Controllability and Observability

Controllability and observability represent two major concepts of modern control system theory. These concepts were introduced by R. Kalman in 1960. They can be roughly defined as follows.

Controllability: *In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable*.

Observability: *In order to see what is going on inside the system under observation, the system must be observable*.

In this lecture we show that the concepts of controllability and observability are related to linear systems of algebraic equations. It is well known that a solvable system of linear algebraic equations has a solution if and only if the rank of the system matrix is full . Observability and controllability tests will be connected to the rank tests of ceratin matrices: the controllability and observability matrices.

5.1 Observability of Discrete Systems

Consider a linear, time invariant, discrete-time system in the state space form

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k), \qquad \mathbf{x}(0) = \mathbf{x}_o = \mathrm{unknown} \tag{5.1}$$

with output measurements

$$\mathbf{y}(k) = \mathbf{C}_d\mathbf{x}(k) \tag{5.2}$$

where $\mathbf{x}(k) \in \Re^n$, $\mathbf{y}(k) \in \Re^p$. $\mathbf{A}_d$ and $\mathbf{C}_d$ are constant matrices of appropriate dimensions. The natural question to be asked is: can we learn everything about the dynamical behavior of the state space variables defined in (5.1) by using only information from the output measurements (5.2). If we know $\mathbf{x}_o$, then the recursion (5.1) apparently gives us complete knowledge about the state variables at any discrete-time instant. Thus, the only thing that we have to determine from the state measurements is the initial state vector $\mathbf{x}(0) = \mathbf{x}_o$.

Since the n-dimensional vector $\mathbf{x}(0)$ has n unknown components, it is expected that n measurements are sufficient to determine $\mathbf{x}_o$. Take $k = 0, 1, ..., n-1$ in (5.1) and (5.2), i.e. generate the following sequence

$$\begin{aligned}
&\mathbf{y}(0) = \mathbf{C}_d\mathbf{x}(0)\\
&\mathbf{y}(1) = \mathbf{C}_d\mathbf{x}(1) = \mathbf{C}_d\mathbf{A}_d\mathbf{x}(0)\\
&\mathbf{y}(2) = \mathbf{C}_d\mathbf{x}(2) = \mathbf{C}_d\mathbf{A}_d\mathbf{x}(\mathbf{1}) = \mathbf{C}_d\mathbf{A}_d^2\mathbf{x}(0)\\
&\vdots\\
&\mathbf{y}(n-1) = \mathbf{C}_d\mathbf{x}(n-1) = \mathbf{C}_d\mathbf{A}_d^{n-1}\mathbf{x}(0)
\end{aligned} \tag{5.3}$$

$$\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \mathbf{y}(2) \\ \vdots \\ \mathbf{y}(n-1) \end{bmatrix}^{(np)\times 1} = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d\mathbf{A}_d \\ \mathbf{C}_d\mathbf{A}_d^2 \\ \vdots \\ \mathbf{C}_d\mathbf{A}_d^{n-1} \end{bmatrix}^{(np)\times n} \times \mathbf{x}(0) \tag{5.4}$$

We know from linear algebra that the system of linear algebraic equations with n unknowns, (5.4), has a unique solution if and only if the system matrix has rank n.

$$\text{rank} \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d\mathbf{A}_d \\ \mathbf{C}_d\mathbf{A}_d^2 \\ \vdots \\ \mathbf{C}_d\mathbf{A}_d^{n-1} \end{bmatrix} = n \tag{5.5}$$

The initial condition $\mathbf{x}_0$ is determined if the so-called *observability matrix*

$$\mathcal{O}(\mathbf{A}_d, \mathbf{C}_d) = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d\mathbf{A}_d \\ \mathbf{C}_d\mathbf{A}_d^2 \\ \vdots \\ \mathbf{C}_d\mathbf{A}_d^{n-1} \end{bmatrix}^{(np)\times n} \tag{5.6}$$

has rank n, that is

$$\text{rank}\mathcal{O} = n \tag{5.7}$$

Theorem 5.1 *The linear discrete-time system* (5.1) *with measurements* (5.2) *is observable if and only if the observability matrix* (5.6) *has rank equal to* n.

