14. S-Parameters

14.1 Scattering Parameters

Linear two-port (and multi-port) networks are characterized by a number of equivalent circuit parameters, such as their transfer matrix, impedance matrix, admittance matrix, and scattering matrix. Fig. 14.1.1 shows a typical two-port network.

![Two-port network](image)

The transfer matrix, also known as the ABCD matrix, relates the voltage and current at port 1 to those at port 2, whereas the impedance matrix relates the two voltages \( V_1, V_2 \) to the two currents \( I_1, I_2 \):

\[
\begin{bmatrix}
V_1 \\
I_1
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
V_2 \\
I_2
\end{bmatrix}
\] (transfer matrix)

\[
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
-\cdot I_2
\end{bmatrix}
\] (impedance matrix)

Thus, the transfer and impedance matrices are the \( 2 \times 2 \) matrices:

\[
T = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix}, \quad Z = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{bmatrix}
\] (14.1.1)

The admittance matrix is simply the inverse of the impedance matrix, \( Y = Z^{-1} \). The scattering matrix relates the outgoing waves \( b_1, b_2 \) to the incoming waves \( a_1, a_2 \) that are incident on the two-port:

\[
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix}, \quad S = \begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
\] (scattering)

(14.1.3)

The matrix elements \( S_{11}, S_{12}, S_{21}, S_{22} \) are referred to as the scattering parameters or the S-parameters. The parameters \( S_{11}, S_{22} \) have the meaning of reflection coefficients, and \( S_{21}, S_{12} \), the meaning of transmission coefficients.

The many properties and uses of the S-parameters in applications are discussed in [1131–1170]. One particularly nice overview is the HP application note AN-95-1 by Anderson [1146] and is available on the web [1506].

We have already seen several examples of transfer, impedance, and scattering matrices. Eq. (11.7.6) or (11.7.7) is an example of a transfer matrix and (11.8.1) is the corresponding impedance matrix. The transfer and scattering matrices of multilayer structures, Eqs. (6.6.23) and (6.6.37), are more complicated examples.

The traveling wave variables \( a_1, b_1 \) at port 1 and \( a_2, b_2 \) at port 2 are defined in terms of \( V_1, I_1 \) and \( V_2, I_2 \) and a real-valued positive reference impedance \( Z_0 \) as follows:

\[
\begin{aligned}
a_1 &= \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}} \\
a_2 &= \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} \\
b_1 &= \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}} \\
b_2 &= \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}}
\end{aligned}
\] (traveling waves) (14.1.4)

The definitions at port 2 appear different from those at port 1, but they are really the same if expressed in terms of the incoming current \( -I_2 \):

\[
\begin{aligned}
a_1 &= \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} = \frac{V_2 + Z_0 (-I_2)}{2\sqrt{Z_0}} \\
b_1 &= \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} = \frac{V_2 - Z_0 (-I_2)}{2\sqrt{Z_0}}
\end{aligned}
\]

The term traveling waves is justified below. Eqs. (14.1.4) may be inverted to express the voltages and currents in terms of the wave variables:

\[
\begin{aligned}
V_1 &= \sqrt{Z_0} (a_1 + b_1) \\
V_2 &= \sqrt{Z_0} (a_2 + b_2) \\
I_1 &= \frac{1}{\sqrt{Z_0}} (a_1 - b_1) \\
I_2 &= \frac{1}{\sqrt{Z_0}} (b_2 - a_2)
\end{aligned}
\] (14.1.5)

In practice, the reference impedance is chosen to be \( Z_0 = 50 \) ohm. At lower frequencies the transfer and impedance matrices are commonly used, but at microwave frequencies they become difficult to measure and therefore, the scattering matrix description is preferred.

The S-parameters can be measured by embedding the two-port network (the device-under-test, or, DUT) in a transmission line whose ends are connected to a network analyzer. Fig. 14.1.2 shows the experimental setup.

A typical network analyzer can measure S-parameters over a large frequency range, for example, the HP 8720D vector network analyzer covers the range from 50 MHz to
14.1. Scattering Parameters

40 GHz. Frequency resolution is typically 1 Hz and the results can be displayed either on a Smith chart or as a conventional gain versus frequency graph.

The network analyzer measures the corresponding S-parameters of the primed variables, that is,

\[ S' = S \]

The S-matrix of the two-port can be obtained then from:

\[ S_{11} S'_{12} = S_{21} S'_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

or, more explicitly,

\[ S_{11} S'_{12} = S_{21} S'_{22} = \begin{bmatrix} e^{j\delta_1} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix} \]

Thus, changing the points along the transmission lines at which the S-parameters are measured introduces only phase changes in the parameters.

Without loss of generality, we may replace the extended circuit of Fig. 14.1.3 with the one shown in Fig. 14.1.4 with the understanding that either we are using the extended two-port parameters \( S' \) or, equivalently, the generator and segment \( l \) have been replaced by their Thévenin equivalents, and the load impedance has been replaced by its propagated version to distance \( l_2 \).

