Ch. 3 Homework Overview (1/7)

P3.1: Find $E[N_Q]$ and $E[W]$ for M/M/1 queue under infinite and finite buffer cases

Assume Poisson arrivals and Poisson server process

\[ \lambda_1 = \frac{1}{10}, \quad \lambda_2 = \frac{4}{5}, \quad \mu = 2 \]
Ch. 3 Homework Overview (2/7)

P3.2: M/M/1/K queue, $\rho = \frac{\lambda}{\mu} = 1$

Verify blocking probability $P_B$

Blocking probability derived in class but can not be applied directly

$$
\Rightarrow P_B = P_K = \frac{1 - \rho}{1 - \rho^{K+1}} \cdot \rho^K
$$

However, balanced equations for steady state probabilities still hold:

$$
\lambda \cdot p_{n-1} = \mu \cdot p_n, \quad 1 \leq n \leq K
$$

$$
\lambda = \mu \Rightarrow p_{n-1} = p_n
$$

Ch. 3 Homework Overview (3/7)

P3.3: Derive expected inter-arrival time ($E[A]$) and expected inter-departure time ($E[D]$) for a M/M/1 system

Define $F_A(a) \equiv$ CDF of $A = P[A \leq a]$

Poisson arrival process

$$
= 1 - \frac{(\lambda \cdot a)^k \cdot e^{-\lambda a}}{k!}
$$

$$
= 1 - e^{-\lambda a}, \quad a \geq 0
$$

This exercise utilizes the interrelationship between Poisson and exponential RVs
Ch. 3 Homework Overview (4/7)

P3.6: M/M/m/K queuing system with \( N_C \) customers
- Sketch state transition diagram; Find \( E[N_Q] \), \( E[T] \) and \( P_B \)
- Error in problem statement:
  \( N_C \geq K = 2 \times m \) \text{ NOT } \( K = m \)

Ch. 3 Homework Overview (5/7)

P3.7: M/M/m/m queuing system for circuit switched voice application
- Compute probability of call blocking (\( P_B \)), expected number of active calls (\( E[N] \)) and expected queuing delay (\( E[W] \))
- Similar to lecture notes

P3.8: Given Poisson arrivals, exponential call holding time and maximum allowable call blocking probability, how many voice circuits are required?
- Use M/M/m/m queuing system model
Ch. 3 Homework Overview (6/7)

P3.9: Source modeling for digitized voice
• Active state (A) and silent state (S)
  • $P_A \equiv$ Probability of being in A
  • $P_S \equiv$ Probability of being in S
• Part (a): In steady state, rate of transition from A to S equals rate of transition from S to A
• Part (b): Number of packets in each talk spurt (i.e., number of packets in a given active interval) is in accordance with a geometric RV distribution with parameter $\alpha$

Ch. 3 Homework Overview (7/7)

P3.12: M/M/$m/\infty$ queue
• Sketch state transition diagram; Find $p_n$ and $P_B$
• See lecture notes

Note: Problems and 3.4 and 3.5 have been removed from Part 1 of the Ch. 3 homework assignment (will revisit these problems if time permits)
Derivation of Little’s Formula (1/5)

\[ \bar{N} = \lambda \cdot \bar{T} \]

- \( N(t) \equiv \text{Number of customers in system at time } t \)
- \( \alpha(t) \equiv \text{Number of customers who arrived in interval } [0,t] \)
- \( \beta(t) \equiv \text{Number of customers who departed in interval } [0,t] \)
  
  Note: \( N(t) = \alpha(t) - \beta(t) \)
- \( T_j \equiv \text{Time spent in the system by the } j^{\text{th}} \text{ arriving customer} \)

Derivation of Little’s Formula (2/5)

- Apply time averages up to time \( t \)

\[
\bar{N}_t = \frac{1}{t} \cdot \int_0^t N(\tau)d\tau \Rightarrow \bar{N} = \lim_{t \to \infty} \bar{N}_t
\]

\[
\bar{\lambda}_t = \frac{\alpha(t)}{t} \Rightarrow \bar{\lambda} = \lim_{t \to \infty} \bar{\lambda}_t
\]

\[
\bar{T}_t = \frac{\sum_{k=0}^{\alpha(t)} T_k}{\alpha(t)} \Rightarrow \bar{T} = \lim_{t \to \infty} \bar{T}_t
\]
Derivation of Little’s Formula (3/5)

• Area between the curves \( \alpha(\tau) \) and \( \beta(\tau) \) is given by:

\[
\int_0^t (\alpha(\tau) - \beta(\tau)) d\tau = \int_0^t N(\tau) d\tau
\]

• At instances when \( N_t = \alpha(\tau) - \beta(\tau) = 0 \), the total time spent by customers in the system is also equal to the area between the curves \( \alpha(\tau) \) and \( \beta(\tau) \):

\[
\sum_{k=1}^{\alpha(t)} T_k = \int_0^t (\alpha(\tau) - \beta(\tau)) d\tau = \int_0^t N(\tau) d\tau
\]

\[
\Rightarrow \int_0^t N(\tau) d\tau = \sum_{k=1}^{\alpha(t)} T_k
\]

