Expansion of a signal in terms of a set of basis signals

For a number of reasons we would like to expand a signal \( x(t) \) in terms of a set of signals, called often as basis signals. Let basis signals be \( \phi_1(t), \phi_2(t), \phi_3(t), \ldots, \phi_N(t) \). Here \( N \) is the number of basis signals and it could be finite or infinite. Suppose we can approximate the signal \( x(t) \) over a time interval \( t_1 \leq t < t_2 \) by a linear combination of the basis signals as

\[
\hat{x}(t) = \sum_{i=1}^{N} X_i \phi_i(t).
\]

The closeness of the approximate signal \( \hat{x}(t) \) to \( x(t) \) in the interval \( t_1 \leq t < t_2 \) will depend on how we choose the coefficients \( X_i, i = 1 \) to \( N \). One measure of closeness often used is mean square error of the expansion,

\[
J_N = \int_{t_1}^{t_2} |x(t) - \hat{x}(t)|^2 \, dt = \int_{t_1}^{t_2} \left| x(t) - \sum_{i=1}^{N} X_i \phi_i(t) \right|^2 \, dt.
\]

The coefficients \( X_i, i = 1 \) to \( N \), are selected or determined by minimizing \( J_N \) with respect to them. In this regard, it turns out to be advantageous to have the basis signals mutually orthogonal.

**Definition:** Two signals \( \phi_k(t) \) and \( \phi_m(t) \) are mutually orthogonal over the interval \( t_1 \leq t < t_2 \) if and only if

\[
\int_{t_1}^{t_2} \phi_k(t)\phi_m^*(t) \, dt = \begin{cases} \lambda_k & \text{if } k = m \\ 0 & \text{if } k \neq m. \end{cases}
\]

Here \( * \) denotes complex conjugation. We note that \( \lambda_k \) is the integral of \( |\phi_k(t)|^2 \) and hence it is real and positive.

A set of basis signals is said to be mutually orthogonal if any two arbitrarily selected in the set are mutually orthogonal, i.e., all the possible selections of two signals from the set must result in orthogonality.

We assume that the given basis signals are mutually orthogonal, and begin to calculate the coefficients \( X_i, i = 1 \) to \( N \). To do so, we rewrite \( J_N \) as follows:

\[
J_N = \int_{t_1}^{t_2} \left| x(t) - \sum_{i=1}^{N} X_i \phi_i(t) \right|^2 \, dt
\]

\[
= \int_{t_1}^{t_2} \left[ x(t) - \sum_{i=1}^{N} X_i \phi_i(t) \right] \left[ x^*(t) - \sum_{k=1}^{N} \lambda_k \phi_k^*(t) \right] \, dt
\]

\[
= \int_{t_1}^{t_2} |x(t)|^2 \, dt - \int_{t_1}^{t_2} x(t) \sum_{k=1}^{N} \phi_k^*(t) \, dt - \int_{t_1}^{t_2} x^*(t) \sum_{i=1}^{N} X_i \phi_i(t) \, dt + \sum_{i=1}^{N} \sum_{k=1}^{N} X_i X_k^* \int_{t_1}^{t_2} \phi_i(t)\phi_k^*(t) \, dt.
\]

We need to simplify \( J_N \) further in order to see how we can minimize it. Because the basis signals are mutually orthogonal, one can rewrite the last term as (show it)

\[
\sum_{i=1}^{N} \sum_{k=1}^{N} X_i X_k^* \int_{t_1}^{t_2} \phi_i(t)\phi_k^*(t) \, dt = \sum_{i=1}^{N} \lambda_i |X_i|^2.
\]
Also, let
\[ A_i = \frac{1}{\lambda_i} \int_{t_1}^{t_2} x(t) \phi_i^*(t) \, dt. \]

In view of the last two equations, \( J_N \) can be rewritten as
\[ J_N = \int_{t_1}^{t_2} |x(t)|^2 \, dt - \sum_{i=1}^{N} \lambda_i A_i X_i^* - \sum_{i=1}^{N} \lambda_i A_i^* X_i + \sum_{i=1}^{N} \lambda_i |X_i|^2. \]

(Note that \( \lambda_i \) is a real number.) We will rewrite the last three terms of \( J_N \) by what is known as completing the squares technique. Consider the following identity (verify it),
\[
\sum_{i=1}^{N} \lambda_i |A_i - X_i|^2 = \sum_{i=1}^{N} \lambda_i |A_i - X_i| [A_i^* - X_i^*] = \sum_{i=1}^{N} \lambda_i |A_i|^2 - \sum_{i=1}^{N} \lambda_i A_i X_i^* - \sum_{i=1}^{N} \lambda_i A_i^* X_i + \sum_{i=1}^{N} \lambda_i |X_i|^2.
\]

