Problem 12.1

\[ M(s) = \frac{Y(s)}{F(s)} = \frac{\frac{K}{s(s+p)}}{1 + \frac{K}{s(s+p)}} = \frac{K}{s^2 + ps + K} \]

The sensitivity of the closed-loop system transfer function with respect to parameter \( p \) is given by

\[ S_p(s) = \frac{\Delta M(s)}{M(s)} = \frac{p}{\Delta p} \frac{\Delta M(s)}{M(s)} = \frac{p}{\Delta p} \frac{1}{M(s)} \left( \frac{K}{s^2 + (p + \Delta p)s + K} - \frac{K}{s^2 + ps + K} \right) \]

\[ = \frac{p}{\Delta p} \left( \frac{-s^2 + ps + K}{s^2 + (p + \Delta p)s + K} \right) \approx \frac{-ps}{s^2 + (p + \Delta p)s + K} \]

In the last step, the approximation is done assuming that \( \Delta p \) is small.

Problem 12.2

The original transfer function \( H(s) = \frac{1000}{s + 10} \) has the static gain equal to \( H(0) = \frac{1000}{10} = 100 \). The closed-loop system transfer function is

\[ M(s) = \frac{Y(s) + \Delta Y(s)}{F(s)} = \frac{100(H(s) + \Delta H(s))}{1 + 100(H(s) + \Delta H(s))} = \frac{100(10 + 5)}{s + 1010 + 500} \]

When the disturbance sign is positive, the static gain is

\[ M^+(0) = \frac{100(15)}{1510} = 0.9934 \]

When the disturbance sign is negative, the static gain is

\[ M^-(0) = \frac{100(5)}{510} = 0.9803 \]

In both cases the tolerance is less than 1% of the original DC gain (100). Hence, the combined tolerance is less than 2%.

Problem 12.3

The sensitivity function for the unity feedback system defined in Problem 12.2, with the transfer function \( H(s) = \frac{1000}{s + 10} \), is given by

\[ S(s) = \frac{1}{1 + G(s)H(s)} = \frac{1}{1 + \frac{1000}{s + 10}} = \frac{s + 10}{s + 1010} \]

COMMENT: STUDENTS WILL NOT BE ASKED TO PLOT THE SENSITIVITY FUNCTION

Problem 12.4

\[ Y(s) = H(s)D(s) + H(s)G_c(s)(F(s) - G(s)Y(s)) \]

\[ = (1 + H(s)G_c(s)G(s))Y(s) = H(s)D(s) + H(s)G_c(s)G(s)F(s) \]

\[ \Rightarrow Y(s) = \frac{H(s)}{1 + H(s)G_c(s)G(s)}F(s) + \frac{H(s)}{1 + H(s)G_c(s)G(s)}D(s) \]

For large values of the controller transfer function \( G_c(s) \) and/or large values of the feedback element \( G(s) \), the disturbance is attenuated on the system output.