lambda'}{n}\right)} \qquad (\text{mod.}\, \lambda')$$

TRANSFORMATIONES COMPLEMENTARIAE.

(a) $\lambda' = k'^n \sin^4 \operatorname{coam}\frac{2iK}{n} \sin^4 \operatorname{coam}\frac{4iK}{n} \cdots \sin^4 \operatorname{coam}\frac{(n-1)iK}{n}$ (mod. $k'$)

$$= \frac{k'^n}{\Delta^4 \operatorname{am}\frac{2K}{n}\,\Delta^4 \operatorname{am}\frac{4K}{n} \cdots \Delta^4 \operatorname{am}\frac{(n-1)K}{n}} \quad (\text{mod.}\, k)$$

(aa) $k' = \lambda'^n \sin^4 \operatorname{coam}\frac{2\Lambda'}{n} \sin^4 \operatorname{coam}\frac{4\Lambda'}{n} \cdots \sin^4 \operatorname{coam}\frac{(n-1)\Lambda'}{n}$ (mod. $\lambda'$)

(b) et (bb) eaedem atque supra.

$$\sin\operatorname{am}(u, k') = x; \quad \sin\operatorname{am}\left(\frac{u}{\mathrm{M}}, \lambda'\right) = y; \quad \sin\operatorname{am}(nu, k') = z$$

(c) $y = \sqrt{\frac{k'^n}{\lambda'}} \sin\operatorname{am} u \sin\operatorname{am}\left(u + \frac{4iK}{n}\right) \sin\operatorname{am}\left(u + \frac{8iK}{n}\right) \cdots \sin\operatorname{am}\left(u + \frac{4(n-1)iK}{n}\right)$ (mod. $k'$)

$$= \frac{\frac{x}{\mathrm{M}}\left(1 + \frac{x^2}{\operatorname{tg}^2 \operatorname{am}\frac{2K}{n}}\right)\left(1 + \frac{x^2}{\operatorname{tg}^2 \operatorname{am}\frac{4K}{n}}\right) \cdots \left(1 + \frac{x^2}{\operatorname{tg}^2 \operatorname{am}\frac{(n-1)K}{n}}\right)}{\left(1 + k'^2 x^2 \operatorname{tg}^2 \operatorname{am}\frac{2K}{n}\right)\left(1 + k'^2 x^2 \operatorname{tg}^2 \operatorname{am}\frac{4K}{n}\right) \cdots \left(1 + k'^2 x^2 \operatorname{tg}^2 \operatorname{am}\frac{(n-1)K}{n}\right)} \quad (\text{mod.}\, k)$$

(cc) $z = (-1)^{\frac{n-1}{2}} \sqrt{\frac{\lambda'^n}{k'}} \sin\operatorname{am}\frac{u}{\mathrm{M}} \sin\operatorname{am}\left(\frac{u}{\mathrm{M}} + \frac{4\Lambda'}{n}\right) \sin\operatorname{am}\left(\frac{u}{\mathrm{M}} + \frac{8\Lambda'}{n}\right) \cdots \sin\operatorname{am}\left(\frac{u}{\mathrm{M}} + \frac{4(n-1)\Lambda'}{n}\right)$ (mod. $\lambda'$)

$$= \frac{n\mathrm{M}y\left(1 - \frac{y^2}{\sin^2 \operatorname{am}\frac{2\Lambda'}{n}}\right)\left(1 - \frac{y^2}{\sin^2 \operatorname{am}\frac{4\Lambda'}{n}}\right) \cdots \left(1 - \frac{y^2}{\sin^2 \operatorname{am}\frac{(n-1)\Lambda'}{n}}\right)}{\left(1 - \lambda'^2 y^2 \sin^2 \operatorname{am}\frac{2\Lambda'}{n}\right)\left(1 - \lambda'^2 y^2 \sin^2 \operatorname{am}\frac{4\Lambda'}{n}\right) \cdots \left(1 - \lambda'^2 y^2 \sin^2 \operatorname{am}\frac{(n-1)\Lambda'}{n}\right)} \quad (\text{mod.}\, \lambda')$$

---

$$\Lambda = \frac{K}{n\mathrm{M}}; \qquad \Lambda' = \frac{K'}{\mathrm{M}}$$

## B. TRANSFORMATIO SECUNDA CUM SUPPLEMENTARIA.

(a) $$\lambda_1 = k^n \sin^4 \operatorname{coam} \frac{2iK'}{n} \sin^4 \operatorname{coam} \frac{4iK'}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)iK'}{n} \qquad (\text{mod.}\, k)$$

$$= \frac{k^n}{\Delta^4 \operatorname{am} \frac{2K'}{n} \Delta^4 \operatorname{am} \frac{4K'}{n} \cdots \Delta^4 \operatorname{am} \frac{(n-1)K'}{n}} \qquad (\text{mod.}\, k')$$