The two line segments of lengths \( l_1, l_2 \) are assumed to have characteristic impedance equal to the reference impedance \( Z_0 \). Then, the wave variables \( a_1, b_1 \) and \( a_2, b_2 \) are recognized as normalized versions of forward and backward traveling waves. Indeed, according to Eq. (11.7.8), we have:

\[
\begin{align*}
a_1 &= \frac{V_1 + Z_0 I_1}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{1+} \\
b_1 &= \frac{V_1 - Z_0 I_1}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{1-}, \\
a_2 &= \frac{V_2 - Z_0 I_2}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{2-} \\
b_2 &= \frac{V_2 + Z_0 I_2}{2\sqrt{Z_0}} = \frac{1}{\sqrt{Z_0}} V_{2+}
\end{align*}
\]  

(14.1.6)

Thus, \( a_1 \) is essentially the incident wave at port 1 and \( b_1 \) the corresponding reflected wave. Similarly, \( a_2 \) is incident from the right onto port 2 and \( b_2 \) is the reflected wave from port 2.

The network analyzer measures the waves \( a_1', b_1' \) and \( a_2', b_2' \) at the generator and load ends of the line segments, as shown in Fig. 14.1.3. From these, the waves at the inputs of the two-port can be determined. Assuming lossless segments and using the propagation matrices (11.7.7), we have:

\[
\begin{bmatrix} a_1' \\ b_1' \end{bmatrix} = \begin{bmatrix} e^{-j\delta_1} & 0 \\ 0 & e^{j\delta_1} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \quad \begin{bmatrix} a_2' \\ b_2' \end{bmatrix} = \begin{bmatrix} e^{-j\delta_2} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}
\]

(14.1.7)

where \( \delta_1 = \beta l_1 \) and \( \delta_2 = \beta l_2 \) are the phase lengths of the segments. Eqs. (14.1.7) can be rearranged into the forms:

\[
\begin{bmatrix} b_1' \\ b_2' \end{bmatrix} = D \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = D \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad D = \begin{bmatrix} e^{j\delta_1} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix}
\]

The network analyzer measures the corresponding S-parameters of the primed variables, that is,

\[
\begin{bmatrix} b_1' \\ b_2' \end{bmatrix} = S_{11}' S_{12}' \begin{bmatrix} a_1' \\ a_2' \end{bmatrix}, \quad S' = \begin{bmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{bmatrix}
\]

(14.1.8)

The S-parameters of the two-port can be obtained then from:

\[
\begin{bmatrix} b_1' \\ b_2' \end{bmatrix} = D S' \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = D S' D \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow S = D S' D
\]

or, more explicitly,

\[
\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} e^{j\delta_1} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix} \begin{bmatrix} S_{11}' & S_{12}' \\ S_{21}' & S_{22}' \end{bmatrix} \begin{bmatrix} e^{j\delta_1} & 0 \\ 0 & e^{j\delta_2} \end{bmatrix} = \begin{bmatrix} S_{11}' e^{2j\delta_1} & S_{12}' e^{2j(\delta_1+\delta_2)} \\ S_{21}' e^{2j(\delta_1+\delta_2)} & S_{22}' e^{2j\delta_2} \end{bmatrix}
\]

(14.1.9)

Thus, changing the points along the transmission lines at which the S-parameters are measured introduces only phase changes in the parameters.

Without loss of generality, we may replace the extended circuit of Fig. 14.1.3 with the one shown in Fig. 14.1.4 with the understanding that either we are using the extended two-port parameters \( S' \) or, equivalently, the generator and segment \( l \) have been replaced by their Thévenin equivalents, and the load impedance has been replaced by its propagated version to distance \( l_2 \).
14.2 Power Flow

The actual measurements of the $S$-parameters are made by connecting to a matched load, $Z_L = Z_0$. Then, there will be no reflected waves from the load, $a_2 = 0$, and the $S$-matrix equations will give:

$$b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 \Rightarrow S_{11} = \frac{b_1}{a_1} \bigg|_{Z_L = Z_0} = \text{reflection coefficient}$$

$$b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 \Rightarrow S_{21} = \frac{b_2}{a_1} \bigg|_{Z_L = Z_0} = \text{transmission coefficient}$$

Reversing the roles of the generator and load, one can measure in the same way the parameters $S_{12}$ and $S_{22}$.

14.3 Parameter Conversions

For a lossless two-port, the power loss is positive, which implies that the matrix $I - S^\dagger S$ must be positive definite. If the two-port is lossless, $P_{\text{loss}} = 0$, the $S$-matrix will be unitary, that is, $S^\dagger S = I$.