Example 5.1:
Consider the following system with measurements

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The observability matrix for this second-order system is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d \mathbf{A}_d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 10 \end{bmatrix}$$

Since the rows of the matrix $\mathcal{O}$ are linearly independent, then $\mathrm{rank}\mathcal{O} = 2 = n$, i.e. the system under consideration is observable. Another way to test the completeness of the rank of square matrices is to find their determinants. In this case

$$\det\mathcal{O} = -4 \neq 0 \;\Leftrightarrow\; \text{full rank} \;=\; n = 2$$

Example 5.2:

Consider a case of an unobservable system, which can be obtained by slightly modifying Example 5.1. The corresponding system and measurement matrices are given by

$$\mathbf{A}_d = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}, \qquad \mathbf{C}_d = [1 \quad 2]$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 2 \\ -5 & -10 \end{bmatrix}$$

so that $\mathrm{rank}\mathcal{O} = 1 < 2$, and the system is unobservable.

5.2 Observability of Continuous Systems

For the purpose of studying its observability, we consider an input-free system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_o = \text{unknown} \tag{5.8}$$

with the corresponding measurements

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{5.9}$$

of dimensions $\mathbf{x}(t) \in \Re^n$, $\mathbf{y}(t) \in \Re^p$, $\mathbf{A} \in \Re^{n \times n}$, and $\mathbf{C} \in \Re^{p \times n}$. Following the same arguments as in the previous section, we can conclude that the knowledge of $\mathbf{x}_o$ is sufficient to determine $\mathbf{x}(t)$ at any time instant, since from (5.8) we have

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) \tag{5.10}$$

The problem that we are faced with is to find $\mathbf{x}(t_0)$ from the available measurements (5.9). We have solved this problem for discrete-time systems by generating the sequence of measurements at discrete-time instants $k = 0, 1, 2, ..., n-1$. Note that a time shift in the discrete-time corresponds to a derivative in the continuous-time. An analogous technique in the continuous-time domain is obtained by taking derivatives of the continuous-time measurements (5.9)

$$\begin{aligned}
\mathbf{y}(t_0) &= \mathbf{C}\mathbf{x}(t_0) \\
\dot{\mathbf{y}}(t_0) &= \mathbf{C}\dot{\mathbf{x}}(t_0) = \mathbf{C}\mathbf{A}\mathbf{x}(t_0) \\
\ddot{\mathbf{y}}(t_0) &= \mathbf{C}\ddot{\mathbf{x}}(t_0) = \mathbf{C}\mathbf{A}^2\mathbf{x}(t_0) \\
&\vdots \\
\mathbf{y}^{(n-1)}(t_0) &= \mathbf{C}\mathbf{x}^{(n-1)}(t_0) = \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(t_0)
\end{aligned} \tag{5.11}$$

Equations (5.11) comprise a system of np linear algebraic equations. They can be put in matrix form as

$$\begin{bmatrix} \mathbf{y}(t_0) \\ \dot{\mathbf{y}}(t_0) \\ \ddot{\mathbf{y}}(t_0) \\ \vdots \\ \mathbf{y}^{(n-1)}(t_0) \end{bmatrix}^{(np)\times 1} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}^{(np)\times n} \times \mathbf{x}(t_0) = \mathcal{O}\mathbf{x}(t_0) = \mathrm{Y}(t_0) \tag{5.12}$$

where $\mathcal{O}$ is the observability matrix already defined in (5.6) and where the definition of $\mathrm{Y}(t_0)$ is obvious. Thus, the initial condition $\mathbf{x}(t_0)$ can be determined uniquely from (5.12) if and only if the observability matrix has full rank, i.e. $\mathrm{rank}\mathcal{O} = n$.

Theorem 5.2 *The linear continuous-time system* (5.8) *with measurements* (5.9) *is observable if and only if the observability matrix has full rank.*

It is important to notice that adding higher-order derivatives in (5.12) cannot increase the rank of the observability matrix since by the Cayley–Hamilton theorem for $k \geq n$ we have

$$\mathbf{A}^k = \sum_{i=0}^{n-1} \alpha_i \mathbf{A}^i \tag{5.13}$$

so that the additional equations would be linearly dependent on the previously defined n equations (5.12).