(aa) $$k = \lambda_1^n \sin^4 \operatorname{coam} \frac{2\Lambda_1}{n} \sin^4 \operatorname{coam} \frac{4\Lambda_1}{n} \cdots \sin^4 \operatorname{coam} \frac{(n-1)\Lambda_1}{n} \qquad (\text{mod.}\, \lambda_1)$$

(b) $$\mathrm{M}_1 = \frac{\sin^2 \operatorname{coam} \frac{2K'}{n} \sin^2 \operatorname{coam} \frac{4K'}{n} \cdots \sin^2 \operatorname{coam} \frac{(n-1)K'}{n}}{\sin^2 \operatorname{am} \frac{2K'}{n} \sin^2 \operatorname{am} \frac{4K'}{n} \cdots \sin^2 \operatorname{am} \frac{(n-1)K'}{n}} \qquad (\text{mod.}\, k')$$

(bb) $$\frac{1}{n\mathrm{M}_1} = \frac{\sin^2 \operatorname{coam} \frac{2\Lambda_1}{n} \sin^2 \operatorname{coam} \frac{4\Lambda_1}{n} \cdots \sin^2 \operatorname{coam} \frac{(n-1)\Lambda_1}{n}}{\sin^2 \operatorname{am} \frac{2\Lambda_1}{n} \sin^2 \operatorname{am} \frac{4\Lambda_1}{n} \cdots \sin^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}} \qquad (\text{mod.}\, \lambda_1)$$

$$\sin \operatorname{am}(u, k) = x; \quad \sin \operatorname{am}\left(\frac{u}{\mathrm{M}_1}, \lambda_1\right) = y; \quad \sin \operatorname{am}(nu, k) = z$$

(c) $$y = \sqrt{\frac{k^n}{\lambda_1}} \sin \operatorname{am} u \sin \operatorname{am}\left(u + \frac{4iK'}{n}\right) \sin \operatorname{am}\left(u + \frac{8iK'}{n}\right) \cdots \sin \operatorname{am}\left(u + \frac{4(n-1)iK'}{n}\right) (\text{mod.}\, k)$$

$$= \frac{\frac{x}{\mathrm{M}_1}\left(1 + \frac{x^2}{\operatorname{tg}^2 \operatorname{am} \frac{2K'}{n}}\right)\left(1 + \frac{x^2}{\operatorname{tg}^2 \operatorname{am} \frac{4K'}{n}}\right) \cdots \left(1 + \frac{x^2}{\operatorname{tg}^2 \operatorname{am} \frac{(n-1)K'}{n}}\right)}{\left(1 + k^2x^2 \operatorname{tg}^2 \operatorname{am} \frac{2K'}{n}\right)\left(1 + k^2x^2 \operatorname{tg}^2 \operatorname{am} \frac{4K'}{n}\right) \cdots \left(1 + k^2x^2 \operatorname{tg}^2 \operatorname{am} \frac{(n-1)K'}{n}\right)} \qquad (\text{mod.}\, k')$$

(cc) $$z = (-1)^{\frac{n-1}{2}} \sqrt{\frac{\lambda_1^n}{k}} \sin \operatorname{am} \frac{u}{\mathrm{M}_1} \sin \operatorname{am}\left(\frac{u}{\mathrm{M}_1} + \frac{4\Lambda_1}{n}\right) \sin \operatorname{am}\left(\frac{u}{\mathrm{M}_1} + \frac{8\Lambda_1}{n}\right) \cdots \sin \operatorname{am}\left(\frac{u}{\mathrm{M}_1} + \frac{4(n-1)\Lambda_1}{n}\right) (\text{mod.}\, \lambda_1)$$

$$= \frac{n\mathrm{M}_1 y \left(1 - \frac{y^2}{\sin^2 \operatorname{am} \frac{2\Lambda_1}{n}}\right)\left(1 - \frac{y^2}{\sin^2 \operatorname{am} \frac{\Lambda_1}{n}}\right) \cdots \left(1 - \frac{y^2}{\sin^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}}\right)}{\left(1 - \lambda_1^2 y^2 \sin^2 \operatorname{am} \frac{2\Lambda_1}{n}\right)\left(1 - \lambda_1^2 y^2 \sin^2 \operatorname{am} \frac{4\Lambda_1}{n}\right) \cdots \left(1 - \lambda_1^2 y^2 \sin^2 \operatorname{am} \frac{(n-1)\Lambda_1}{n}\right)} \qquad (\text{mod.}\, \lambda_1)$$