Thus
\[
- \sum_{i=1}^{N} \lambda_i A_i X_i^* - \sum_{i=1}^{N} \lambda_i A_i^* X_i + \sum_{i=1}^{N} \lambda_i |X_i|^2 = \sum_{i=1}^{N} \lambda_i |A_i - X_i|^2 \sum_{i=1}^{N} \lambda_i |A_i|^2 - \sum_{i=1}^{N} \lambda_i |A_i|^2.
\]

Substituting the above equation in the expression for \( J_N \) we get,
\[
J_N = \int_{t_1}^{t_2} |x(t)|^2 \, dt + \sum_{i=1}^{N} \lambda_i |A_i - X_i|^2 - \sum_{i=1}^{N} \lambda_i |A_i|^2.
\]

From the above expression for \( J_N \) we see that \( J_N \) is minimum with respect to the coefficients \( X_i, i = 1, 2, \cdots, N \), if and only if \( X_i = A_i \). This tells us how to calculate the coefficient \( X_i \),
\[
X_i = A_i = \frac{1}{\lambda_i} \int_{t_1}^{t_2} x(t) \phi_i^*(t) \, dt \quad i = 1, 2, \cdots, N.
\]

Note that with the selection of \( X_i \) as given by the above equation, the mean square error of the expansion, namely \( J_N \), is given by
\[
J_N = \int_{t_1}^{t_2} |x(t)|^2 \, dt - \sum_{i=1}^{N} \lambda_i |A_i|^2.
\]

Note that \( X_i \) that minimizes \( J_N \) depends only on \( x(t) \) and the basis function \( \phi_i(t) \). In particular, \( X_i \) does not depend on any other \( \phi_k(t) \) for \( k \neq i \). This property is called the finality property of orthogonal expansions. Among other attributes, this property lets us expand or shrink the basis i.e. increase or decrease the number of basis signals \( N \) without affecting the values of coefficients already selected.

One may wish to increase \( N \) in order to possibly decrease the expansion error \( J_N \). A set of basis functions by using which \( J_N \) tends to zero as \( N \) tends to infinity is called the complete basis. When the basis is complete, we have
\[
\lim_{N \to \infty} J_N = \int_{t_1}^{t_2} |x(t)|^2 \, dt - \sum_{i=1}^{\infty} \lambda_i |A_i|^2 = 0.
\]

Hence the signal energy is given by
\[
\int_{t_1}^{t_2} |x(t)|^2 \, dt = \sum_{i=1}^{\infty} \lambda_i |A_i|^2.
\]

The above equation is known as Parseval’s theorem.
Example: Consider $x(t) = 1 + t^2$ as shown over the interval $0 \leq t < 2$. Also, consider two basis functions $\phi_1(t) = 1$ and $\phi_2(t) = t - 1$ as shown. Expand $x(t)$ in terms of $\phi_1(t)$ and $\phi_2(t)$ and calculate the minimum mean square error $J_2$.

The graph shows the closeness of $\hat{x}(t)$ to $x(t)$.

Figures and graphs are missing in this file.
Exponential and Trigonometric Fourier Series

Both Exponential and Trigonometric Fourier Series play a central role in a number of areas of Electrical Engineering including Communication Theory, Digital Signal Processing, and Control Theory. Exponential and Trigonometric Fourier Series are two different forms of the same type of expansion. This will become obvious later on when we take into account the celebrated Euler’s formula \( e^{i\theta} = \cos \theta + j \sin \theta \). Almost all types of transforms of signals (Fourier Transforms, Laplace Transforms, Discrete-time Fourier Transforms, Discrete Fourier Transforms, Fast Fourier Transforms etc.) can be directly traced for their beginnings to Exponential Fourier Series. In other words, the study of Fourier Series paves the way to study different kinds of transforms of signals. Thus the study of Fourier Series is significant not only for its direct use but also because it sets the path for other studies.