5.3 Controllability of Discrete Systems

Consider a linear discrete-time invariant control system defined by

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k), \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{5.14}$$

The system controllability is roughly defined as an ability to do whatever we want with our system, or in more technical terms, the ability to transfer our system from any initial state $\mathbf{x}(0) = \mathbf{x}_o$ to any desired final state $\mathbf{x}(k_1) = \mathbf{x}_f$ in a finite time, i.e. for $k_1 < \infty$ (it makes no sense to achieve that goal at $k_1 = \infty$). Thus, the question to be answered is: can we find a control sequence $\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(n-1)$, such that $\mathbf{x}(k) = \mathbf{x}_f$?

Let us start with a simplified problem, namely let us assume that the input $\mathbf{u}(k)$ is a scalar, i.e. the input matrix $\mathbf{B}_d$ is a vector denoted by $\mathbf{b}_d$. Thus, we have

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{b}_d u(k), \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{5.15}$$

Taking $k = 0, 1, 2, ..., n$ in (5.15), we obtain the following set of equations

$$\begin{aligned}
&\mathbf{x}(1) = \mathbf{A}_d\mathbf{x}(0) + \mathbf{b}_d u(0) \\
&\mathbf{x}(2) = \mathbf{A}_d\mathbf{x}(1) + \mathbf{b}_d u(1) = \mathbf{A}_d^2\mathbf{x}(0) + \mathbf{A}_d\mathbf{b}_d u(0) + \mathbf{b}_d u(1) \\
&\vdots \\
&\mathbf{x}(n) = \mathbf{A}_d^n\mathbf{x}(0) + \mathbf{A}_d^{n-1}\mathbf{b}_d u(0) + \cdots + \mathbf{b}_d u(n-1)
\end{aligned} \tag{5.16}$$

$$\mathbf{x}(n) - \mathbf{A}_d^n \mathbf{x}(0) = \left[\mathbf{b}_d \vdots \mathbf{A}_d \mathbf{b}_d \vdots \cdots \vdots \mathbf{A}_d^{n-1} \mathbf{b}_d \right] \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} \tag{5.17}$$

Note that $\left[\mathbf{b}_d \vdots \mathbf{A}_d \mathbf{b}_d \vdots \cdots \vdots \mathbf{A}_d^{n-1} \mathbf{b}_d \right]$ is a square matrix. We call it the *controllability matrix* and denote it by $\mathcal{C}$. If the controllability matrix $\mathcal{C}$ is nonsingular, equation (5.17) produces the unique solution for the input sequence given by

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} = \mathcal{C}^{-1}(\mathbf{x}(n) - \mathbf{A}_d^n \mathbf{x}(0)) \tag{5.18}$$

Thus, for any $\mathbf{x}(n) = \mathbf{x}_f$, the expression (5.18) determines the input sequence that transfers the initial state $\mathbf{x}_o$ to the desired state $\mathbf{x}_f$ in n steps. It follows that the controllability condition, in this case, is equivalent to nonsingularity of the controllability matrix $\mathcal{C}$.

In a general case, when the input $\mathbf{u}(k)$ is a vector of dimension r, the repetition of the same procedure as in (5.15)–(5.17) leads to

$$\mathbf{x}(n) - \mathbf{A}_d^n\mathbf{x}(0) = \left[\mathbf{B}_d \vdots \mathbf{A}_d\mathbf{B}_d \vdots \cdots \vdots \mathbf{A}_d^{n-1}\mathbf{B}_d\right] \begin{bmatrix} \mathbf{u}(n-1) \\ \mathbf{u}(n-2) \\ \vdots \\ \mathbf{u}(1) \\ \mathbf{u}(0) \end{bmatrix} \tag{5.19}$$

The controllability matrix, in this case, defined by

$$\mathcal{C}(\mathbf{A}_d, \mathbf{B}_d) = \left[\mathbf{B}_d \vdots \mathbf{A}_d\mathbf{B}_d \vdots \cdots \vdots \mathbf{A}_d^{n-1}\mathbf{B}_d\right] \tag{5.20}$$

is of dimension $n \times r \cdot n$. The system of n linear algebraic equations in $r \cdot n$ unknowns for n r-dimensional vector components of $\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(n-1)$, is

$$\mathcal{C}^{n\times(nr)} \begin{bmatrix} \mathbf{u}(n-1) \\ \mathbf{u}(n-2) \\ \vdots \\ \mathbf{u}(1) \\ \mathbf{u}(0) \end{bmatrix}^{(nr)\times 1} = \mathbf{x}(n) - \mathbf{A}_d^n\mathbf{x}(0) = \mathbf{x}_f - \mathbf{A}_d^n\mathbf{x}(0) \tag{5.21}$$

will have a solution for any $\mathbf{x}_f$ if and only if the matrix $\mathcal{C}$ has full rank, i.e. $\mathrm{rank}\mathcal{C} = n$.