TRANSFORMATIONES COMPLEMENTARIAE.

$$\text{(a)}\quad \lambda'_1 = k'^n \sin^4 \operatorname{coam}\frac{2K'}{n}\sin^4 \operatorname{coam}\frac{4K'}{n}\dots\sin^4\operatorname{coam}\frac{(n-1)K'}{n} \qquad (\text{mod.}\, k')$$

$$\text{(aa)}\quad k' = \lambda'^n_1 \sin^4 \operatorname{coam}\frac{2i\Lambda_1}{n}\sin^4 \operatorname{coam}\frac{4i\Lambda_1}{n}\dots\sin^4\operatorname{coam}\frac{(n-1)i\Lambda_1}{n} \qquad (\text{mod.}\, \lambda'_1)$$

$$= \frac{\lambda'^n_1}{\Delta^4 \operatorname{am}\frac{2\Lambda_1}{n}\,\Delta^4 \operatorname{am}\frac{4\Lambda_1}{n}\dots\Delta^4\operatorname{am}\frac{(n-1)\Lambda_1}{n}} \qquad (\text{mod.}\, \lambda_1)$$

(b) et (bb) eaedem atque supra.

$$\sin\operatorname{am}(u,k') = x;\quad \sin\operatorname{am}\left(\frac{u}{M_1},\lambda'_1\right) = y;\quad \sin\operatorname{am}(nu,k') = z$$

$$\text{(c)}\quad y = (-1)^{\frac{n-1}{2}}\sqrt{\frac{k'^n}{\lambda'_1}}\sin\operatorname{am}u\sin\operatorname{am}\left(u+\frac{4K'}{n}\right)\sin\operatorname{am}\left(u+\frac{8K'}{n}\right)\cdot\cdot\sin\operatorname{am}\left(u+\frac{4(n-1)K'}{n}\right) (\text{mod.}\, k')$$

$$= \frac{\frac{x}{M_1}\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{2K'}{n}}\right)\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{4K'}{n}}\right)\cdots\left(1-\frac{x^2}{\sin^2\operatorname{am}\frac{(n-1)K'}{n}}\right)}{\left(1-k'^2x^2\sin^2\operatorname{am}\frac{2K'}{n}\right)\left(1-k'^2x^2\sin^2\operatorname{am}\frac{4K'}{n}\right)\cdot\cdot\left(1-k'^2x^2\sin^2\operatorname{am}\frac{(n-1)K'}{n}\right)} \qquad (\text{mod.}\, k')$$

$$\text{(cc)}\quad z = \sqrt{\frac{\lambda'^n_1}{k'}}\sin\operatorname{am}\frac{u}{M_1}\sin\operatorname{am}\left(\frac{u}{M_1}+\frac{4i\Lambda_1}{n}\right)\sin\operatorname{am}\left(\frac{u}{M_1}+\frac{8i\Lambda_1}{n}\right)\cdot\cdot\sin\operatorname{am}\left(\frac{u}{M_1}+\frac{4(n-1)i\Lambda_1}{n}\right) (\text{mod.}\, \lambda'_1)$$

$$= \frac{nM_1y\left(1+\frac{y^2}{\operatorname{tg}^2\operatorname{am}\frac{2\Lambda_1}{n}}\right)\left(1+\frac{y^2}{\operatorname{tg}^2\operatorname{am}\frac{4\Lambda_1}{n}}\right)\cdots\left(1+\frac{y^2}{\operatorname{tg}^2\operatorname{am}\frac{(n-1)\Lambda_1}{n}}\right)}{\left(1+\lambda'^2_1y^2\operatorname{tg}^2\operatorname{am}\frac{2\Lambda_1}{n}\right)\left(1+\lambda'^2_1y^2\operatorname{tg}^2\operatorname{am}\frac{4\Lambda_1}{n}\right)\cdot\cdot\left(1+\lambda'^2_1y^2\operatorname{tg}^2\operatorname{am}\frac{(n-1)\Lambda_1}{n}\right)} \qquad (\text{mod.}\, \lambda_1)$$

---

$$\Lambda_1 = \frac{K}{M_1};\qquad \Lambda'_1 = \frac{K'}{nM_1}$$

## FORMULAE ANALYTICAE GENERALES PRO MULTIPLICATIONE FUNCTIONUM ELLIPTICARUM.

28.