We already saw examples of such unitary scattering matrices in the cases of the equal travel-time multilayer dielectric structures and their equivalent quarter wavelength multisection transformers.

$$V_1 = AV_2 + BI_2 \Rightarrow V_1 = A \bigg( \frac{1}{C} I_1 - \frac{D}{C} I_2 \bigg) + BI_2 = \frac{A}{C} I_1 - \frac{AD - BC}{C} I_2$$

$$I_1 = CV_2 + DI_2 \Rightarrow V_2 = \frac{1}{C} I_1 - \frac{D}{C} I_2$$

The coefficients of $I_1, I_2$ are the impedance matrix elements. The steps are reversible, and we summarize the final relationships below:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \frac{1}{C} \begin{bmatrix} A & AD - BC \\ C & D \end{bmatrix}$$

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{1}{Z_{21}} \begin{bmatrix} Z_{11} & Z_{12} Z_{22} - Z_{12} Z_{21} \\ 1 & Z_{22} \end{bmatrix}$$

We note the determinants $\det(T) = AD - BC$ and $\det(Z) = Z_{11} Z_{22} - Z_{12} Z_{21}$. The relationship between the scattering and impedance matrices is also straightforward to derive. We define the $2 \times 1$ vectors:

$$\mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} I_1 \\ -I_2 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then, the definitions (14.1.4) can be written compactly as:

$$\mathbf{a} = \frac{1}{2\sqrt{Z_0}} (\mathbf{V} + \mathbf{Z}_0 \mathbf{I}) = \frac{1}{2\sqrt{Z_0}} (\mathbf{Z} + \mathbf{Z}_0 \mathbf{I})$$

$$\mathbf{b} = \frac{1}{2\sqrt{Z_0}} (\mathbf{V} - \mathbf{Z}_0 \mathbf{I}) = \frac{1}{2\sqrt{Z_0}} (\mathbf{Z} - \mathbf{Z}_0 \mathbf{I})$$

where we used the impedance matrix relationship $\mathbf{V} = \mathbf{Z} \mathbf{I}$ and defined the $2 \times 2$ unit matrix $\mathbf{I}$. It follows then,

$$\frac{1}{2\sqrt{Z_0}} \mathbf{I} = (\mathbf{Z} + \mathbf{Z}_0 \mathbf{I})^{-1} \mathbf{a} \Rightarrow \mathbf{b} = \frac{1}{2\sqrt{Z_0}} (\mathbf{Z} - \mathbf{Z}_0 \mathbf{I}) \mathbf{I} = (\mathbf{Z} - \mathbf{Z}_0 \mathbf{I}) (\mathbf{Z} + \mathbf{Z}_0 \mathbf{I})^{-1} \mathbf{a}$$

Thus, the scattering matrix $S$ will be related to the impedance matrix $Z$ by

$$S = (\mathbf{Z} - \mathbf{Z}_0 \mathbf{I}) (\mathbf{Z} + \mathbf{Z}_0 \mathbf{I})^{-1} \Rightarrow \mathbf{Z} = (\mathbf{I} - S)^{-1} (\mathbf{I} + S) \mathbf{Z}_0$$

(14.3.4)
14.4 Input and Output Reflection Coefficients

Explicitly, we have:

\[
S = \begin{bmatrix}
Z_{11} - Z_0 & Z_{12} \\
2Z_{21} & Z_{22} - Z_0
\end{bmatrix}
\begin{bmatrix}
Z_{11} + Z_0 & Z_{12} \\
Z_{21} & Z_{22} + Z_0
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
Z_{11} - Z_0 & Z_{12} \\
Z_{21} & Z_{22} - Z_0
\end{bmatrix}
\frac{1}{D_z}
\begin{bmatrix}
Z_{22} + Z_0 & -Z_{12} \\
-Z_{21} & Z_{11} + Z_0
\end{bmatrix}
\]

where \( D_z = \text{det}(Z + Z_0) = (Z_{11} + Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21} \). Multiplying the matrix factors, we obtain:

\[
S = \frac{1}{D_z}
\begin{bmatrix}
(Z_{11} - Z_0)(Z_{22} + Z_0) - Z_{12}Z_{21} & 2Z_{12}Z_0 \\
2Z_{21}Z_0 & (Z_{11} + Z_0)(Z_{22} - Z_0) - Z_{12}Z_{21}
\end{bmatrix}
\]

Similarly, the inverse relationship gives:

\[
Z = \frac{Z_0}{D_x}
\begin{bmatrix}
(1 + S_{11})(1 - S_{22}) + S_{12}S_{21} & 2S_{12} \\
2S_{21} & (1 + S_{11})(1 + S_{22}) + S_{12}S_{21}
\end{bmatrix}
\]

where \( D_x = \text{det}(I - S) = (1 - S_{11})(1 - S_{22}) - S_{12}S_{21} \). Expressing the impedance parameters in terms of the transfer matrix parameters, we also find:

\[
S = \frac{1}{D_a}
\begin{bmatrix}
A + \frac{B}{Z_0} - CZ_0 - D & 2(AD - BC) \\
2 & -A + \frac{B}{Z_0} + CZ_0 + D
\end{bmatrix}
\]

where \( D_a = A + \frac{B}{Z_0} + CZ_0 + D \).

