Chapter 3 (Khaere "Nonlinear systems")

Feb. 6, 1998 (3)

Lyapunov Stability

This chapter - stability of equilibrium points
≡ stability in the sense of Lyapunov

3.1 Autonomous Systems

\[ \dot{x} = f(x) \quad f: \mathbb{R}^n \to \mathbb{R}^n \] is locally Lipschitz

\[ 0 = f(x) \Rightarrow x \text{ - equilibrium point } \in D \]

Eq. point is stable if all trajectories starting at nearby points stay nearby. If in addition they tend to the eq. point as \( t \to \infty \) \equiv asymptotic stability.

Without loss of generality, we can take \( x = 0 \) assuming that OED.

The change of variables

\[ y = x - x \]

\[ \Rightarrow \quad \dot{y} = \dot{x} = f(x) - f(y + x) = g(y) \quad g(0) = 0 \]

Ref. 3-1 The equilibrium point \( x = 0 \) is

stable if for each \( \varepsilon > 0 \) there is \( \delta = \delta(\varepsilon) > 0 \) such that

\[ \| x(0) \| < \delta \Rightarrow \| x(t) \| < \varepsilon \quad \forall t \geq 0 \]

It is asymptotically stable.

\[ \| x(0) \| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0 \]

It is unstable if it is not stable.

It is known from physics that if system energy decays \Rightarrow equilibrium

Lyapunov showed that certain other functions could be used instead of energy to determine stability of an equilibrium point.

Lyapunov's 1892 doctoral dissertation was republished in March of 1992 issue of International Journal of Control (C) 1992
Let \( \mathbf{v} : D \to \mathbb{R}^n \) be a continuously differentiable function defined on a domain \( D \subseteq \mathbb{R}^n \) that contains the origin. The derivative of \( \mathbf{v} \) along the trajectories of \( x = \mathbf{f}(x) \) denoted by \( \dot{\mathbf{v}}(x) \) is given by

\[
\frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}}(x) = \sum_{i=1}^{n} \frac{\partial \mathbf{v}}{\partial x_i} \frac{dx_i}{dt} = \sum_{i=1}^{n} \frac{\partial \mathbf{v}}{\partial x_i} f_i(x) \\
= \begin{bmatrix} \frac{\partial \mathbf{v}}{\partial x_1} & \frac{\partial \mathbf{v}}{\partial x_2} & \cdots & \frac{\partial \mathbf{v}}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} = \frac{\partial \mathbf{v}}{\partial x} \cdot \mathbf{f}(x)
\]

**Lyapunov Stability Theorem**

**Theorem 3.1** Let \( x = 0 \) be an equilibrium point for \( x = \mathbf{f}(x) \). Let \( \mathbf{v} : D \to \mathbb{R}^n \) be a continuously differentiable function on a neighborhood \( D \) of \( x = 0 \), such that

1) \( \mathbf{v}(0) = 0 \) and \( \mathbf{v}(x) > 0 \) on \( D - \{0\} \)
2) \( \dot{\mathbf{v}}(x) \leq 0 \) on \( D - \{0\} \)

Then, \( x = 0 \) is stable. Moreover, if

\( \dot{\mathbf{v}}(x) < 0 \) on \( D - \{0\} \)

then \( x = 0 \) is asymptotically stable.

Note: That there is no systematic method for finding \( \mathbf{v}(x) \). This should be considered for every nonlinear system as a separate problem.
Proof given \( e > 0 \) choose \( r \in (0, e] \) such that
\[
B_r = \{ x \in \mathbb{R}^n | \| x \| < r \} \subseteq C_D
\]

Let \( \alpha = \min \{ \nabla V(x) \} \). Then \( \alpha > 0 \) since \( V(x) > 0 \). Take \( \beta \in (0, \alpha) \), and let
\[
S_{\beta} = \{ x \in B_r | V(x) \leq \beta \}
\]

\[
V(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \quad \forall t \geq 0
\]
thus, \( S_{\beta} \) has the property that any trajectory starting on \( S_{\beta} \) at \( t = 0 \) stays in \( S_{\beta} \) for \( t > 0 \) (means closed and bounded)