Often Exponential and Trigonometric Fourier Series are discussed in connection with periodic signals. The expansions are constructed to be valid on one period, and then by extending the expansion to other periods they become valid for the complete time domain. Let \( T \) be the period of a periodic signal. This period in general occupies the interval \( t_2 = t_1 + T \) where \( t_1 \) is arbitrary. Often \( t_1 \) is taken as either 0 or \( -\frac{T}{2} \). Let

\[
\begin{align*}
f_0 &= \frac{1}{T} = \text{fundamental frequency,} \\
\omega_0 &= 2\pi f_0 = \text{fundamental radian frequency.}
\end{align*}
\]

**Exponential Fourier Series:** As the name implies, here we choose the basis functions \( \phi_i(t), i = -\infty, \cdots, -1, 0, 1, \cdots, \infty \), as exponential functions,

\[
\phi_i(t) = e^{i\omega_0 t}, \quad i = -\infty, \cdots, -2, -1, 0, 1, 2, \cdots, \infty.
\]

We note that, except for \( i = 0 \), the integral \( \int_{t_1}^{t_2} e^{i\omega_0 t} dt \) is zero. Let us next check the mutual orthogonality of these basis functions. For all \( -\infty < m < \infty \) and \( -\infty < k < \infty \), we get

\[
\int_{t_1}^{t_2} \phi_m(t)\phi^*_k(t)dt = \int_{t_1}^{t_2} e^{i(m-k)\omega_0 t}dt
\]

\[
= \begin{cases} 
\int_{t_1}^{t_2} 1 dt = T & \text{if } k = m \\
\frac{1}{i(m-k)\omega_0} \left[ e^{i(m-k)\omega_0 t_2} \right]_{t_1}^{t_2} = 0 & \text{if } k \neq m.
\end{cases}
\]

The last step follows because \( \omega_0 t_2 = \omega_0(T + t_1) = (2\pi + \omega_0 t_1) \) and thus

\[
\left[ e^{i(m-k)\omega_0 t} \right]_{t_1}^{t_2} = \left[ e^{i(m-k)\omega_0 t_2} - e^{i(m-k)\omega_0 t_1} \right] = e^{i(m-k)\omega_0 t_1} \left[ e^{i(m-k)2\pi} - 1 \right] = 0.
\]

One can look at the above result in another way. For \( k \neq m \), we note that the region of integration in the integral \( \int_{t_1}^{t_2} \phi_m(t)\phi^*_k(t)dt \) is over an integral number of periods of integrand and thus the integral is equal to the area of integrand over an integral number of periods; and this area is indeed zero.
Now that the basis functions are mutually orthogonal, we can write the Exponential Fourier Series expansion of a periodic function \( x(t) \) with a period \( T \) as

\[
x(t) = \sum_{i=-\infty}^{\infty} X_i e^{j \omega_0 t} \quad \text{for all} \quad -\infty < t < \infty
\]

\[
X_i = \frac{1}{T} \int_{t_1}^{t_2} x(t) e^{-j \omega_0 t} \, dt \quad \text{for all integers} \quad i, \quad -\infty < i < \infty.
\]

In particular,

\[
X_0 = \frac{1}{T} \int_{t_1}^{t_2} x(t) \, dt = \text{Average value of} \ x(t).
\]

We observe that in general the value of any coefficient \( X_i \) is complex.

Let us suppose we use only a finite number of terms in the expansion of \( x(t) \), that is consider

\[
\hat{x}(t) = \sum_{i=-N}^{N} X_i e^{j \omega_0 t}.
\]

The calculation of coefficient \( X_i \) is as given earlier because of the finality property of orthogonal expansions. One can easily show (try it by invoking the mutual orthogonality property of basis functions) that

\[
\int_{t_1}^{t_2} |\hat{x}(t)|^2 \, dt = \sum_{i=-N}^{N} T |X_i|^2.
\]

Also, when only finite number of terms are used, the mean square error in expansion is given by

\[
\int_{t_1}^{t_2} |x(t) - \hat{x}(t)|^2 \, dt = \sum_{i=-\infty}^{-(N+1)} T |X_i|^2 + \sum_{i=(N+1)}^{\infty} T |X_i|^2.
\]

This can be proved (not done here) by showing that the exponential functions used here as basis indeed form a complete basis, and then invoking the mutual orthogonality property of basis functions.

**Trigonometric Fourier Series:** The Exponential Fourier series expansion given above can be rewritten to yield Trigonometric Fourier Series expansion. We will do this only for real valued \( x(t) \). That is, we consider \( x(t) \) to be a real signal. In this case, one can easily show that the coefficients \( X_i \) and \( X_{-i} \) are complex conjugates of one another, \( X_{-i} = X_i^* \) (show this). Let us also observe that the sum of any complex number \( c \) and its complex conjugate \( c^* \) equals twice the real value of \( c \).