### 14.4 Input and Output Reflection Coefficients

When the two-port is connected to a generator and load as in Fig. 14.1.4, the impedance and scattering matrix equations take the simpler forms:

\[
\begin{align*}
V_1 &= Z_{in}I_1 & b_1 &= \Gamma_{in}a_1 \\
V_2 &= Z_{Li}I_2 & a_2 &= \Gamma_Lb_2
\end{align*}
\]

where \( Z_{in} \) is the input impedance at port 1, and \( \Gamma_{in}, \Gamma_L \) are the reflection coefficients at port 1 and at the load:

\[
\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}, \quad \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}
\]

The input impedance and input reflection coefficient can be expressed in terms of the \( Z \) and \( S \)-parameters, as follows:

\[
Z_{in} = \frac{Z_{11} - Z_{12}Z_{21}}{Z_{22} + Z_0} \quad \Leftrightarrow \quad \Gamma_{in} = \frac{S_{11} + S_{12}S_{21}I_L}{1 - S_{22}I_L}
\]

The equivalence of these two expressions can be shown by using the parameter conversion formulas of Eqs. (14.3.5) and (14.3.6), or they can be shown indirectly, as follows. Starting with \( V_2 = Z_{Li}I_2 \) and using the second impedance matrix equation, we can solve for \( I_2 \) in terms of \( I_1 \):

\[
V_2 = Z_{21}I_1 - Z_{22}I_2 = Z_{Li}I_2 \quad \Rightarrow \quad I_2 = \frac{Z_{21}}{Z_{22} + Z_L}I_1
\]

Then, the first impedance matrix equation implies:

\[
V_1 = Z_{11}I_1 - Z_{12}I_2 = \left( \frac{Z_{11} - Z_{12}Z_{21}}{Z_{22} + Z_L} \right) I_1 = Z_{in}I_1
\]

Starting again with \( V_2 = Z_{Li}I_2 \) we find for the traveling waves at port 2:

\[
a_2 = \frac{V_2 - Z_0I_2}{2\sqrt{Z_0}} = \frac{Z_L - Z_0}{2\sqrt{Z_0}} \quad \Rightarrow \quad a_2 = \frac{Z_L - Z_0}{Z_L + Z_0}b_2 = \Gamma_Lb_2
\]

Using \( V_1 = Z_{in}I_1 \), a similar argument implies for the waves at port 1:

\[
a_1 = \frac{V_1 + Z_0I_1}{2\sqrt{Z_0}} = \frac{Z_{in} + Z_0}{2\sqrt{Z_0}} \quad \Rightarrow \quad b_1 = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}a_1 = \Gamma_{in}a_1
\]

It follows then from the scattering matrix equations that:

\[
b_2 = S_{21}a_1 + S_{22}a_2 = S_{21}a_1 + S_{22}\Gamma_Lb_2 \quad \Rightarrow \quad b_2 = \frac{S_{21}}{1 - S_{22}\Gamma_L}a_1
\]

which implies for \( b_1 \):

\[
b_1 = S_{11}a_1 + S_{12}a_2 = S_{11}a_1 + S_{12}\Gamma_Lb_2 = \left( S_{11} + S_{12}S_{21}\Gamma_L \right) a_1 = \Gamma_{in}a_1
\]

Reversing the roles of generator and load, we obtain the impedance and reflection coefficients from the output side of the two-port:

\[
Z_{out} = Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11} + Z_G} \quad \Leftrightarrow \quad \Gamma_{out} = S_{22} + \frac{S_{12}S_{21}I_G}{1 - S_{11}I_G}
\]

where

\[
\Gamma_{out} = \frac{Z_{out} - Z_0}{Z_{out} + Z_0}, \quad \Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0}
\]
14.5 Stability Circles

The input and output impedances allow one to replace the original two-port circuit of Fig. 14.1.4 by simpler equivalent circuits. For example, the two-port and the load can be replaced by the input impedance \( Z_{in} \) connected at port 1, as shown in Fig. 14.4.1.

Similarly, the generator and the two-port can be replaced by a Thévenin equivalent circuit connected at port 2. By determining the open-circuit voltage and short-circuit current at port 2, we find the corresponding Thévenin parameters in terms of the impedance parameters:

\[
V_{th} = \frac{Z_{21}V_G}{Z_{11} + Z_G}, \quad Z_{th} = Z_{out} = Z_{22} - \frac{Z_{11}Z_{21}}{Z_{11} + Z_G}
\]  

(14.4.8)

14.5 Stability Circles

In discussing the stability conditions of a two-port in terms of \( S \)-parameters, the following definitions of constants are often used:

\[
\Delta = \det(S) = S_{11}S_{22} - S_{12}S_{21}
\]

\[
K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|} \quad \text{(Rollett stability factor)}
\]

\[
\mu_1 = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|} \quad \text{(Edwards-Sinsky stability parameter)}
\]

\[
\mu_2 = \frac{1 - |S_{22}|^2}{|S_{11} - \Delta S_{22}^*| + |S_{12}S_{21}|} \quad \text{(14.5.1)}
\]

\[
B_1 = 1 + |S_{11}|^2 - |S_{22}|^2 - |\Delta|^2
\]

\[
B_2 = 1 + |S_{22}|^2 - |S_{11}|^2 - |\Delta|^2
\]

\[
C_1 = S_{11} - \Delta S_{22}^*, \quad D_1 = |S_{11}|^2 - |\Delta|^2
\]

\[
C_2 = S_{22} - \Delta S_{11}^*, \quad D_2 = |S_{22}|^2 - |\Delta|^2
\]

