NAME OF THE STUDENT: Solution
Last four digits of Student ID #: 

This is a closed-book closed-notes exam. Do all your work on the sheets provided. If more space is required, work on the back of the sheets and indicate accordingly.

Do not ask questions (unless you find a typo). Part of the exam is understanding the problems.

Please make sure that there are 8 pages in this booklet excluding this cover page. Pages 5 and 6 contain semi-log sheets for practice of Bode plots. The final Bode plots must be given on pages 3 and 4.

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Total points earned by the student =
Problem 1a (10 points); Time to do the problem not more than 6 minutes.

Determine the transfer function \( H(s) \) of the circuit shown. If you do it correctly, you will get \( H(s) \) of the form

\[
H(s) = \frac{\omega_0^2}{s^2 + \beta s + \omega_0^2}.
\]

Identify the values of \( \beta \) and \( \omega_0^2 \) in terms of \( R, L, \) and \( C \).

By voltage division rule, we have

\[
H(s) = \frac{\frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{1}{\frac{1}{s^2LC} + sRC + 1} = \frac{1}{\frac{1}{LC} + \frac{R}{L} s + \frac{1}{LC}}.
\]

We can rewrite the transfer function \( H(s) \) as

\[
H(s) = \frac{\omega_0^2}{s^2 + \beta s + \omega_0^2},
\]

where

\[
\omega_0^2 = \frac{1}{LC} \quad \text{and} \quad \beta = \frac{R}{L}.
\]
Problem 1b (15 points); Time to do the problem not more than 14 minutes.

Determine the transfer function $H(s)$ of the ideal Op-Amp circuit shown where the output is $V_o$ and the input is $V_g$. To simplify algebra, assume that $R_1 = R_2 = R_3 = 1 \Omega$. Express $H(s)$ in the form

$$H(s) = \frac{-K\beta s}{s^2 + \beta s + \omega_o^2}$$

and identify the values of $K$, $\beta$, and $\omega_o^2$ in terms of the parameter $C$.

We will use $R_1 = R_2 = R_3 = R$. Let G be the reference node, and consider the node voltages as marked. Then the node equation at node $V_1$ can be written as

$$\frac{V_1 - V_g}{R} + sCV_1 + \frac{V_3}{R} + sC(V_1 - V_o) = 0.$$  

We can write a second node equation at N as

$$-\frac{V_o}{R} - sCV_1 = 0.$$  

From the above equation, we get

$$V_1 = -\frac{1}{sCR}V_o.$$  

Substituting the above in the very first node equation, we get

$$\left[\frac{1}{sCR^2} + \frac{1}{R} + \frac{1}{sCR^2} + \frac{1}{R} + sC\right]V_o = -\frac{V_g}{R}.$$  

Multiplying throughout by $\frac{1}{C}$ and simplifying, we get

$$\left[s^2 + \frac{2s}{RC} + \frac{2}{C^2R^2}\right]V_o = -\frac{sV_g}{RC}.$$  

Let

$$\omega_o^2 = \frac{2}{C^2R^2}, \quad \beta = \frac{2}{RC}, \quad K = \frac{1}{2} \quad \text{so that} \quad K\beta = \frac{1}{RC}.$$  

Then the transfer function is given by

$$\frac{V_o}{V_g} = H(s) = \frac{-K\beta s}{s^2 + \beta s + \omega_o^2}.$$  

We note that if $R_1 = R_2 = R_3 = 1 \Omega,$

$$\omega_o^2 = \frac{2}{C^2}, \quad \beta = \frac{2}{C}, \quad K = \frac{1}{2}.$$
Problem 3 (20 points); Time to do the problem not more than 15 minutes.
Using its definition, determine the half-power frequency (3 dB frequency) of \( |H(j\omega)| \) where

\[
H(s) = \frac{1}{(s + 1)^3}.
\]

If you use any formula, you must first derive it. We note that the maximum of \( |H(j\omega)| \) with respect to \( \omega \) occurs at \( \omega = 0 \) and the maximum magnitude equals 1.

Let us note that the half-power frequency (3 dB frequency) of \( |H(j\omega)| \) is the frequency \( \omega \) when \( |H(j\omega)| \) equals \( \frac{1}{\sqrt{2}} \). This yields

\[
\left( \frac{1}{\sqrt{1 + \omega^2}} \right)^3 = \left( \frac{1}{1 + \omega^2} \right)^{\frac{3}{2}} = \frac{1}{\sqrt{2}} = \left[ \frac{1}{2} \right]^{\frac{3}{2}}
\]

This implies that

\[
\frac{1}{1 + \omega^2} = \left[ \frac{1}{2} \right]^{\frac{3}{2}} = \frac{1}{2^{\frac{3}{2}}}.
\]

This implies further that

\[
1 + \omega^2 = 2^{\frac{3}{2}}.
\]

Hence

\[
\omega^2 = 2^{\frac{3}{2}} - 1 \Rightarrow \omega = \sqrt{2^{\frac{3}{2}} - 1}.
\]

If

\[
H(s) = \frac{1}{(s + 1)^n},
\]

we can see easily that the half-power frequency (3 dB frequency) is given by

\[
\omega = \sqrt{2^{\frac{n}{2}} - 1}.
\]
Problem 4 (25 points); Time to do the problem not more than 15 minutes.

It is known that

\[ H(s)H(-s) = \frac{1}{1 - s^6}. \]

Determine \( H(s) \) such that it has poles only in the left half of \( s \)-plane. Show all your work clearly. It is known that

\[ (s - 1/\theta^o)(s - 1/-\theta^o) = s^2 - (1/\theta^o + 1/-\theta^o)s + 1 = s^2 - 2 \cos(\theta)s + 1. \]

![Diagram of unit circle with roots marked]

We need to compute the roots of \( 1 - s^6 = 0 \). The six roots of \( 1 - s^6 = 0 \) are given by

\[ s^6 = 1 / 0^o = 1 / 360n^o, \]

where \( n \) is an integer. Therefore, the six roots are given by

\[
\begin{align*}
    s_1 &= 1 / 0^o = 1 \\
    s_2 &= 1 / 60^o \\
    s_3 &= 1 / 120^o \\
    s_4 &= 1 / 180^o = -1 \\
    s_5 &= 1 / -120^o \\
    s_6 &= 1 / -60^o \\
\end{align*}
\]

The roots are shown on the unit circle in the figure given above. The roots \( s_3, s_4, \) and \( s_5 \) are in the left-half \( s \) plane and correspond to stable roots of \( H(s) \). Hence \( H(s) \) is given by

\[
H(s) = \frac{1}{(s + 1)(s - 1/120^o)(s - 1/-120^o)} = \frac{1}{(s + 1)(s^2 - 2 \cos(120^o)s + 1)} = \frac{1}{(s + 1)(s^2 + s + 1)}. 
\]