Consider a dish antenna sitting on a rotating base that can be rotated azimuthally by a drive motor to track a flying aircraft. The dynamics of the rotating structure is described by the equations:

$$J\ddot{\theta}(t) = -\beta\dot{\theta}(t) + N(t) + N_{\text{dist}}(t)$$

where $\theta(t)$ is the azimuthal angle, $N(t)$ is the torque applied by the drive motor, $N_{\text{dist}}(t)$ is a torque due to disturbances such as wind gusts or steady wind noise, J is the moment of inertia of the structure, and β is a frictional constant that quantifies an opposing frictional torque that is proportional to the angular velocity $\dot{\theta}$.

A control system must be designed to apply an appropriate torque $N(t)$ such that the angle $\theta(t)$ will follow a desired reference angle $\theta_{\text{ref}}(t)$, that is, $\theta(t) \to \theta_{\text{ref}}(t)$. For example, if one wishes to point the antenna towards a given angle θ_1, then, $\theta_{\text{ref}}(t) = \theta_1 u(t)$. To point initially towards θ_1 and t_0 seconds later to point towards θ_2, one would choose $\theta_{\text{ref}}(t) = \theta_1 \text{upulse}(t, t_0) + \theta_2 u(t - t_0)$. Similarly, to track a uniformly moving aircraft, one would choose the ramp function $\theta_{\text{ref}}(t) = \dot{\theta}_0 t u(t)$, or, more accurately as explained in class, $\theta_{\text{ref}}(t) = \arctan(\dot{\theta}_0 t) u(t)$.
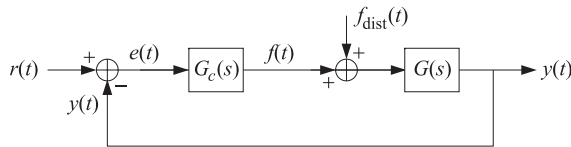
By some redefinitions, the above system can be replaced by the standardized form where the output $y(t)$ represents $\theta(t)$ and $f(t), f_{\text{dist}}(t)$ represent the torque inputs $N(t), N_{\text{dist}}(t)$:

$$\ddot{y}(t) = -a\dot{y}(t) + f(t) + f_{\text{dist}}(t) \quad \Leftrightarrow \quad Y(s) = G(s)\left[F(s) + F_{\text{dist}}(s)\right]$$

where the system transfer function is

$$G(s) = \frac{1}{s(s+a)}$$

The control system is implemented as the feedback system shown below,

where the overall reference input $r(t)$ represents the desired reference angle $\theta_{\text{ref}}(t)$, and the controller $G_c(s)$ is designed to generate the appropriate torque input $f(t)$ to make the system follow the reference input, i.e., $y(t) \to r(t)$, or for the error signal, $e(t) = r(t) - y(t) \to 0$.

In this lab, you will use a PID controller and experiment with its settings to achieve a desired response and also investigate the robustness of the control system in the presence of disturbance inputs. You willalso study the performance of the corresponding digital control system, with the system discretized using the zero-order hold method, and look at the effect of selecting different values of the sampling time interval. The PID controller has the transfer function:

$$G_c(s) = k_p + \frac{k_i}{s} + k_d s$$

The overall transfer relationships from the two inputs $R(s)$, $F_{\text{dist}}(s)$ to the two outputs $Y(s)$ and $E(s)$ are as follows:

$$\begin{aligned} Y(s) &= H(s)R(s) + H_{\text{dist}}(s)F_{\text{dist}}(s) \\ E(s) &= H_{\text{err}}(s)R(s) - H_{\text{dist}}(s)F_{\text{dist}}(s) \end{aligned}$$

$$H(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)} = \frac{k_d s^2 + k_p s + k_i}{s^3 + (a + k_d)s^2 + k_p s + k_i} = \text{closed-loop}$$

$$H_{\text{err}}(s) = \frac{1}{1 + G_c(s)G(s)} = \frac{s^2(s+a)}{s^3 + (a + k_d)s^2 + k_p s + k_i} = \text{error}$$

$$H_{\text{dist}}(s) = \frac{G(s)}{1 + G_c(s)G(s)} = \frac{s}{s^3 + (a + k_d)s^2 + k_p s + k_i} = \text{disturbance}$$