The quantity \( K \) is the Rollett stability factor [1142], and \( \mu_1, \mu_2 \), the Edwards-Sinsky stability parameters [1145]. The following identities hold among these constants:

\[
B_1^2 - 4|C_1|^2 = B_2^2 - 4|C_2|^2 = 4|S_{12}S_{21}|^2 (K^2 - 1)
\]

\[
|C_1|^2 = |S_{12}S_{21}|^2 + (1 - |S_{22}|^2)D_1
\]

\[
|C_2|^2 = |S_{12}S_{21}|^2 + (1 - |S_{11}|^2)D_2
\]

For example, noting that \( S_{12}S_{21} = S_{11}S_{22} - \Delta \), the last of Eqs. (14.5.2) is a direct consequence of the identity:

\[
|A - BC|^2 - |B - AC^*|^2 = (1 - |C|^2) (|A|^2 - |B|^2)
\]  

(14.5.3)

We define also the following parameters, which will be recognized as the centers and radii of the source and load stability circles:

\[
c_G = \frac{C_1}{D_1}, \quad r_G = \frac{|S_{12}S_{21}|}{|D_1|} \quad \text{(source stability circle)}
\]

\[
c_L = \frac{C_2}{D_2}, \quad r_L = \frac{|S_{12}S_{21}|}{|D_2|} \quad \text{(load stability circle)}
\]  

(14.5.4)

They satisfy the following relationships, which are consequences of the last two of Eqs. (14.5.2) and the definitions (14.5.4) and (14.5.5):

\[
1 - |S_{11}|^2 = (|c_L|^2 - r_L^2)D_2
\]  

(14.5.6)

\[
1 - |S_{22}|^2 = (|c_G|^2 - r_G^2)D_1
\]  

(14.5.7)

We note also that using Eqs. (14.5.6), the stability parameters \( \mu_1, \mu_2 \) can be written as:

\[
\mu_1 = (|c_L| - r_L) \text{sign}(D_2)
\]  

(14.5.8)

\[
\mu_2 = (|c_G| - r_G) \text{sign}(D_1)
\]  

(14.5.9)

For example, we have:

\[
\mu_1 = \frac{1 - |S_{11}|^2}{|C_2| + |S_{12}S_{21}|} = \frac{D_2 (|c_L|^2 - r_L^2)}{|D_2| |c_L| + |D_2| r_L} = \frac{D_2 (|c_L|^2 - r_L^2)}{|D_2| (|c_L| + r_L)} = \frac{D_2 (|c_L| - r_L)}{|D_2|}
\]

We finally note that the input and output reflection coefficients can be written in the alternative forms:

\[
\Gamma_{in} = S_{11} + \frac{S_{12}S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} = S_{11} - \Delta \Gamma_L
\]  

(14.5.10)

\[
\Gamma_{out} = S_{22} + \frac{S_{12}S_{21} \Gamma_G}{1 - S_{11} \Gamma_G} = S_{22} - \Delta \Gamma_G
\]  

(14.5.11)

Next, we discuss the stability conditions. The two-port is unconditionally stable if any generator and load impedances with positive resistive parts \( R_G, R_L \), will always lead to input and output impedances with positive resistive parts \( R_{in}, R_{out} \).

Equivalently, unconditional stability requires that any load and generator with \( |\Gamma_L| < 1 \) and \( |\Gamma_G| < 1 \) will result into \( |\Gamma_{in}| < 1 \) and \( |\Gamma_{out}| < 1 \).

The two-port is termed potentially or conditionally unstable if there are \( |\Gamma_L| < 1 \) and \( |\Gamma_G| < 1 \) resulting into \( |\Gamma_{in}| \geq 1 \) and/or \( |\Gamma_{out}| \geq 1 \).
The load stability region is the set of all $|I_L| < 1$, and the source stability region, the set of all $|I_G| < 1$.

In the unconditionally stable case, the load and source stability regions contain the entire unit-circles $|I_L| < 1$ or $|I_G| < 1$. However, in the potentially unstable case, only portions of the unit-circles may lie within the stability regions and such $I_G$, $I_L$ will lead to a stable input and output impedances.