\( S_{\beta} \) is a compact set then 5thm.2.4 \( \Rightarrow \)
\( \dot{x} = f(x) \) has a unique solution for all \( t > 0 \)
when ever \( x(0) \in S_{\beta} \)

Since \( V(x) \) is continuous and \( V(0) = 0 \) there is \( \delta > 0 \) such that
\[
\| x \| \leq \delta \Rightarrow V(x) < \beta
\]

\( B_{\delta} \subseteq S_{\beta} \subseteq B_r \)
\( x(0) \in B_{\delta} \Rightarrow x(0) \in S_{\beta} \Rightarrow x(t) \in S_{\beta} \Rightarrow x(t) \in B_r \)

Therefore \( \| x(0) \| < \delta \Rightarrow \| x(t) \| < \epsilon, \quad \forall t > 0 \Rightarrow \) eq. point \( x^* \)
A continuously differentiable function \( V(x) \) satisfying 
1) \( V(0) = 0, \) \( V(x) > 0 \) and 
2) \( \dot{V}(x) < 0 \) is called a Lyapunov function.

\( V(x) = c, \) \( c > 0 \) is called a Lyapunov surface.

A function \( V(x) \) satisfying \( V(x) > 0 \) and \( V(0) = 0 \) is said to be positive definite.

\( V(x) \geq 0 \) for \( x \neq 0 \) positive semidefinite
\( V(x) \) is negative (semi)definite if \( -V(x) \) is positive (semi)definite.

If none of the above \( V(x) \) is undefined,

**Lyapunov's Theorem.** The origin is stable if there is a continuously differentiable positive definite function \( V(x) \) so that \( \dot{V}(x) \) is negative semidefinite and it is asymptotically stable if \( \dot{V}(x) \) is negative definite.

**Quadratic forms:** Let \( P = P^T \) be a real matrix

\[
V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \quad x^T P x \geq 0 \Rightarrow P \geq 0
\]

\[
x^T P x \geq 0 \Rightarrow P \geq 0
\]

**Example 3.1.** \( P > 0 \) off all the leading principal minors of \( P \) are positive definite.

\[
V(x) = a x_1^2 + 2 x_1 x_3 + a x_2^2 + 4 x_2 x_3 + a x_3^2
\]

\[
= (x_1 \ x_2 \ x_3) \begin{pmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x^T P x
\]

\[
\text{det}(a) = a, \quad \text{det} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a^2, \quad \text{det} \begin{pmatrix} a \end{pmatrix} = a (a^2 - 6)
\]

\[
a^2 - 5 > 0 \text{ or } a^2 > 5 \Rightarrow P > 0
\]
Thm 2.4 Let $f(x)$ be locally Lipschitz on a domain $D \subset \mathbb{R}^n$, and let $V$ be a compact subset of $D$. Let $x_0 \in V$ and suppose it is known that every solution of $\dot{x} = f(x)$, $x(0) = x_0$ lies entirely in $V$. Then, there is a unique solution that is defined for all $t \geq 0$. 

(4.7) Thm 2.2
(Ex. 3.3) pendulum equation without friction
\[ \ddot{x}_1 = x_2 \]
\[ \dot{x}_2 = - \left( \frac{g}{L} \right) \sin x_1 \]
A natural Lyapunov function candidate is the energy function
\[ V(x) = \left( \frac{g}{L} \right) \left( 1 - \cos x_1 \right) + \frac{1}{2} x_2^2 \]
\[ V(0) = 0 \quad \forall x > 0 \quad \text{over the domain } -2\pi < x < 2\pi \]
\[ \dot{V}(x) = \frac{g}{L} \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 \]
\[ = \frac{g}{L} x_2 \sin x_1 - \frac{g}{L} x_2 \sin x_1 = 0 \Rightarrow \text{origin is stable} \]

(Ex. 3.4) adding friction
\[ \ddot{x}_1 = x_2 \]
\[ \dot{x}_2 = - \left( \frac{g}{L} \right) \sin x_1 - \frac{\mu}{m} x_2 \]
Let us try the same Lyapunov function as in Example 3.3
\[ V(x) = \left( \frac{g}{L} \right) \left( 1 - \cos x_1 \right) + \frac{1}{2} x_2^2 \]
\[ \dot{V}(x) = \left( \frac{g}{L} \right) \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 = - \frac{\mu}{m} x_2^2 \leq 0 \Rightarrow \text{stable} \]

However, we know that the origin is asymptotically stable. This Lyapunov function does not show this fact.