Theorem 5.3 *The linear discrete-time system* (5.14) *is controllable if and only if*

$$\mathrm{rank}\mathcal{C} = n \tag{5.22}$$

where the controllability matrix $\mathcal{C}$ *is defined by* (5.20).

5.4 Controllability of Continuous Systems

Studying the concept of controllability in the continuous-time domain is more challenging than in the discrete-time domain. At the beginning of this section we will first apply the same strategy as in Section 5.3 in order to indicate difficulties that we are faced with in the continuous-time domain. Then, we will show how to find a control input that will transfer our system from any initial state to any final state.

A linear continuous-time system with a scalar input is represented by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x}(t_0) = \mathbf{x}_o \tag{5.23}$$

Following the discussion and derivations from Section 5.3, we have, for a scalar input, the following set of equations

$$\begin{aligned}
\dot{\mathbf{x}} &= \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\
\ddot{\mathbf{x}} &= \frac{d^2}{dt^2}\mathbf{x} = \mathbf{A}^2\mathbf{x} + \mathbf{A}\mathbf{b}u + \mathbf{b}\dot{u} \\
&\vdots \\
\mathbf{x}^{(n)} &= \frac{d^n}{dt^n}\mathbf{x} = \mathbf{A}^n\mathbf{x} + \mathbf{A}^{n-1}\mathbf{b}u + \mathbf{A}^{n-2}\mathbf{b}\dot{u} + \cdots + \mathbf{b}u^{(n-1)}
\end{aligned} \tag{5.24}$$

$$\mathbf{x}^{(n)}(t) - \mathbf{A}^n\mathbf{x}(t) = \mathcal{C}\begin{bmatrix} u^{(n-1)}(t) \\ u^{(n-2)}(t) \\ \vdots \\ \dot{u}(t) \\ u(t) \end{bmatrix} \tag{5.25}$$

Note that (5.25) is valid for any $t \in [t_0, t_f]$ with t_f free but finite. Thus, the nonsingularity of the controllability matrix $\mathcal{C}$ implies the existence of the scalar input function $u(t)$ and its $n-1$ derivatives, for any $t < t_f < \infty$.

For a vector input system dual to (5.23), the above discussion produces the same relation as (5.25) with the controllability matrix $\mathcal{C}$ given by (5.20) and with the input vector $\mathbf{u}(t) \in \Re^r$, that is

$$\mathcal{C}^{n\times m\cdot n}\begin{bmatrix} \mathbf{u}^{(n-1)}(t) \\ \mathbf{u}^{(n-2)}(t) \\ \vdots \\ \dot{\mathbf{u}}(t) \\ \mathbf{u}(t) \end{bmatrix}^{r\cdot n\times 1} = \mathbf{x}^{(n)}(t) - \mathbf{A}^n\mathbf{x}(t) = \gamma(t) \tag{5.26}$$

It is well known from linear algebra that in order to have a solution of (5.26), it is sufficient that

$$\text{rank}\mathcal{C} = \text{rank}\Big[\,\mathcal{C} \,\vdots\, \gamma(t)\,\Big] \tag{5.27}$$

Also, a solution of (5.26) exists for any $\gamma(t)$—any desired state at t—if and only if

$$\text{rank}\mathcal{C} = n \tag{5.28}$$

From Section 3.2 we know that the solution of the state space equation is

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

At the final time t_1 we have

$$\mathbf{x}(t_1) = \mathbf{x}_f = e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

or

$$e^{-\mathbf{A}t_1}\mathbf{x}_f - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \int_{t_0}^{t_1} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Using the Cayley–Hamilton theorem, that is

$$e^{-\mathbf{A}\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)\mathbf{A}^i \tag{5.29}$$

where $\alpha_i(\tau), \;\; i = 0, 1, ..., n-1$, are scalar time functions, we have

$$e^{-\mathbf{A}t_1}\mathbf{x}_f - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \sum_{i=0}^{n-1} \mathbf{A}^i\mathbf{B} \int_{t_0}^{t_1} \alpha_i(\tau)\mathbf{u}(\tau)d\tau$$