Let us rewrite now Exponential Fourier series as

\[
x(t) = \sum_{i=-\infty}^{\infty} X_i e^{j \omega_0 t}
\]

\[
= X_0 + \left[ X_1 e^{j \omega_0 t} + X_{-1} e^{-j \omega_0 t} \right] + \left[ X_2 e^{j 2 \omega_0 t} + X_{-2} e^{-j 2 \omega_0 t} \right] + \cdots + \left[ X_i e^{j \omega_0 t} + X_{-i} e^{-j \omega_0 t} \right] + \cdots
\]

\[
= X_0 + \left[ X_1 e^{j \omega_0 t} + X_1^* e^{-j \omega_0 t} \right] + \left[ X_2 e^{j 2 \omega_0 t} + X_2^* e^{-j 2 \omega_0 t} \right] + \cdots + \left[ X_i e^{j \omega_0 t} + X_i^* e^{-j \omega_0 t} \right] + \cdots
\]

\[
= X_0 + \sum_{i=1}^{\infty} \left[ X_i e^{j \omega_0 t} + X_i^* e^{-j \omega_0 t} \right].
\]
Let us next denote \( X_i \) in its rectangular form as

\[
X_i = \alpha_i + j \beta_i.
\]

We have the following algebraic reductions:

\[
X_i e^{j \omega t} + X_i^* e^{-j \omega t} = 2 \text{ Real value of } X_i e^{j \omega t}
\]

\[
eq 2 \text{ Real value of } (\alpha_i + j \beta_i)(\cos(i \omega t) + j \sin(i \omega t))
\]

\[
eq 2 \alpha_i \cos(i \omega t) - 2 \beta_i \sin(i \omega t).
\]

Using the above reductions, we get

\[
x(t) = X_0 + \sum_{i=1}^{\infty} \left[ X_i e^{j \omega t} + X_i^* e^{-j \omega t} \right]
\]

\[
= X_0 + \sum_{i=1}^{\infty} 2 \alpha_i \cos(i \omega t) - \sum_{i=1}^{\infty} 2 \beta_i \sin(i \omega t)
\]

\[
= A_0 + \sum_{i=1}^{\infty} A_i \cos(i \omega t) + \sum_{i=1}^{\infty} B_i \sin(i \omega t)
\]

where (verify them)

\[
A_0 = X_0 = \text{ Average value of } x(t),
\]

\[
A_i = 2 \alpha_i = 2 \text{ Real part of } X_i
\]

\[
= \frac{2}{T} \int_{t_1}^{t_2} x(t) \cos(i \omega t) dt \quad \text{for all integers } i, \ 1 \leq i < \infty,
\]

\[
B_i = -2 \beta_i = 2 \text{ Imaginary part of } X_i
\]

\[
= \frac{2}{T} \int_{t_1}^{t_2} x(t) \sin(i \omega t) dt \quad \text{for all integers } i, \ 1 \leq i < \infty.
\]

Rightfully, the above expansion is called \textit{Trigonometric Fourier Series}.

One can derive the above Trigonometric Fourier Series directly. For this purpose, one needs to select first the appropriate basis signals as

\[
\phi_0(t) = 1, \ \phi_i(t) = \cos(i \omega_0 t), \ \ i = 1, 2, \cdots \infty
\]

and

\[
\psi_i(t) = \sin(i \omega_0 t), \quad i = 1, 2, \cdots \infty.
\]

We need to prove next that the above basis signals are mutually orthogonal. Then the formulae for the coefficients follow directly.
The following calculations show that the above basis signals are mutually orthogonal:

\[
\int_{t_1}^{t_2} \phi_0(t) \phi_i^*(t) dt = \int_{t_1}^{t_2} \cos(i \omega_0 t) dt = 0, \quad 1 \leq i < \infty.
\]

\[
\int_{t_1}^{t_2} \phi_0(t) \psi_i^*(t) dt = \int_{t_1}^{t_2} \sin(i \omega_0 t) dt = 0, \quad 1 \leq i < \infty.
\]