Note that the Lyapunov theorem gives only a sufficient condition which means if a Lyapunov function exists, the origin is stable (asymptotically stable). The fact that we are not able to find a Lyapunov function does not mean that the origin is stable (asymptotically stable).

\[ V(x) = \left( \frac{g}{L} \right) \left( 1 - \cos x_1 \right) + \frac{1}{2} \begin{bmatrix} x_1^T \end{bmatrix} \begin{bmatrix} \frac{4}{m} & p_{12} \\ p_{12} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad 0 < p_{12} < \frac{\mu}{m} \]

shows the asymptotic stability of the ex. pt. at the origin, in the domain \( D = \{ x \in \mathbb{R}^2, |x_1| < \pi \} \).
(Examples) PD, PSD or ID (indefinite)

1. $a, b, c > 0$

a) $V_1 = a x_1^2$

$$V_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

b) $V_2 = a x_1 + b x_2^2$

$$V_2 = \text{ID}$$

c) $V_3 = a x_1 x_2 + b x_2^2 = b (x_2 + \frac{a}{2b} x_1)^2 - \frac{a^2}{4b} x_1^2$

d) $V_4 = -a x_1 x_2 + b x_2^2 = \text{ID}$

e) $V_5 = a x_1^2 + b x_1 x_2 + c x_2^2$

$$V_5 = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$a > 0 \quad \quad ac - \frac{b^2}{4} > 0 \quad \Rightarrow \quad ac > \frac{b^2}{4} \quad \Rightarrow \quad V_5 = \text{PD}$$

$$ac = \frac{b^2}{4a} \quad \Rightarrow \quad V_5 = \text{PSD} \quad \quad b^2 > 4ac \quad \Rightarrow \quad V_5 = \text{ID}$$

2. $V_6 = (a x_1 - b x_2)^2 + c x_1 x_2$

$$= a^2 x_1^2 + (c - 2ab) x_1 x_2 + b^2 x_2^2$$

$$= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} a^2 & \frac{c}{2} - ab \\ \frac{c}{2} - ab & b^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$m_{11} = a^2 > 0$

$m_{22} = c \left( c - \frac{4ab}{c} \right) > 0$

$c > 0 \quad \quad c < 4ab \quad \Rightarrow \quad \text{PD}$

$c = 4ab \quad \Rightarrow \quad \text{PSD}$
8) \( V_7 = (ax_1 - bx_2)^2 + cx_2^2 \)

9) \( V_8 = (ax_1 - bx_2)^2 + cx_2^2 \)

10) \( V_9 = (ax_1 - bx_2)^6 + cx_4^4 > 0 \)

2. **Fundamental Lyapunov Functions**

   (EX)

   \[
   x_1 = ax_1^2 + bx_2^2 \\
   x_2 = \frac{x_1^2}{x_1} - x_1^4
   \]

   Try \( V = \frac{1}{2} ax_1^2 + \frac{1}{2} bx_2^2 \) \( \text{a} \geq 0, b > 0 \)

   \[
   V = ax_1 \cdot x_1 + bx_2 \cdot x_2 \\
   = ax_1 (-x_1 + x_1 x_2) + bx_2 (-\frac{x_1^2}{x_1} - x_1^4) \\
   = -ax_1^4 + ax_2^2 x_2 - bx_2 x_1^2 - bx_2 x_1^4 \\
   = -ax_1^4 + (a - \frac{b}{x_1}) x_2^2 x_2 - bx_2 x_1^4
   \]