$$e^{-\mathbf{A}t_1}\mathbf{x}_f - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \left[\mathbf{B}\vdots\mathbf{AB}\vdots\ \cdots\ \vdots\mathbf{A}^{n-1}\mathbf{B}\right]\begin{bmatrix}\int_{t_0}^{t_1}\alpha_0(\tau)\mathbf{u}(\tau)d\tau\\ \int_{t_0}^{t_1}\alpha_1(\tau)\mathbf{u}(\tau)d\tau\\ \vdots\\ \int_{t_0}^{t_1}\alpha_{n-1}(\tau)\mathbf{u}(\tau)d\tau\end{bmatrix}$$

On the left-hand side of this equation all quantities are known, i.e. we have a constant vector. On the right-hand side the controllability matrix is multiplied by a vector whose components are functions of the required control input. Thus, we have a functional equation in the form

$$\mathbf{const}^{n\times 1} = \mathcal{C}(\mathbf{A},\mathbf{B})^{n\times rn}\begin{bmatrix}\mathbf{f}_1(\mathbf{u}(\tau))\\ \mathbf{f}_2(\mathbf{u}(\tau))\\ \vdots\\ \mathbf{f}_{n-1}(\mathbf{u}(\tau))\end{bmatrix}^{rn\times 1}, \quad \tau\in(t_0,t_1) \tag{5.30}$$

A solution of this equation exists if and only if $\operatorname{rank}\mathcal{C}(\mathbf{A},\mathbf{B}) = n$, which is the condition already established in (5.28). In general, it is very hard to solve this equation. One of the many possible solutions of (5.30) will be given in Section 5.8 in terms of the controllability Grammian.

Theorem 5.4 *The linear continuous-time system is controllable if and only if the controllability matrix* $\mathcal{C}$ *has full rank, i.e.* $\operatorname{rank}\mathcal{C} = n$.

Example 5.3: Given the linear continuous-time system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & -2 \\ 3 & -4 & 5 \\ -6 & 7 & 8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & -1 \\ 2 & -3 \\ 4 & -5 \end{bmatrix} \mathbf{u}$$

The controllability matrix for this third-order system is given by

$$\begin{aligned} \mathcal{C} &= \left[\mathbf{B} \vdots \mathbf{AB} \vdots \mathbf{A}^2\mathbf{B} \right] \\ &= \begin{bmatrix} 0 & -1 & \vdots & -6 & 7 & \vdots & \\ 2 & -3 & \vdots & 12 & -10 & \vdots & \mathbf{A}^2\mathbf{B} \\ 4 & -5 & \vdots & 46 & -55 & \vdots & \end{bmatrix} \end{aligned}$$

Since the first three columns are linearly independent we can conclude that $\mathrm{rank}\mathcal{C} = 3$. Hence there is no need to compute $\mathbf{A}^2\mathbf{B}$ since it is well known from linear algebra that the row rank of the given matrix is equal to its column rank. Thus, $\mathrm{rank}\mathcal{C} = 3 = n$ implies that the system under consideration is controllable.

5.5 Additional Controllability/Observability Topics

Invariance Under Nonsingular Transformations

We will show that both system controllability and observability are invariant under similarity transformation.

Consider the vector input form of (5.23) and the similarity transformation

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x} \tag{5.31}$$

such that

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u}$$

where $\hat{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$ and $\hat{\mathbf{B}} = \mathbf{P}\mathbf{B}$. Then the following theorem holds.

Theorem 5.5 *The pair* $(\mathbf{A}, \mathbf{B})$ *is controllable if and only if the pair* $\left(\hat{\mathbf{A}}, \hat{\mathbf{B}}\right)$ *is controllable.*

