1 Inverse of a Two-sided z-transform

In general the z-transform of a general two sided signal has a ROC that contains all points interior to an annular ring in the z-plane centered at the origin. The lower radius of the ring equals the maximum of all magnitudes of poles arising from the causal part, and the upper radius equals the minimum of all magnitudes of poles arising from the anti-causal part. Hence the poles arising from the casual part lie interior to the circle with lower radius while the poles arising from the anti-casual part lie exterior to the circle with upper radius. This is illustrated in Figure.

In view of the above discussion, a particular expression for \( X(z) \) can have different time domain functions depending upon the ROC. The following example illustrates this aspect.

Example 1.1

Consider a z-transform \( X(z) \) of a signal \( x[n] \) as given below,

\[
X(z) = \frac{112 - 44z^{-1} - 90z^{-2} + 15z^{-3}}{32 - 48z^{-1} - 2z^{-2} + 15z^{-3}}
\]

We can rewrite the denominator in terms of its roots, and then expand \( X(z) \) into its partial fraction expansions. This yields

\[
X(z) = \frac{80 + 4z^{-1} - 88z^{-2}}{32(1 + 0.5z^{-1})(1 - 0.75z^{-1})(1 - 1.25z^{-1})} + \frac{1}{1 + 0.5z^{-1}} + \frac{2}{1 - 0.75z^{-1}} + \frac{1.5}{1 - 1.25z^{-1}}
\]

Different possible ROCs and the corresponding time-domain signals are as given below.

Case 1, ROC: \(|z| < 0.5\). Figure shows the pattern of poles and the ROC. All the three poles -0.5, 0.75, and 1.25 are exterior to the ROC, i.e. to the circle with radius 0.5. Hence, all the poles arise from an anti-causal signal. The inverse of \( X(z) \) is then given by

\[
x[n] = \delta[n] + (-0.5)^nu[-n-1] - 2(0.75)^nu[-n-1] - 1.5(1.25)^nu[-n-1].
\]