   \[
   \hat{V} = -a x_1^4 - bx_2 x_1^4 = -(a + bx_2) x_1^4
   \]

   \[
   \hat{V} = -a (1 + bx_2) x_1^4
   \]

   need \( 1 + bx_2 > 0 \) true \( \|x_2\| \leq \frac{1}{a} \)

   hence \( a = 1, b = 4 \) \( \Rightarrow \hat{V} = \frac{1}{2} x_1^2 + 2 x_2^2 \)
\[ (E^2) \]
\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -(x_1^5 + 3x_1^3) / x_2^2 \]
\[ \dot{v} = x_2 \left( \frac{\partial V}{\partial x_1} \right) - \left( \frac{x_1^5 + 3x_1^3}{x_2^2} \right) \left( \frac{\partial V}{\partial x_2} \right) \]
\[ \frac{\partial V}{\partial x_1} = a \left( x_1^5 + 3x_1^3 \right) \quad \text{and} \quad \frac{\partial V}{\partial x_2} = b \cdot x_2^3 \]
looks promising
\[ \dot{v} = a \cdot x_2 \left( x_1^5 + 3x_1^3 \right) - b \left( x_1^5 + 3x_1^3 \right) \cdot x_2 = 0 \]
results in

\[ a = b = 1 \]
\[ v_{x_1} = x_1^5 + 3x_1^3 \quad \Rightarrow \quad v = \frac{1}{6}x_1^6 + \frac{3}{4}x_1^4 + c \]
\[ v_{x_2} = x_2^3 \quad \Rightarrow \quad v = \frac{1}{4}x_2^4 + c \]

\[ V = \frac{1}{6}x_1^6 + \frac{3}{4}x_1^4 + \frac{1}{4}x_2^4 + c \]
Region of attraction is the set of all points $x(t)$ such that $\lim_{t \to \infty} \phi(t; x) = 0$.

If the region of attraction is $\mathbb{R}^m = \text{globally asymptotically stable equilibrium}$

Barbashin-Krasovskii Theorem:

THEOREM 3.2. Let $x = 0$ be an equilibrium point for $x = f(x)$, let $V : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function such that

- $V(0) = 0$ and $V(x) > 0$, $\forall x \neq 0$
- $\|x\| \to \infty \Rightarrow V(x) \to \infty$ (radially bounded $V$)
- $V(x) < 0$, $\forall x \neq 0$

Then $x = 0$ is globally asymptotically stable.

Instability Theorem

THEOREM 3.3. Chetaev's Theorem

Let $x = 0$ be an equilibrium point for (3.1), let $V : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function such that $V(0) = 0$ and $V(x) > 0$ for some $x_0$ with arbitrary small $\|x_0\|$. Define a set $U$

$$U = \{ x \in B_r \mid V(x) > 0 \} \quad B_r = \{ x \in \mathbb{R}^m \mid \|x\| \leq r \}$$

and suppose that $V(x) > 0$ on $U$. Then $x = 0$ is unstable.
Exercise 3.3 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

(1) \[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2^2 \\
\dot{x}_2 &= -x_2
\end{align*}
\]

(2) \[
\begin{align*}
\dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\
\dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1)
\end{align*}
\]

(3) \[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1^2 x_2 \\
\dot{x}_2 &= -x_2 + x_1
\end{align*}
\]

(4) \[
\begin{align*}
\dot{x}_1 &= -x_1 - x_2 \\
\dot{x}_2 &= x_1 - x_2^3
\end{align*}
\]

(1) \[
\begin{align*}
\dot{x}_1 &= -x_1 + x_2^2 \\
\dot{x}_2 &= -x_2
\end{align*}
\]

Try \( V(x) = \frac{1}{2}(x_1^2 + x_2^2) \).