This theorem can be proved as follows

$$\begin{aligned}
\mathcal{C}\left(\hat{\mathbf{A}}, \hat{\mathbf{B}}\right) &= \left[\hat{\mathbf{B}} \,\vdots\, \hat{\mathbf{A}}\hat{\mathbf{B}} \,\vdots\, \cdots \,\vdots\, \hat{\mathbf{A}}^{n-1}\hat{\mathbf{B}}\right] \\
&= \left[\mathbf{P}\mathbf{B} \,\vdots\, \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{P}\mathbf{B} \,\vdots\, \cdots \,\vdots\, \mathbf{P}\mathbf{A}^{n-1}\mathbf{P}^{-1}\mathbf{P}\mathbf{B}\right] \\
&= \mathbf{P}\left[\mathbf{B} \,\vdots\, \mathbf{A}\mathbf{B} \,\vdots\, \cdots \,\vdots\, \mathbf{A}^{n-1}\mathbf{B}\right] = \mathbf{P}\mathcal{C}(\mathbf{A}, \mathbf{B})
\end{aligned}$$

Since $\mathbf{P}$ is a nonsingular matrix (it cannot change the rank of the product $\mathbf{P}\mathcal{C}$), we get

$$\text{rank}\mathcal{C}\left(\hat{\mathbf{A}},\hat{\mathbf{B}}\right)=\text{rank}\mathcal{C}(\mathbf{A},\mathbf{B})$$

A similar theorem is valid for observability. The similarity transformation (5.32) applied to (5.8) and (5.9) produces

$$\begin{aligned}\dot{\hat{\mathbf{x}}}&=\hat{\mathbf{A}}\hat{\mathbf{x}}\\ \mathbf{y}&=\hat{\mathbf{C}}\hat{\mathbf{x}}\end{aligned}$$

where

$$\hat{\mathbf{C}}=\mathbf{C}\mathbf{P}^{-1}$$

Then, we have the following theorem

Theorem 5.6 *The pair* $(\mathbf{A},\mathbf{C})$ *is observable if and only if the pair* $\left(\hat{\mathbf{A}},\hat{\mathbf{C}}\right)$ *is observable.*

The proof of this theorem is as follows

$$\mathcal{O}\left(\hat{\mathbf{A}},\hat{\mathbf{C}}\right)=\begin{bmatrix}\hat{\mathbf{C}}\\ \hat{\mathbf{C}}\hat{\mathbf{A}}\\ \hat{\mathbf{C}}\hat{\mathbf{A}}^2\\ \vdots\\ \hat{\mathbf{C}}\hat{\mathbf{A}}^{n-1}\end{bmatrix}=\begin{bmatrix}\mathbf{C}\mathbf{P}^{-1}\\ \mathbf{C}\mathbf{P}^{-1}\mathbf{P}\mathbf{A}\mathbf{P}^{-1}\\ \mathbf{C}\mathbf{P}^{-1}\mathbf{P}\mathbf{A}^2\mathbf{P}^{-1}\\ \vdots\\ \mathbf{C}\mathbf{P}^{-1}\mathbf{P}\mathbf{A}^{n-1}\mathbf{P}^{-1}\end{bmatrix}=\begin{bmatrix}\mathbf{C}\\ \mathbf{C}\mathbf{A}\\ \mathbf{C}\mathbf{A}^2\\ \vdots\\ \mathbf{C}\mathbf{A}^{n-1}\end{bmatrix}\mathbf{P}^{-1}$$

$$\mathcal{O}\left(\hat{\mathbf{A}},\hat{\mathbf{C}}\right)=\mathcal{O}(\mathbf{A},\mathbf{C})\mathbf{P}^{-1}$$

The nonsingularity of $\mathbf{P}$ implies

$$\mathrm{rank}\mathcal{O}\left(\hat{\mathbf{A}},\hat{\mathbf{C}}\right)=\mathrm{rank}\mathcal{O}(\mathbf{A},\mathbf{C})$$

which proves the stated observability invariance.

Frequency Domain Controllability and Observability Test

Controllability and observability have been introduced in the state space domain as pure time domain concepts. It is interesting to point out that in the frequency domain there exists a very powerful and simple theorem that gives a single condition for both the controllability and the observability of a system. It is given below.

Let $H(s)$ be the transfer function of a single-input single-output system

$$H(s)=\mathbf{c}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}$$

Note that $H(s)$ is defined by a ratio of two polynomials containing the corresponding system poles and zeros. The following controllability–observability theorem is given without a proof.

