332.346 DIGITAL SIGNAL PROCESSING – Spring 2002
Quiz 1

⇒ See both sides of the sheet.

The following are some useful formulae:

\[ a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \text{ for } |z| > |a|. \]  \hspace{1cm} (1)

\[ -a^n u[-n - 1] \leftrightarrow \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \text{ for } |z| < |a|. \]  \hspace{1cm} (2)

\[ \sum_{k=0}^{\infty} a^k = \frac{1}{1 - a} \text{ provided } |a| < 1. \]  \hspace{1cm} (3)

\[ \sum_{k=0}^{N-1} a^k = \frac{1 - a^N}{1 - a}. \]  \hspace{1cm} (4)

**Problem 1:** Find the z-transforms of the following signals along with their ROCs. State explicitly what properties of z-transforms if any you used.

**Problem 1a; 2 points:** \( x_1[n] = (0.75)^n u[n] - (3)^n u[-n - 1] \).

**Solution:** Using the formulae (1) and (2) with \( a = 0.75 \) and \( a = 3 \) respectively, we get

\[ X_1(z) = \frac{z}{z - 0.75} + \frac{z}{z - 3} \text{ for } |z| > 0.75 \text{ and } |z| < 3. \]

The ROC is given by \( 0.75 < |z| < 3 \).

**Problem 1b; 3.5 points:** \( x_2[n] = 2^{n+1} \cos\left(\frac{\pi}{2}n\right) = 2^nu[n] + 2^n e^{-j\frac{\pi}{2}n}u[n] = (2j)^nu[n] + (-2j)^nu[n] \).

Several equivalent ways of describing the signal \( x_2[n] \) are given above. You can use any form you like, however you are allowed to use only the formulae given above. If you use any other formula, you must derive it. The final result you get must show explicitly that the coefficients of \( X_2(z) \) do not depend on \( j \).

**Solution:** Consider \( x_2[n] = (2j)^nu[n] + (-2j)^nu[n] \). Using the formula (1) twice, first with \( a = 2j \) and then with \( a = -2j \), we get

\[ X_2(z) = \frac{z}{z - 2j} + \frac{z}{z + 2j} \text{ for } |z| > |2j| \text{ and } |z| > |-2j|. \]

We can simplify it as

\[ X_2(z) = z(z + 2j + z - 2j) \frac{2z^2}{(z - 2j)(z + 2j)} = \frac{2z^2}{(z^2 + 4)} \text{ for } |z| > |2|. \]

The ROC is given by \( |z| > 2 \).
Problem 2; 4.5 points: Consider the two-sided $z$-transform,

$$X(z) = 2 + \frac{1}{1 - 0.1z^{-1}} + \frac{2}{1 - 1.25z^{-1}} = 2 + \frac{z}{z - 0.1} + \frac{2z}{z - 1.25}.$$ 

There are THREE possible ROCs for this $X(z)$. Choose one possible ROC at a time, and then determine the corresponding $x[n]$.

Case 1, one possible ROC: Consider the ROC as $|z| < |0.1|$. In this case, both the poles arise from an anti-causal signal. Thus

$$x[n] = 2\delta[n] - (0.1)^nu[-n - 1] - 2(1.25)^nu[-n - 1].$$

Case 2, Another possible ROC: Consider the ROC as $|z| > |1.25|$. In this case, both the poles arise from a causal signal. Thus

$$x[n] = 2\delta[n] + (0.1)^nu[n] + 2(1.25)^nu[n].$$

Case 3, yet another possible ROC: Consider the ROC as $0.1 < |z| < |1.25|$. In this case, the pole at 0.1 arises from a causal signal and the pole at 1.25 arises from an anti-causal signal. Thus

$$x[n] = 2\delta[n] + (0.1)^nu[n] - 2(1.25)^nu[-n - 1].$$