\( \dot{V}(x) = x_1(-x_1 + x_2^2) - x_1^2 = -x_1^2 - x_2^2 + x_1 x_2^2 \)

In the neighborhood of the origin, the term \(-(x_1^2 + x_2^2)\) dominates. Hence, the origin is asymptotically stable. Moreover

\[
\begin{align*}
x_2(t) &= e^{-t} x_2(0) \\
\Rightarrow x_1(t) &= e^{-t} x_1(0) + \int_0^t e^{-t} e^{-2s} ds x_2^2(0) \\
&= e^{-t} x_1(0) + \left[ e^{-t} - e^{-2t} \right] x_2^2(0)
\end{align*}
\]

For all \( x_0, \ x(t) \to 0 \) as \( t \to \infty \), which implies that the origin is globally asymptotically stable.

(2) \[
\begin{align*}
\dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1) \\
\dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1)
\end{align*}
\]

\[
\begin{align*}
V(x) &= ax_1^2 + bx_2^2, \quad a > 0, b > 0 \\
\dot{V}(x) &= 2a x_1(x_1 - x_2)(x_1^2 + x_2^2 - 1) + 2b x_2(x_1 + x_2)(x_1^2 + x_2^2 - 1) \\
&= 2[ax_1(x_1 - x_2) + bx_2(x_1 + x_2)](x_1^2 + x_2^2 - 1)
\end{align*}
\]

Let \( a = b \).

\[
\dot{V}(x) = -2a(x_1^2 + x_2^2)[1 - (x_1^2 + x_2^2)]
\]

For \( x_1^2 + x_2^2 < 1 \), \( \dot{V}(x) \) is negative definite. Hence, the origin is asymptotically stable. It is not globally asymptotically stable since there are other equilibrium points on the unit circle.

(3) \[
\begin{align*}
\dot{x}_1 &= -x_1 + x_1^2 x_2 \\
\dot{x}_2 &= -x_2 + x_1
\end{align*}
\]

\[
\begin{align*}
V(x) &= \frac{1}{2}(ax_1^2 + bx_2^2), \quad a > 0, b > 0 \\
\dot{V}(x) &= ax_1(-x_1 + x_1^2 x_2) + bx_2(-x_2 + x_1) \\
&= -ax_1^2 + bx_1 x_2 - bx_2^2 + ax_1^2 x_2 \\
&= -x^T Q x + ax_1^2 x_2
\end{align*}
\]

where \( Q = \begin{bmatrix} a & -0.5b \\ -0.5b & b \end{bmatrix} \). The matrix \( Q \) is positive definite when \( ab - b^2/4 > 0 \). Choose \( b = a = 1 \).

Near the origin, the quadratic term \(-x^T Q x\) dominates the fourth-order term \( x_1^2 x_2 \). Thus, \( \dot{V}(x) \) is negative definite and the origin is asymptotically stable. It is not globally asymptotically stable since there are other equilibrium points at \((1, 1)\) and \((-1, -1)\).
\[
\begin{aligned}
\dot{x}_1 &= -x_1 - x_2 \\
\dot{x}_2 &= x_1 - x_2^2
\end{aligned}
\]

Consider \( V(x) = ax_1^2 + bx_2^2 \), with \( a > 0 \) and \( b > 0 \), as a Lyapunov function candidate.

\[
\dot{V}(x) = 2ax_1(-x_1 - x_2) + 2bx_2(x_1 - x_2^2) = -2ax_1^2 - 2bx_2^2 - 2ax_1x_2 + 2bx_1x_2
\]

Take \( a = b \).

\[
\dot{V}(x) = -2ax_1^2 - 2bx_2^2 < 0, \quad \forall \ x \in \mathbb{R}^2, \ x \neq 0
\]

\( V(x) \) is radially unbounded. Hence, the origin is globally asymptotically stable.

---

**Exercise 3.4** Using \( V(x) = x_1^2 + x_2^2 \), study stability of the origin of the system

\[
\begin{aligned}
\dot{x}_1 &= x_1(k^2 - x_1^2 - x_2^2) + x_2(x_1^2 + x_2^2 + k^2) \\
\dot{x}_2 &= -x_1(k^2 + x_1^2 + x_2^2) + x_2(k^2 - x_1^2 - x_2^2)
\end{aligned}
\]

when (a) \( k = 0 \) and (b) \( k \neq 0 \).