The corresponding digital control system, and its equivalent transfer function description in which the D/A converter has been replaced by a zero-order hold, are shown below:
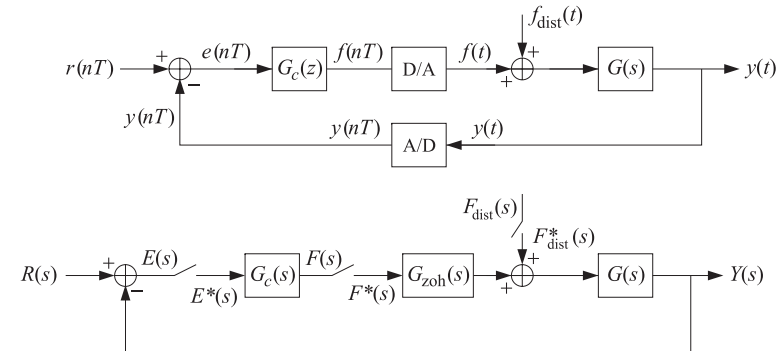
The discrete-time operation can be obtained by sampling all the signals in this figure at a sampling interval T. Assuming the discrete controller has transfer function $G_c(z) = G_c^*(s)$, the corresponding z-domain transfer functions of the feedback system are obtained as follows:

$$Y(s) = G(s) \left[G_{\text{zoh}}(s) F^*(s) + F_{\text{dist}}^*(s) \right], \quad G_{\text{zoh}}(s) = \frac{1 - e^{-sT}}{s}$$

$$F(s) = G_c(s) E^*(s) = G_c(s) \left[R^*(s) - Y^*(s) \right]$$

and taking the starred-Laplace transforms,

$$Y^*(s) = \left[G(s) G_{\text{zoh}}(s) \right]^* F^*(s) + G^*(s) F_{\text{dist}}^*(s)$$

$$F^*(s) = G_c^*(s) \left[R^*(s) - Y^*(s) \right]$$

Denoting $G_d(z) = \left[G(s) G_{\text{zoh}}(s) \right]^*$ and $G(z) = G^*(s)$, $Y(z) = Y^*(s)$, etc., we may rewrite the above as

$$Y(z) = G_d(z) F(z) + G(z) F_{\text{dist}}(z)$$

$$F(z) = G_c(z) \left[R(z) - Y(z) \right]$$

which leads to the feedback discrete-time transfer functions:

$$Y(z) = H_d(z) R(z) + H_{\text{dist}}(z) F_{\text{dist}}(z)$$

$$H_d(z) = \frac{G_c(z) G_d(z)}{1 + G_c(z) G_d(z)}, \quad H_{\text{dist}}(z) = \frac{G(z)}{1 + G_c(z) G_d(z)} \tag{1}$$

The discretized versions of $G_c(z), G_d(z), G(z)$ were derived in class:

$$G_d(z) = \frac{(aT + e^{-aT} - 1)z + (1 - e^{-aT} - aTe^{-aT})}{a^2(z - 1)(z - e^{-aT})} = \text{ZOH system}$$

$$G(z) = \frac{(1 - e^{-aT})z^{-1}}{a(1 - z^{-1})(1 - e^{-aT}z^{-1})} \tag{2}$$

$$G_c(z) = k_p + \frac{k_i T}{2}\left(\frac{1 + z^{-1}}{1 - z^{-1}}\right) + \frac{k_d}{T}(1 - z^{-1}) = \text{discrete PID controller}$$

Regarding the choice of the PID controller parameters k_p, k_i, k_d, we recall from class the following: (a) k_i must be nonzero in order to guarantee zero steady-state error for the given system, i.e., $e(t) \to 0$, for both the step and ramp responses, (b) increasing k_i will increase the overshoot and the settling time and decrease the rise time, (b) increasing k_d will decrease the overshoot and the settling time, and (c) increasing k_p will decrease the rise time but increase the overshoot. With above background, please carry out the following experiments:

a. Starting with the parameter values $a = 2$, $k_p = 10$, $k_i = 5$, $k_d = 3$, construct the transfer function objects for the system, controller, closed-loop feedback system, and disturbance transfer function using, for example, the MATLAB code:

```
a = 2; kp=20; ki=5; kd = 3;
s = tf('s');
G = 1/(s*(s+a));
Gc = kp + ki/s + kd*s;
H = minreal(Gc*G/(1+Gc*G));
Hdist = minreal(G/(1+Gc*G));
```

Plot the step response of H and then increase the PID parameters by doubling their values one at a time, plot the step response, and observe the dependence on such changes.

Experiment with other choices of the controller settings k_p, k_i, k_d and come up with values that meet your expectations for a good step response. Explain your reasons for your choice and use these values in the rest of the lab.

b. Define t as a vector of 1001 equally-spaced time samples spanning the interval $0 \le t \le 20$. For these values of t, generate the following three reference input signals describing the typical reference angle situations mentioned at the beginning of this handout:

$$r(t) = \text{upulse}(t, 10) + 2u(t - 10)$$

$$r(t) = 0.1\, t\, u(t)$$

$$r(t) = \arctan(0.1\, t)\, u(t)$$

$$r(t) = \begin{cases} 0.04\, t, & 0 \le t \le 10 \\ -2 + 0.69\, t - 0.07\, t^2 + 0.0025\, t^3, & 10 \le t \le 14 \\ 0.8 + 0.2\, (t - 14), & 14 \le t \le 20 \end{cases}$$

The fourth case, emulates a situation where the aircraft is moving at constant speed until $t = 10$ and then between $t = 10$ and $t = 14$, it accelerates to a new speed. The expression between $10 \le t \le 14$ is the cubic Hermite interpolation polynomial (see z-transform page on the course web page) that interpolates smoothly between the two speeds.