The connection of the stability regions to the stability circles is brought about by the following identities, which can be proved easily using Eqs. (14.5.1)–(14.5.8):

$$1 - |I_{\text{in}}|^2 = \left|\frac{I_L - c_L}{1 - S_{22}I_L} - r_L^2\right|D_2$$

$$1 - |I_{\text{out}}|^2 = \left|\frac{I_G - c_G}{1 - S_{11}I_G} - r_G^2\right|D_1$$

For example, the first can be shown starting with Eq. (14.5.8) and using the definitions (14.5.5) and the relationship (14.5.6):

$$1 - |I_{\text{in}}|^2 = 1 - \left[\frac{S_{11} - \Delta I_L}{1 - S_{22}I_L}\right]^2 = \frac{\left|\frac{S_{11} - \Delta I_L}{1 - S_{22}I_L}\right|^2 - 1 - S_{22}I_L}{1 - S_{22}I_L^2}$$

$$= 0$$

For $D_2 > 0$, this is equivalent to (14.5.9).

It follows from Eq. (14.5.9) that the load stability region is defined by the conditions:

$$1 - |I_{\text{in}}|^2 > 0 \quad \Rightarrow \quad \left|\frac{I_L - c_L}{1 - S_{22}I_L} - r_L^2\right|D_2 > 0$$

Depending on the sign of $D_2$, these are equivalent to the outside or the inside of the load stability circle of center $c_L$ and radius $r_L$:

$$|I_L - c_L| > r_L, \quad \text{if} \quad D_2 > 0$$

$$|I_L - c_L| < r_L, \quad \text{if} \quad D_2 < 0$$

(load stability region) (14.5.10)

The boundary of the circle $|I_L - c_L| = r_L$ corresponds to $|I_{\text{in}}| = 1$. The complement of these regions corresponds to the unstable region with $|I_{\text{in}}| > 1$. Similarly, we find for the source stability region:

$$|I_G - c_G| > r_G, \quad \text{if} \quad D_1 > 0$$

$$|I_G - c_G| < r_G, \quad \text{if} \quad D_1 < 0$$

(source stability region) (14.5.11)

In order to have unconditional stability, the stability regions must contain the unit-circle in its entirety. If $D_2 > 0$, the unit-circle and load stability circle must not overlap at all, as shown in Fig. 14.5.1. Geometrically, the distance between the points $O$ and $A$ in the figure is $(OA) = |c_L| - r_L$. The non-overlapping of the circles requires the condition $(OA) > 1$, or $|c_L| - r_L > 1$.

If $D_2 < 0$, the stability region is the inside of the stability circle, and therefore, the unit-circle must lie within that circle. This requires that $(OA) = r_L - |c_L| > 1$, as shown in Fig. 14.5.1.

These two conditions can be combined into sign $(D_2) \left(|c_L| - r_L\right) > 1$. But, that is equivalent to $\mu_1 > 1$ according to Eq. (14.5.7). Geometrically, the parameter $\mu_1$ represents the distance $(O A)$. Thus, the condition for the unconditional stability of the input is equivalent to:

$$\mu_1 > 1$$

(14.5.12)

It has been shown by Edwards and Sinsky [1145] that this single condition (or, alternatively, the single condition $\mu_2 > 1$) is necessary and sufficient for the unconditional stability of both the input and output impedances of the two-port. Clearly, the source stability regions will be similar to those of Fig. 14.5.1. If the stability condition is not satisfied, that is, $\mu_1 < 1$, then only that portion of the unit-circle that lies within the stability region will be stable and will lead to stable input and output impedances. Fig. 14.5.2 illustrates such a potentially unstable case.

If $D_2 > 0$, then $\mu_1 > 1$ is equivalent to $|c_L| - r_L > 1$, and if $D_2 < 0$, it is equivalent to $r_L - |c_L| < 1$. In either case, the unit-circle is partially overlapping with the stability circle, as shown in Fig. 14.5.2. The portion of the unit-circle that does not lie within the stability region will correspond to an unstable $Z_m$.

There exist several other unconditional stability criteria that are equivalent to the single criterion $\mu_1 > 1$. They all require that the Rollett stability factor $K$ be greater than unity, $K > 1$, as well as one other condition. Any one of the following criteria are necessary and sufficient for unconditional stability [1143]:
14. S-Parameters

The MATLAB function `sparm` calculates the stability parameters $\mu_1, K, |\Delta|, B_1, B_2$, as well as the parameters $C_1, C_2, D_1, D_2$. It has usage:

```matlab
[K, mu, D, B1, B2, C1, C2, D1, D2] = sparm(S); % stability parameters
```

The function `sgcirc` calculates the centers and radii of the source and load stability circles. It also calculates gain circles to be discussed later on. Its usage is:

```matlab
[cL, rL] = sgcirc(S, 'l'); % load stability circle
[cG, rG] = sgcirc(S, 's'); % source or Zout stability circle
```

The MATLAB function `smith` draws a basic Smith chart, and the function `smithcir` draws the stability circles:

```matlab
smith(n); % draw four basic types of Smith charts, n = 1:2, 3, 4
smith; % default Smith chart corresponding to n = 3
smithcir(c, r, max, width); % draw circle of center c and radius r
smithcir(c, r, max); % equivalent to linewidth width = 1
smithcir(c, r); % draw full circle with linewidth width = 1
```

The parameter `max` controls the portion of the stability circle that is visible outside the Smith chart. For example, `max = 1.1` will display only that portion of the circle that has $|F| < 1.1$.