---

\[
\begin{aligned}
\dot{x}_1 &= x_1(k^2 - x_1^2 - x_2^2) + x_2(x_1^2 + x_2^2 + k^2) \\
\dot{x}_2 &= -x_1(k^2 + x_1^2 + x_2^2) + x_2(k^2 - x_1^2 - x_2^2)
\end{aligned}
\]

Consider \( V(x) = x_1^2 + x_2^2 \) as a Lyapunov function candidate.

\[
\dot{V}(x) = 2x_1^2(k^2 - x_1^2 - x_2^2) + 2x_1x_2(x_1^2 + x_2^2 + k^2) \\
-2x_1x_2(x_1^2 + x_2^2 + k^2) + 2x_2^2(k^2 - x_1^2 - x_2^2)
\]

\[
= 2(x_1^2 + x_2^2)(k^2 - x_1^2 - x_2^2)
\]

(a) \( k = 0 \Rightarrow \dot{V}(x) = -2(x_1^2 + x_2^2)^2 \)

The origin is globally asymptotically stable.

(b) \( k \neq 0 \Rightarrow \dot{V}(x) > 0, \quad \text{for} \ 0 < x_1^2 + x_2^2 < k^2 \)

By Chetaev's theorem, the origin is unstable.

---

**Exercise 3.8** Consider the system

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 - \text{sat}(2x_1 + x_2)
\end{aligned}
\]

(a) Show that the origin is asymptotically stable.

(b) Show that all trajectories starting in the first quadrant to the right of the curve \( x_1x_2 = c \) (with sufficiently large \( c > 0 \)) cannot reach the origin.

(c) Show that the origin is not globally asymptotically stable.

**Hint:** In part (b), consider \( V(x) = x_1x_2 \); calculate \( \dot{V}(x) \) and show that on the curve \( V(x) = c \) the derivative \( \dot{V}(x) > 0 \) when \( c \) is large enough.

---

\* 3.8

(a) For \( |2x_1 + x_2| \leq 1 \), we have

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x
\]

This matrix is Hurwitz. Hence, the origin is asymptotically stable.
Let us now evaluate $\dot{V}(z)$ on the curve $z_1 z_2 = c$ and suppose $c > 0$ is chosen large enough to ensure that the curve $z_1 z_2 = c$ lies outside the strip $|2z_1 + z_2| \leq 1$. We have

$$\dot{V}(z) = z_1^2 + \frac{c^2}{z_1^2} - z_1$$

It can be seen that for $c \geq 1$, $\dot{V}(z)$ is positive for all $z_1 \geq 0$. Hence, all trajectories starting in the first quadrant to the right of the curve $z_1 z_2 = c$ cannot cross the curve. Consequently, they cannot reach the origin.

(c) It follows from (b) that the origin is not globally asymptotically stable.

Exercise 3.28 Show that the system

$$\begin{align*}
z_1 &= \frac{1}{1 + x_3} - x_1 \\
z_2 &= x_1 - 2x_2 \\
z_3 &= x_2 - 3x_3
\end{align*}$$

has a unique equilibrium point in the region $z_i \geq 0$, $i = 1, 2, 3$, and investigate stability of this point using linearization.

\[3.28\]

Substitution of $z_1$ and $z_3$ in the first equation yields

$$2x_2^2 + 6x_2 - 3 = 0 \Rightarrow x_2 = \frac{-3 \pm \sqrt{15}}{2}$$

Thus there is only one equilibrium point in the region $z_1 \geq 0$; namely,

$$z_1 = \sqrt{15} - 3, \quad z_2 = \frac{\sqrt{15} - 3}{2}, \quad z_3 = \frac{\sqrt{15} - 3}{6}$$

The eigenvalues are $-1.3671 \pm j 0.449$ and $-3.2657$. Hence, the equilibrium point is asymptotically stable.
3.3 Linear Systems and Linearization

\[ \dot{x} = Ax \]
\[ V(x) = x^T P x \quad P = P^T > 0 \quad \text{Lyapunov function} \]
\[ V(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x \leq 0 \]
\[ = -x^T Q x, \quad Q > 0 \]
\[ PA + A^T P = -Q \]

Lyapunov equation (asymptotic stability) (Hurwitz matrix)

(Thm. 3.6) A matrix \( A \) is a stability matrix, i.e., \( \Re \lambda < 0 \) for all eigenvalues of \( A \), if and only if for any given \( Q = Q^T > 0 \) there is a positive definite symmetric matrix \( P \) that satisfies the Lyapunov equation.