For all \(1 \leq m < \infty\) and \(1 \leq k < \infty\), we get

\[
\int_{t_1}^{t_2} \phi_m(t) \phi_k^*(t) dt = \int_{t_1}^{t_2} \cos(m \omega_0 t) \cos(k \omega_0 t) dt
\]

\[
= \frac{1}{2} \int_{t_1}^{t_2} \left[ \cos((m + k) \omega_0 t) + \cos((m - k) \omega_0 t) \right] dt
\]

\[
= \begin{cases} 
\int_{t_1}^{t_2} \frac{1}{2} dt = \frac{T}{2} & \text{if} \ k = m \\
0 & \text{if} \ k \neq m,
\end{cases}
\]

\[
\int_{t_1}^{t_2} \psi_m(t) \psi_k^*(t) dt = \int_{t_1}^{t_2} \sin(m \omega_0 t) \sin(k \omega_0 t) dt
\]

\[
= \frac{1}{2} \int_{t_1}^{t_2} \left[ \cos((m - k) \omega_0 t) - \cos((m + k) \omega_0 t) \right] dt
\]

\[
= \begin{cases} 
\int_{t_1}^{t_2} \frac{1}{2} dt = \frac{T}{2} & \text{if} \ k = m \\
0 & \text{if} \ k \neq m,
\end{cases}
\]

\[
\int_{t_1}^{t_2} \phi_m(t) \psi_k^*(t) dt = \int_{t_1}^{t_2} \cos(m \omega_0 t) \sin(k \omega_0 t) dt
\]

\[
= \frac{1}{2} \int_{t_1}^{t_2} \left[ \sin((k + m) \omega_0 t) + \sin((k - m) \omega_0 t) \right] dt
\]

\[
= 0 \quad \text{for all} \ k \text{ and } m.
\]

Now it is easy to write the Trigonometric Fourier Series as

\[x(t) = A_0 + \sum_{i=1}^{\infty} A_i \cos(i \omega_0 t) + \sum_{i=1}^{\infty} B_i \sin(i \omega_0 t),\]

where

\[
A_0 = \frac{1}{T} \int_{t_1}^{t_2} x(t) dt = \text{Average value of } x(t),
\]

\[
A_i = \frac{2}{T} \int_{t_1}^{t_2} x(t) \cos(i \omega_0 t) dt \quad \text{for all integers } i, \ 1 \leq i < \infty,
\]

\[
B_i = \frac{2}{T} \int_{t_1}^{t_2} x(t) \sin(i \omega_0 t) dt \quad \text{for all integers } i, \ 1 \leq i < \infty.
\]
Trigonometric Fourier Series as given above is in what is known as rectangular form. It can also be written in another form called polar form by combining a cosine and a sine function together into a cosine function along with a certain phase angle. In this regard, we note the following trigonometric identity,

\[ C \cos(\omega_0 t + \theta) = C \cos \theta \cos \omega_0 t - C \sin \theta \sin \omega_0 t \]
\[ = A \cos \omega_0 t + B \sin \omega_0 t \]

where

\[ A = C \cos \theta \quad \text{and} \quad B = -C \sin \theta \]

or equivalently

\[ C = \sqrt{A^2 + B^2} \quad \text{and} \quad \tan \theta = \frac{-B}{A}. \]

The above identity allows us to rewrite the Trigonometric Fourier Series as

\[ x(t) = A_0 + \sum_{i=1}^{\infty} A_i \cos(i\omega_0 t) + \sum_{i=1}^{\infty} B_i \sin(i\omega_0 t) \]
\[ = C_0 + \sum_{i=1}^{\infty} C_i \cos(i\omega_0 t + \theta_i) \]

where

\[ C_0 = A_0 \]
\[ C_i = \sqrt{A_i^2 + B_i^2} \]
\[ \tan \theta_i = \frac{-B_i}{A_i}. \]

There exists as well a direct relationship between Exponential and Trigonometric Fourier Series in polar form. To see this, let

\[ X_i = \text{The } i\text{-th Exponential Fourier Coefficient} = |X_i|e^{j\theta_i} = |X_i|\angle \theta_i. \]

Then we have

\[ x(t) = X_0 + \sum_{i=1}^{\infty} [X_i e^{j\omega_0 t} + X_i^* e^{-j\omega_0 t}] = X_0 + \sum_{i=1}^{\infty} \left[ |X_i| e^{j(\omega_0 t + \theta_i)} + |X_i| e^{-j(\omega_0 t + \theta_i)} \right] \]
\[ = X_0 + \sum_{i=1}^{\infty} 2 |X_i| \cos(i\omega_0 t + \theta_i) \]
\[ = C_0 + \sum_{i=1}^{\infty} C_i \cos(i\omega_0 t + \theta_i) \]

where obviously

\[ C_0 = X_0, \quad C_i = 2 |X_i|, \quad \text{and} \quad \theta_i = \text{angle of } X_i, \quad 1 \leq i < \infty. \]

**Examples:** Typical periodic signals that are commonly used are tabulated as shown. This is missing in this file.