**Example 14.5.1:** The Hewlett-Packard AT-41511 NPN bipolar transistor has the following $S$-parameters at 1 GHz and 2 GHz [1507]:

- $S_{11} = 0.48 \angle -149^o$, $S_{21} = 5.189 \angle 89^o$, $S_{12} = 0.073 \angle 43^o$, $S_{22} = 0.49 \angle -39^o$
- $S_{11} = 0.46 \angle 162^o$, $S_{21} = 2.774 \angle 59^o$, $S_{12} = 0.103 \angle 45^o$, $S_{22} = 0.42 \angle -47^o$

Determine the stability parameters, stability circles, and stability regions.

**Solution:** The transistor is potentially unstable at 1 GHz, but unconditionally stable at 2 GHz. The source and load stability circles at 1 GHz are shown in Fig. 14.5.3.

![Fig. 14.5.3 Load and source stability circles at 1 GHz.](image-url)
The computed stability parameters at 1 GHz were:

\[
[K, \mu_1, |\Delta|, B_1, B_2, D_1, D_2] = [0.781, 0.847, 0.250, 0.928, 0.947, 0.168, 0.178]
\]

The transistor is potentially unstable because \( K < 1 \) even though \( |\Delta| < 1 \), \( B_1 > 0 \), and \( B_2 > 0 \). The load and source stability circle centers and radii were:

\[
c_L = 2.978 \angle 51.75^\circ, \quad r_L = 2.131
\]

\[
c_G = 3.098 \angle 162.24^\circ, \quad r_G = 2.254
\]

Because both \( D_1 \) and \( D_2 \) are positive, both stability regions will be the portion of the Smith chart that lies outside the stability circles. For 2 GHz, we find:

\[
[K, \mu_1, |\Delta|, B_1, B_2, D_1, D_2] = [1.089, 1.056, 0.103, 1.025, 0.954, 0.201, 0.166]
\]

\[
c_L = 2.779 \angle 50.12^\circ, \quad r_L = 1.723
\]

\[
c_G = 2.473 \angle 159.36^\circ, \quad r_G = 1.421
\]

The transistor is stable at 2 GHz, with both load and source stability circles being completely outside the unit-circle. \( \Box \)

Problem 14.2 presents an example for which the \( D_2 \) parameter is negative, so that the stability regions will be the insides of the stability circles. At one frequency, the unit-circle is partially overlapping with the stability circle, while at another frequency, it lies entirely within the stability circle.

### 14.6 Power Gains

The amplification (or attenuation) properties of the two-port can be deduced by comparing the power \( P_m \) going into the two-port to the power \( P_L \) coming out of the two-port and going into the load. These were given in Eq. (14.2.1) and we rewrite them as:

\[
P_m = \frac{1}{2} \text{Re}[V_1^* I_1] = \frac{1}{2} \Re S_{21} |I_1|^2 \quad \text{(power into two-port)}
\]

\[
P_L = \frac{1}{2} \text{Re}[V_2^* I_2] = \frac{1}{2} \Re S_{11} |I_2|^2 \quad \text{(power out of two-port and into load)}
\]

where we used \( V_1 = Z_m I_1 \), \( V_2 = Z_L I_2 \), and defined the real parts of the input and load impedances by \( R_m = \Re \left( Z_m \right) \) and \( R_L = \Re \left( Z_L \right) \). Using the equivalent circuits of Fig. 14.4.1, we may write \( I_1, I_2 \) in terms of the generator voltage \( V_G \) and obtain:

\[
P_m = \frac{1}{2} |V_G|^2 R_m \left| \frac{Z_m + Z_G}{Z_m + Z_G} \right|^2
\]

\[
P_L = \frac{1}{2} |V_G|^2 R_L \left| \frac{Z_G + Z_{out} + Z_L}{Z_{out} + Z_L} \right|^2
\]  \hspace{1cm} (14.6.2)

Using the identities of Problem 14.1, \( P_L \) can also be written in the alternative forms:

\[
P_L = \frac{1}{2} \left| \frac{V_G}{Z_{out} + Z_L} \right|^2 \left| \frac{Z_{out} + Z_L}{Z_{out} + Z_L} \right|^2 = \frac{1}{2} \left| \frac{V_G}{Z_{out} + Z_L} \right|^2
\]

\[
= \frac{1}{2} \left| \frac{V_G}{Z_{out} + Z_L} \right|^2 \left| \frac{Z_{out} + Z_L}{Z_{out} + Z_L} \right|^2
\]  \hspace{1cm} (14.6.3)

The maximum power that can be delivered by the generator to a connected load is called the available power of the generator, \( P_{avG} \), and is obtained when the load is conjugate-matched to the generator, that is, \( P_{avG} = P_m \) when \( Z_{in} = Z_G \).