Derivation of Little’s Formula (4/5)

\[
\int_0^t N(\tau) d\tau = \sum_{k=1}^{\alpha(t)} T_k
\]

• Dividing both sides of above relation by \( t \) yields:

\[
\frac{1}{t} \int_0^t N(\tau) d\tau = \frac{1}{t} \sum_{k=1}^{\alpha(t)} T_k = \frac{\alpha(t)}{t} \cdot \frac{\sum_{k=1}^{\alpha(t)} T_k}{\alpha(t)}
\]

• Recalling earlier relations for \( N_t, \lambda_t \) and \( T_t \):

\[
N_t = \frac{1}{t} \int_0^t N(\tau) d\tau \quad \lambda_t = \frac{\alpha(t)}{t} \quad T_t = \sum_{k=0}^{\alpha(t)} T_k
\]

\[
\Rightarrow N_t = \lambda_t \cdot T_t \text{ for } t \text{ such that } N_t = 0
\]
Derivation of Little’s Formula (5/5)

- Considering now for instances when \( N(t) \neq 0 \):
  \[
  \frac{\beta(t)}{t} \cdot \sum_{k=1}^{\beta(t)} T_k \leq N_i \leq \lambda_i \cdot T_i
  \]

- Assuming \( N_i \to \overline{N} \), \( \lambda_i \to \lambda \) and \( T_i \to \overline{T} \)

  and that the departure rate \( \beta(t)/t \to \lambda \) as \( t \to \infty \),

  then Little’s formula is obtained

  \[
  \Rightarrow \overline{N} = \lambda \cdot \overline{T}
  \]

Little’s Formula and Network-Wide Delay (1/3)

E.g., a network of queues:

- Indicates a point or node at which packets are routed, or switched, onto one of two or more alternative links
Little’s Formula and Network-Wide Delay (2/3)

- $\alpha \equiv$ Aggregate traffic arrival rate into the network
- $S$ corresponds to a source node that originates the external traffic inputted to the network
- $D$ corresponds to an external sink that receives departing traffic
- $\alpha_{k,j} \equiv$ Probability that a packet that has been processed at node $k$ (or inputted by $S$) is routed/switched to node $j$ (or outputted to $D$)
- $\lambda_j \equiv$ Total traffic inputted to the $j^{th}$ buffer/queue
  $\lambda_S = \lambda_D = \alpha$

Little’s Formula and Network-Wide Delay (3/3)

- Little’s formula applies also at each individual queue in the network
- For the $j^{th}$ queue: $E[T_j] = E[n_j] / \lambda_j$
- Letting $\alpha$ be the aggregate arrival rate to the network, $E[T]$ is given by:

$$\bar{T} = \frac{E[N]}{\alpha} = \frac{1}{\alpha} \cdot \sum_{j=1}^{J} E[n_j] = \frac{1}{\alpha} \cdot \sum_{j=1}^{J} \frac{\lambda_j}{\mu_j - \lambda_j}$$
Jackson Networks (1/3)

- Special instance of a network of queues (interconnected by links and switches)
  - Poisson arrivals
  - Independent service times at all buffers
  - Switching of packet/customers based on random routing

Networks of Queues (2/3)

- Joint distribution of buffer contents that may be expressed in product form:
  \[ P[n] = P[n_1, n_2, ..., n_J] = \prod_{k=1}^{J} P_k[n_k] \]

- Local balanced equations may be applied at each node (Eq. 3.44-4.48) yielding:
  \[ P[n] = \prod_{j=1}^{J-1} \rho_j^{n_j} \cdot P[0] \quad (3.49) \]
  \[ \rho_j = \frac{\lambda_j}{\mu_j} \text{ and } P[0] \text{ is the probability of all queues being empty} \]
Networks of Queues (3/3)

• As for other queuing systems, solve for $P[0]$:

$$
\sum_{n_1=0}^{\infty} \ldots \sum_{n_J=0}^{\infty} \prod_{j=1}^{J} \rho_j^{n_j} \cdot P[0] = 1 \Rightarrow P[0] = \left( \sum_{n_1=0}^{\infty} \ldots \sum_{n_J=0}^{\infty} \prod_{j=1}^{J} \rho_j^{n_j} \right)^{-1}
$$

• Interchanging $+$ and $\times$ of denominator:

$$
P[0] = \left\{ \prod_{j=1}^{J} \left( \sum_{n_j=0}^{\infty} \rho_j^{n_j} \right) \right\}^{-1} = \left\{ \prod_{j=1}^{J} \frac{1}{1-\rho_j} \right\}^{-1} = \prod_{j=1}^{J} (1-\rho_j)
$$

$$
\Rightarrow P[n_1, n_2, \ldots, n_J] = \prod_{j=1}^{J} (1-\rho_j) \cdot \rho_j^{n_j} \quad (3.50)
$$