E binis transformationibus supplementariis componere licet ipsas pro multiplicatione formulas, seu formulas, quibus functiones ellipticae argumenti $nu$ per functiones ellipticas argumenti $u$ exprimuntur. Quod ut exemplo demonstretur, multiplicationem e transformatione prima eiusque supplementaria componamus. Quem in finem revocetur formula:

$$\sin \operatorname{am}\left(\frac{u}{M}, \lambda\right) = (-1)^{\frac{n-1}{2}} \sqrt{\frac{k^n}{\lambda}} \sin \operatorname{am} u \sin \operatorname{am}\left(u + \frac{4K}{n}\right) \sin \operatorname{am}\left(u + \frac{8K}{n}\right) \cdots \sin \operatorname{am}\left(u + \frac{4(n-1)K}{n}\right),$$

quam etiam hunc in modum repraesentare licet:

$$(-1)^{\frac{n-1}{2}} \sin \operatorname{am}\left(\frac{u}{M}, \lambda\right) = \sqrt{\frac{k^n}{\lambda}} \prod \sin \operatorname{am}\left(u + \frac{2mK}{n}\right),$$

designante $m$ numeros $0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}$. In hac formula loco $u$ ponamus $u + \frac{2m'iK'}{n}$, unde $\frac{u}{M}$ abit in $\frac{u}{M} + \frac{2m'iK'}{nM} = \frac{u}{M} + \frac{2m'i\Lambda'}{n}$: prodit

$$(-1)^{\frac{n-1}{2}} \sin \operatorname{am}\left(\frac{u}{M} + \frac{2m'i\Lambda'}{n}, \lambda\right) = \sqrt{\frac{k^n}{\lambda}} \prod \sin \operatorname{am}\left(u + \frac{2mK + 2m'iK'}{n}\right).$$

Iam ubi et ipsi $m'$ tribuuntur valores $0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}$, ita ut utrisque $m, m'$ isti conveniant valores, facto producto obtinemus:

$$(-1)^{\frac{n-1}{2}} \prod \sin \operatorname{am}\left(\frac{u}{M} + \frac{2m'i\Lambda'}{n}, \lambda\right) = \sqrt{\frac{k^{nn}}{\lambda^n}} \prod \sin \operatorname{am}\left(u + \frac{2mK + 2m'iK'}{n}\right),$$

ubi in altero producto numero $m'$, in altero utrique $m, m'$ valores $0, \pm 1, \pm 2, \ldots, \pm \frac{n-1}{2}$ tribuendi sunt.

At vidimus in §° praecedente, esse:

$$\sin \operatorname{am}(nu, k) = \sqrt{\frac{\lambda^n}{k}} \sin \operatorname{am} \frac{u}{M} \sin \operatorname{am}\left(\frac{u}{M} + \frac{4i\Lambda'}{n}\right) \sin \operatorname{am}\left(\frac{u}{M} + \frac{8i\Lambda'}{n}\right) \cdots \sin \operatorname{am}\left(\frac{u}{M} + \frac{4(n-1)i\Lambda'}{n}\right) (\text{mod.}\, \lambda),$$

quam ita quoque repraesentare licet formulam:

$$\sin\operatorname{am}(nu, k) = \sqrt{\frac{\lambda^n}{k}} \prod \sin\operatorname{am}\left(\frac{u}{M} + \frac{2m'i\Lambda'}{n}, \lambda\right),$$

unde iam:

$$(1.)\qquad \sin\operatorname{am} nu = (-1)^{\frac{n-1}{2}} \sqrt{k^{nn-1}} \prod \sin\operatorname{am}\left(u + \frac{2mK + 2m'iK'}{n}\right).$$

Eodem modo invenitur:

$$(2.)\qquad \cos\operatorname{am} nu = \sqrt{\left(\frac{k}{k'}\right)^{nn-1}} \prod \cos\operatorname{am}\left(u + \frac{2mK + 2m'iK'}{n}\right)$$

$$(3.)\qquad \Delta\operatorname{am} nu = \sqrt{\left(\frac{1}{k'}\right)^{nn-1}} \prod \Delta\operatorname{am}\left(u + \frac{2mK + 2m'iK'}{n}\right).$$