Theorem 5.7 *If there are no zero-pole cancellations in the transfer function of a single-input single-output system, then the system is both controllable and observable. If the zero-pole cancellation occurs in* $H(s)$*, then the system is either uncontrollable or unobservable or both uncontrollable and unobservable.*

Example 5.4: Consider a linear continuous-time dynamic system represented by its transfer function

$$H(s) = \frac{(s+3)}{(s+1)(s+2)(s+3)} = \frac{s+3}{s^3+6s^2+11s+6}$$

Theorem 5.7 indicates that any state space model for this system is either uncontrollable or/and unobservable. To get the complete answer we have to go to a state space form and examine the controllability and observability matrices. One of the possible many state space forms of $H(s)$ is as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It is easy to show that the controllability and observability matrices are given by

$$\mathcal{C} = \begin{bmatrix} 1 & -6 & 25 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \\ -3 & -11 & -6 \end{bmatrix}$$

Since

$$\det \mathcal{C} = 1 \neq 0 \;\Rightarrow\; \text{rank}\mathcal{C} = 3 = n$$

and

$$\det \mathcal{O} = 0 \;\Rightarrow\; \mathrm{rank}\mathcal{O} < 3 = n$$

this system is controllable, but unobservable.

Note that, due to a zero-pole cancellation at $s = -3$, the system transfer function $H(s)$ is reducible to

$$H(s) = H_r(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2 + 3s + 2}$$

so that the equivalent system of order $n = 2$ has the corresponding state space form

$$\begin{bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 1] \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix}$$

For this reduced-order system we have

$$\mathcal{C} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and therefore the system is both controllable and observable.

Interestingly enough, the last two mathematical models of dynamic systems of order $n = 3$ and $n = 2$ represent exactly the same physical system. Apparently, the second one ($n = 2$) is preferred since it can be realized with only two integrators.

It can be concluded from Example 5.4 that Theorem 5.7 gives an answer to the problem of dynamic system reducibility. It follows that a single-input single-output dynamic system is irreducible if and only if it is both controllable and observable. Such a system realization is called the *minimal realization*. If the system is either uncontrollable and/or unobservable it can be represented by a system whose order has been reduced by removing uncontrollable and/or unobservable modes. It can be seen from Example 5.4 that the reduced system with $n = 2$ is both controllable and observable, and hence it cannot be further reduced. This is also obvious from the transfer function $H_r(s)$.

Theorem 5.7 can be generalized to multi-input multi-output systems, where it plays very important role in the procedure of testing whether or not a given system is in the minimal realization form. The procedure requires the notion of the characteristic polynomial for proper rational matrices which is beyond the scope of this book. Interested readers may find all details and definitions in Chen (1984).

It is important to point out that the similarity transformation does not change the transfer function as was shown in Section 3.4.

Controllability and Observability of Special Forms

In some cases, it is easy to draw conclusions about system controllability and/or observability by examining directly the state space equations. In those cases there is no need to find the corresponding controllability and observability matrices and check their ranks.

Consider the phase variable canonical form with

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

This form is both controllable and observable due to an elegant chain connection of the state variables. The variable $x_1(t)$ is directly measured, so that $x_2(t)$ is known from $x_2(t) = \dot{x}_1(t)$. Also, $x_3(t) = \dot{x}_2(t) = \ddot{x}_1(t)$, and so on, $x_n(t) = x_1^{(n-1)}(t)$. Thus, this form is observable. The controllability follows from the fact that all state variables are affected by the control input, i.e. x_n is affected directly by $u(t)$ and then $\dot{x}_{n-1}(t)$ by $x_n(u(t))$ and so on. The control input is able to indirectly move all state variables into the desired positions so that the system is controllable. This can be formally verified by forming the corresponding controllability matrix and checking its rank. This is left as an exercise for students (see Problem 5.13).

Another example is the modal canonical form. Assuming that all eigenvalues of the system matrix are distinct, we have

$$\dot{\mathbf{x}} = \Lambda \mathbf{x} + \Gamma \mathbf{u}$$
$$\mathbf{y} = \mathfrak{D} \mathbf{x}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

$$\mathfrak{D} = [\delta_1 \ \delta_2 \ \cdots \ \delta_n]$$

We are apparently faced with n completely decoupled first-order systems. Obviously, for controllability all γ_i, $i = 1, \ldots, n$, must be different from zero, so that each state variable can be controlled by the input $\mathbf{u}(t)$. Similarly, $\delta_i \neq 0$, $i = 1, \ldots, n$, ensures observability since, due to the state decomposition, each system must be observed independently.