Ex. 3.13
\[ A = \begin{pmatrix} 0 & -1 \\ 1 & -4 \end{pmatrix}, \quad Q = I_2, \quad P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \]

\begin{align*}
& (1) \quad 2p_{12} = -1 \\
& (2) \quad -p_{11} - p_{12} + p_{22} = 0 \quad \Rightarrow \quad P = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 4 \end{pmatrix} \\
& (3) \quad -2p_{12} - 2p_{22} = -4
\end{align*}

However, there is no computational advantage in solving the Lyapunov equation over calculating the eigenvalues of \( A \), since the most efficient methods for solving the algebraic Lyapunov equation (Bartels-Stewart algorithm) first find the eigenvalues of \( AC \) by putting \( A \) onto Schur form
Thm. 3.6a. \( \text{If } x_1 \leq 0 \text{ and } \| x(t) \| \leq \text{const} \)

(bounded linear system motion is stable in the sense of Lyapunov) if for \( a > 0 \) there exists a matrix \( P > 0 \), for solving (*) \( A^T P + P A = -Q \) such that \( P > 0 \).

Stability of multiple eigenvalues on the imaginary axis \( \Rightarrow \) Jordan form

Lyapunov's Indirect Method - First Method of Lyapunov

Thm. 3.7. Let \( x = 0 \) be an equilibrium point for the nonlinear system

\[
  x = f(x)
\]

where \( f : D \rightarrow \mathbb{R}^n \) is continuously differentiable and \( D \) is a neighborhood of the origin. Let

\[
  A = \frac{\partial f}{\partial x} \bigg|_{x=0}
\]

Then,

1) The origin is asymptotically stable if \( \det A < 0 \) for all eigenvalues of \( A \).

2) The origin is unstable if \( \det A > 0 \) for one or more of the eigenvalues of \( A \).

Proof. No, too long, bad sample. (Pages 127–130).

(Ex. 3.15) The pendulum equation

\[
  \begin{aligned}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -\left(\frac{g}{l}\right) \sin x_1 - \left(\frac{\mu}{m}\right) x_2
  \end{aligned}
\]

two eq. points \((0, 0)\) and \((\pi, 0)\)

\[
  \frac{\partial f}{\partial x} = \begin{bmatrix}
  0 & 1 \\
  -\frac{g}{l} \cos x_1 & -\frac{\mu}{m}
  \end{bmatrix}
\]

at \((0, 0)\)

\[
  A = \begin{pmatrix}
  0 & 1 \\
  -\frac{g}{l} & -\frac{\mu}{m}
  \end{pmatrix} \Rightarrow \lambda_{1,2} = -\frac{\mu}{2m} \pm \frac{1}{2} \sqrt{\left(\frac{\mu}{m}\right)^2 - \frac{g}{l}}
\]

both stable

at \((\pi, 0)\)

\[
  A = \begin{pmatrix}
  0 & 1 \\
  \frac{g}{l} & -\frac{\mu}{m}
  \end{pmatrix} \Rightarrow \lambda_{1,2} = -\frac{\mu}{2m} \pm \frac{1}{2} \sqrt{\left(\frac{g}{l}\right)^2 + \frac{\mu}{2}}
\]

unstable
Exercise 3.30 For each of the following systems, use linearization to show that the origin is asymptotically stable. Then, show that the origin is globally asymptotically stable.