Quae facile etiam in hanc formam rediguntur formulae:

$$(4.)\qquad \sin\operatorname{am} nu = n \sin\operatorname{am} u \prod \frac{1 - \dfrac{\sin^2\operatorname{am} u}{\sin^2\operatorname{am}\frac{2mK + 2m'iK'}{n}}}{1 - k^2 \sin^2\operatorname{am}\frac{2mK + 2m'iK'}{n} \sin^2\operatorname{am} u}$$

$$(5.)\qquad \cos\operatorname{am} nu = \cos\operatorname{am} u \prod \frac{1 - \dfrac{\sin^2\operatorname{am} u}{\sin^2\operatorname{coam}\frac{2mK + 2m'iK'}{n}}}{1 - k^2 \sin^2\operatorname{am}\frac{2mK + 2m'iK'}{n} \sin^2\operatorname{am} u}$$

$$(6.)\qquad \Delta\operatorname{am} nu = \Delta\operatorname{am} u \prod \frac{1 - k^2 \sin^2\operatorname{coam}\frac{2mK + 2m'iK'}{n} \sin^2\operatorname{am} u}{1 - k^2 \sin^2\operatorname{am}\frac{2mK + 2m'iK'}{n} \sin^2\operatorname{am} u}.$$

Quibus addere placet sequentes:

$$(7.)\qquad \prod \sin^2\operatorname{am}\frac{2mK + 2m'iK'}{n} = \frac{(-1)^{\frac{n-1}{2}}\, n}{k^{\frac{nn-1}{2}}}$$

$$(8.)\qquad \prod \cos^2\operatorname{am}\frac{2mK + 2m'iK'}{n} = \left(\frac{k'}{k}\right)^{\frac{nn-1}{2}}$$

$$(9.)\qquad \prod \Delta^2\operatorname{am}\frac{2mK + 2m'iK'}{n} = k'^{\frac{nn-1}{2}}.$$

In sex formulis postremis numero $m$ valores tantum positivi $0, 1, 2, 3, \ldots, \frac{n-1}{2}$ conveniunt, ita tamen, ut quoties $m = 0$, et ipsi $m'$ valores tantum positivi $1, 2, 3, \ldots, \frac{n-1}{2}$ tribuantur. Et has et alias pro multiplicatione formulas iam prius Cl. Abel mutatis mutandis proposuit, unde nobis breviores esse licuit.

## DE AEQUATIONUM MODULARIUM AFFECTIBUS.

29.

Quia eodem modo $\lambda$ a $k$ atque $k$ a $\lambda_1$ nec non $\lambda'_1$ a $k'$, $k'$ a $\lambda'$ pendet: patet, ubi secundum eandem legem modulorum scalas condas, qui in se invicem transformari possunt, alteram modulum $k$, alteram complementum eius $k'$ continentem, in iis terminos fore eodem ordine se excipientes:

$$\ldots, \quad \lambda, \quad k, \quad \lambda_1, \quad \ldots$$
$$\ldots, \quad \lambda'_1, \quad k', \quad \lambda', \quad \ldots,$$

id quod in transformationibus secundi et tertii ordinis iam prius a Cl°. Legendre observatum et facto calculo confirmatum est. Similia cum de omnibus modulis transformatis et imaginariis valeant, patet, designante $\lambda$ modulum transformatum quemlibet, aequationes algebraicas inter $k$ et $\lambda$, seu inter $u = \sqrt[4]{k}$ et $v = \sqrt[4]{\lambda}$, quas *aequationes modulares* nuncupavimus, immutatas manere,

1.) ubi $k$ et $\lambda$ inter se commutentur,
2.) ubi $k'$ loco $k$, $\lambda'$ loco $\lambda$ ponatur.

Alterum iam supra in aequationibus modularibus, quae ad transformationes tertii et quinti ordinis pertinent:

$$(1.) \qquad u^4 - v^4 + 2uv(1 - u^2v^2) = 0$$
$$(2.) \qquad u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0,$$

observavimus eiusque observationis ope expressiones algebraicas pro transformationibus supplementariis exhibuimus. Ut alterum quoque his exemplis probetur, aequationes illas in alias transformemus inter $kk = u^8$ et $\lambda\lambda = v^8$, quod non sine calculo prolixo fit. Quo subducto obtinentur aequationes:

$$(1.) \qquad (k^2 - \lambda^2)^4 = 128\, k^2\lambda^2(1 - k^2)(1 - \lambda^2)(2 - k^2 - \lambda^2 + 2\, k^2\lambda^2)$$
$$(2.) \qquad (k^2 - \lambda^2)^6 = 512\, k^2\lambda^2(1 - k^2)(1 - \lambda^2)(L - L'k^2 + L''k^4 - L'''k^6),$$