For each $r(t)$, compute the corresponding output of the closed-loop system H, using the function lsim:

```
y = lsim(H,r,t);
```

On the same graph, plot both $y(t)$ and $r(t)$ with different colors, observing whether the controlled system is capable of following the desired input reference setting.

For the particular case of the ramp input $r(t) = 0.1tu(t)$, set temporarily $k_i = 0$, and recompute the system output $y(t)$ noting that the steady-state error $e(t)$ is no longer zero, although the slope of the output does follow the slope of the reference input. After this part, set k_i back to its non-zero value.

c. Because of the difficulty in implementing the derivative term $k_d s$ of the PID controller, the following variant is often used:

$$G_c(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{\tau s + 1}$$

where τ is a very small quantity. Set $\tau = 0.05$ and repeat part (b).

In addition, determine the transfer function $H_f(s)$ from the overall input $r(t)$ to the controller's torque output $f(t)$ and for all four choices of $r(t)$, compute the applied torque $f(t)$ and plot it versus t. This will give you a sense of the actual input being applied to the controlled system $G(s)$. Set $\tau = 0$ after this part is complete.

d. Next, you will investigate how the controlled system responds to a disturbance. Consider two types of disturbances, one imitating a wind gust lasting for a brief period of time, say, $4 \le t \le 6$, and the other imitating steady wind noise. They can be generated by the following MATLAB code (for the same length-1001 vector of t's that you defined in part (b)):

```
fdist = 2*upulse(t-4,2);                          % wind gust
seed=2009; randn('state',seed); fdist = randn(1,1001);   % steady wind noise
```

For each type of disturbance, compute the corresponding system output using the disturbance transfer function H_{dist}, and add it to the previously obtained output from each of the four reference signals $r(t)$ to get the total system output:

```
ydist = lsim(Hdist,fdist,t);
y = lsim(H,r,t);
ytot = y + ydist;
```

For each of the resulting eight cases (2 disturbances × 4 reference signals), plot the signals $y_{\text{tot}}(t)$ and $r(t)$ on the same graph, observing how the system recovers (or not) from the disturbance.

e. Next, you will study the behavior of the discrete PID control system, described by Eqs. (1) and (2). For this part, you may ignore the disturbance input. A reasonable initial choice for the discretization sampling time interval T_s is to choose it to be a small fraction of the effective time constant of the closed-loop system H. The time constant is the the inverse of the smallest damping constant and can be obtained with the help of the function `pzmap`:

```
p = pzmap(H);                  % poles of H
teff = 1/abs(max(real(p)));    % effective time constant
Ts = teff/20;                  % initial choice of Ts
```

The zero-order hold discretization of the system $G(s)$ is given by Eq. (2), but it can be obtained also using the `c2d` function:

5

```
T = Ts;          % to be changed later to T = Ts/2, 2*Ts, 4*Ts, 6*Ts
Gd = c2d(G,T);   % ZOH discretization
```

The discrete PID controller $G_c(z)$ of Eq. (2) and the discrete feedback transfer function $H_d(z)$ can be constructed by the code:

```
z = tf('z');
Gc = kp + ki*T*(z+1)/(z-1)/2 + kd*(z-1)/z/T;
Hd = feedback(Gc*Gd,1);
```

The time vector t must now be resampled at multiples of the chosen interval T, that is, $t_n = nT$, and in order for it to span the interval $0 \le t \le 20$, we must redefine:

```
tn = (0:T:20);
```

Using this new vector of t's, construct the discrete-time reference inputs:

$$r_n = \text{upulse}(t_n, 10) + 2u(t_n - 10)$$

$$r_n = 0.1\, t_n$$

$$r_n = \arctan(0.1\, t_n)$$

and compute the output y_n of the discrete closed-loop system:

```
yn = lsim(Hd,rn);
%yn = filter(Hd.num{1}, Hd.den{1}, rn);   % alternative evaluation of yn
```

On the same graph, plot y_n versus the sampled time t_n, together with the output $y(t)$ of the continuous-time system computed in the previous parts using the original length-1001 time vector t, that is,

```
figure; plot(tn,yn,'r-', t,y,'b-');
```

Compare the outputs of the discrete and continuous time systems.

f. Repeat part (d) using the alternative choices of the interval T:

$$T = \frac{1}{2}T_s, \quad T = 2T_s, \quad T = 4T_s, \quad T = 6T_s$$

And discuss the improvement or deterioration of the expected response.

6

Typical Outputs

tracking step changes

ramp tracking

pulse disturbance

noisy disturbance

tracking ramp changes

pulse disturbance

discrete control, $T = T_s$

discrete control, $T = 4T_s$