\begin{align*}
(1) \quad \dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= (x_1 + x_2) \sin x_1 - 3x_2 \\
(2) \quad \dot{x}_1 &= -x_1^3 + x_2 \\
\dot{x}_2 &= -ax_1 - bx_2, \quad a, b > 0
\end{align*}

\textit{Exercise 3.30 (1)}

\[\begin{align*}
0 &= -x_1 + x_2 \Rightarrow x_1 = x_2 \\
0 &= (x_1 + x_2) \sin x_1 - 3x_2
\end{align*}\]

Thus

\[x_1(2 \sin x_1 - 3) = 0 \Rightarrow x_1 = 0 \Rightarrow x_2 = 0\]

Hence, the origin is the unique equilibrium point.

\[\frac{\partial f}{\partial x} = \begin{bmatrix} -1 \\
\sin x_1 + (x_1 + x_2) \cos x_1 \\
\sin x_1 - 3 \end{bmatrix}\]

\[A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix} -1 & 1 \\
0 & -3 \end{bmatrix}\]

\[A\text{ is Hurwitz; hence, the origin is asymptotically stable. To show global asymptotic stability, let } V(x) = \frac{1}{2}(x_1^2 + x_2^2).\]

\[\dot{V}(x) = -x_1^2 + x_1x_2(1 + \sin x_1) - (3 - \sin x_1)x_2^2 \leq -x_1^2 + 2|x_1| |x_2| - 2x_2^2\]

\[= -\left[ \begin{bmatrix} |x_1| \\
|x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\
-1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\
|x_2| \end{bmatrix} < 0, \quad \forall x \neq 0\]

Hence, the origin is globally asymptotically stable.

\[\textit{Exercise 3.30 (2)}\]

\[\begin{align*}
0 &= -x_1^3 + x_2 \\
0 &= -ax_1 - bx_2 \Rightarrow x_2 = -\frac{b}{a}x_1 \Rightarrow -x_1(x_1^2 + a/b) = 0 \Rightarrow x_1 = 0
\end{align*}\]

Hence, the origin is the unique equilibrium point.

\[\frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\
-a & -b \end{bmatrix}, \quad A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix} 0 & 1 \\
-a & -b \end{bmatrix}\]

\[A\text{ is Hurwitz; hence, the origin is asymptotically stable. To show global asymptotic stability, let } V(x) = \frac{1}{2}(x_1^2 + ax_2^2), \quad a > 0.\]

\[\dot{V}(x) = -x_1^2 + x_1x_2(1 - a\alpha) - bax_2^2\]

Taking \(\alpha = 1/a\), we obtain

\[\dot{V}(x) = -x_1^2 - \frac{b}{a}x_2^2 < 0, \quad \forall x \neq 0\]

Hence, the origin is globally asymptotically stable.

\textit{Exercise 3.31 For each for the following systems, investigate stability of the origin.}

\textit{Exercise 3.31 (1)}

\[\begin{align*}
\dot{x}_1 &= -x_1 + x_2^2 \\
\dot{x}_2 &= -x_2 + x_3 \\
\dot{x}_3 &= x_3 - x_1^2
\end{align*}\]

\[\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + 2x_1 & 0 & 0 \\
0 & -1 & 2x_3 \\
-2x_1 & 0 & 1 \end{bmatrix}, \quad A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix} -1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \end{bmatrix}\]

\[A\text{ has an eigenvalue at 1; hence, the origin is unstable.}\]

\[\begin{align*}
\dot{x}_1 &= -2x_1 + x_1^3 \\
\dot{x}_2 &= -x_2 + x_1^2 \\
\dot{x}_3 &= -x_3
\end{align*}\]

\[\frac{\partial f}{\partial x} = \begin{bmatrix} -2 + 3x_1^2 & 0 & 0 \\
2x_1 & -1 & 0 \\
0 & 0 & -1 \end{bmatrix}, \quad A = \frac{\partial f}{\partial x}_{x=0} = \begin{bmatrix} -2 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \end{bmatrix}\]

The eigenvalues of \(A\) are \(-2, -1, -1\); hence, the origin is asymptotically stable.