The Role of Observability in Analog Computer Simulation

In addition to applications in control system theory and practice, the concept of observability is useful for analog computer simulation. Consider the problem of solving an nth-order differential equation given by

$$y^{(n)} + \sum_{i=1}^{n} a_{n-i} y^{(n-i)} = \sum_{i=0}^{m} b_{m-i} u^{(m-i)}$$

with known initial conditions for $y(0), \dot{y}(0), ..., y^{(n-1)}(0)$. This system can be solved by an analog computer by using n integrators. The outputs of these n integrators represent the state variables $x_1, x_2, ..., x_n$, so that this system has the state space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \qquad \mathbf{x}(0) = \text{unknown}$$
$$y = \mathbf{c}\mathbf{x}$$

However, the initial condition for $\mathbf{x}(0)$ is not given. In other words, the initial conditions for the considered system of n integrators are unknown. They can be determined from $y(0), \dot{y}(0), ..., y^{(n-1)}(0)$ by following the observability derivations performed in Section 5.2, namely

$$
\begin{aligned}
&y(0) = \mathbf{c}\mathbf{x}(0) \\
&\dot{y}(0) = \mathbf{c}\dot{\mathbf{x}}(0) = \mathbf{c}\mathbf{A}\mathbf{x}(0) + \mathbf{c}\mathbf{b}u(0) \\
&\ddot{y}(0) = \mathbf{c}\ddot{\mathbf{x}}(0) = \mathbf{c}\mathbf{A}^2\mathbf{x}(0) + \mathbf{c}\mathbf{A}\mathbf{b}u(0) + \mathbf{c}\mathbf{b}\dot{u}(0) \\
&\vdots \\
&y^{(n-1)}(0) = \mathbf{c}\mathbf{x}^{(n-1)}(0) = \mathbf{c}\mathbf{A}^{n-1}\mathbf{x}(0) + \mathbf{c}\mathbf{A}^{n-2}\mathbf{b}u(0) \\
&\qquad\qquad + \mathbf{c}\mathbf{A}^{n-3}\mathbf{b}\dot{u}(0) + \cdots + \mathbf{c}\mathbf{A}\mathbf{b}u^{(n-3)}(0) + \mathbf{c}\mathbf{b}u^{(n-2)}(0)
\end{aligned}
$$

This system can be written in matrix form as follows

$$
\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \mathcal{O} \cdot \mathbf{x}(0) + \mathcal{T} \begin{bmatrix} 0 \\ u(0) \\ \vdots \\ u^{(n-2)}(0) \end{bmatrix} \tag{5.32}
$$

where $\mathcal{O}$ is the observability matrix and $\mathcal{T}$ is a known matrix. Since $u(0), \dot{u}(0), ..., u^{(n-1)}(0)$ are known, it follows that a unique solution for $\mathbf{x}(0)$ exists if and only if the observability matrix, which is square in this case, is invertible, i.e. the pair $(\mathbf{A}, \mathbf{c})$ is observable.

Example 5.5: Consider a system represented by the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \frac{du}{dt} + u, \quad y(0) = 2, \ \ \dot{y}(0) = 1, \quad u(t) = e^{-4t}, \ \ t \geq 0$$

Its state space form is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{c}\mathbf{x} = [1 \quad 1\,]\mathbf{x}$$

The initial condition for the state space variables is obtained from (5.33) as

$$\begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} \mathbf{y}(0) \\ \dot{\mathbf{y}}(0) \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{cb}u(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

leading to

$$\begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$

This means that if analog computer simulation is used to solve the above second-order differential equation, the initial conditions for integrators should be set to -6 and 8.

Stabilizability and Detectability

So far we have defined and studied observability and controllability of the complete state vector. We have seen that the system is controllable (observable) if all components of the state vector are controllable (observable). The natural question to be asked is: do we really need to control and observe all state variables? In some applications, it is sufficient to take care only of the unstable components of the state vector. This leads to the definition of stabilizability and detectability.

Definition 5.1 *A linear system (continuous or discrete) is stabilizable if all unstable modes are controllable.*

Definition 5.2 *A linear system (continuous or discrete) is detectable if all unstable modes are observable.*

The concepts of stabilizability and detectability play very important roles in optimal control theory, and hence are studied in detail in advanced control theory courses. For the purpose of this course, it is enough to know their meanings.