Similarly, the available power from the two-port network, \( P_{avN} \), is the maximum power that can be delivered by the Thévenin-equivalent circuit of Fig. 14.4.1 to a connected load, that is, \( P_{avN} = P_L \) when \( Z_L = Z_{out}= Z_G \). It follows then from Eq. (14.6.2) that the available powers will be:

\[
P_{avG} = \text{max} \left( P_m \right) = \frac{|V_G|^2}{8 R_G}
\]

\[
P_{avN} = \text{max} \left( P_L \right) = \frac{|V_G|^2}{8 R_{out}}
\]  \hspace{1cm} (14.6.4)

Using Eq. (14.4.8), \( P_{avN} \) can also be written as:

\[
P_{avN} = \frac{|V_G|^2}{8 R_{out}} \left| \frac{Z_{out} + Z_G}{Z_{out} + Z_G} \right|^2
\]  \hspace{1cm} (14.6.5)

The powers can be expressed completely in terms of the S-parameters of the two-port and the input and output reflection coefficients. With the help of the identities of Problem 14.1, we find the alternative expressions for \( P_m \) and \( P_L \):

\[
P_m = \frac{|V_G|^2}{8 Z_0} \frac{(1 - |T_m|^2)(1 - |T_G|^2)}{1 - \Re T_m T_G^*}
\]

\[
P_L = \frac{|V_G|^2}{8 Z_0} \frac{(1 - |T_L|^2)(1 - |T_G|^2) |S_{21}|^2}{|1 - \Re T_m T_G^*(1 - S_{22} T_L)|^2}
\]

\[
= \frac{|V_G|^2}{8 Z_0} \frac{(1 - |T_L|^2)(1 - |T_G|^2) |S_{21}|^2}{|1 - \Re T_m T_G^*(1 - S_{11} T_G)|^2}
\]

\[
= \frac{|V_G|^2}{8 Z_0} \frac{(1 - |T_L|^2)(1 - |T_G|^2) |S_{21}|^2}{|1 - \Re T_m T_G^* - (S_{11} T_G)(S_{22} T_G - S_{21} T_G T_L)|^2}
\]  \hspace{1cm} (14.6.6)

Similarly, we have for \( P_{avG} \) and \( P_{avN} \):
The transducer gain $G_T$ is, perhaps, the most representative measure of gain for the two-port because it incorporates the effects of both the load and generator impedances, whereas $G_a$ depends only on the generator impedance and $G_p$ only on the load impedance.

If the generator and load impedances are matched to the reference impedance $Z_0$, so that $Z_G = Z_L = Z_0$ and $\Gamma_G = \Gamma_L = 0$, and $\Gamma_m = S_{11}, \Gamma_{out} = S_{22}$, then the power gains reduce to:

$$G_T = |S_{21}|^2, \quad G_a = \frac{|S_{21}|^2}{1 - |S_{22}|^2}, \quad G_p = \frac{|S_{21}|^2}{1 - |S_{11}|^2}$$  \hspace{1cm} (14.6.12)

A unilateral two-port has by definition zero reverse transmission coefficient, that is, $S_{12} = 0$. In this case, the input and output reflection coefficients simplify into:

$$\Gamma_m = S_{11}, \quad \Gamma_{out} = S_{22}$$ \hspace{1cm} (unilateral two-port) \hspace{1cm} (14.6.13)

The expressions of the power gains simplify somewhat in this case:

$$G_{T_u} = \frac{1 - |\Gamma_G|^2}{1 - S_{11}\Gamma_G^*}\frac{|S_{21}|^2}{1 - |S_{22}\Gamma_L|^2}$$  \hspace{1cm} (unilateral gains) \hspace{1cm} (14.6.14)

For both the bilateral and unilateral cases, the gains $G_a, G_p$ are obtainable from $G_T$ by setting $\Gamma_L = \Gamma_{out}^*$ and $\Gamma_m = \Gamma_G^*$, respectively, as was the case for $P_{avN}$ and $P_{avG}$.

The relative power ratios $P_{av}/P_{avG}$ and $P_{out}/P_{avN}$ measure the mismatching between the generator and the two-port and between the load and the two-port. Using the definitions for the power gains, we obtain the input and output mismatch factors:

$$M_{in} = \frac{P_{in}}{P_{avG}} = \frac{G_T}{|Z_m + Z_G|^2} = \frac{1 - |\Gamma_m|^2}{1 - |S_{11}\Gamma_G|^2}$$ \hspace{1cm} (14.6.15)

$$M_{out} = \frac{P_{out}}{P_{avN}} = \frac{G_T}{|Z_{out} + Z_L|^2} = \frac{1 - |\Gamma_{out}|^2}{1 - |S_{11}\Gamma_L|^2}$$ \hspace{1cm} (14.6.16)

The mismatch factors are always less than or equal to unity (for positive $R_m$ and $R_{out}$.) Clearly, $M_{in} = 1$ under the conjugate-match condition $Z_m = Z_G^*$ or $\Gamma_m = \Gamma_G^*$, and $M_{out} = 1$ if $Z_L = Z_{out}^*$ or $\Gamma_L = \Gamma_{out}^*$. The mismatch factors can also be written in the following forms, which show more explicitly the mismatch properties:

$$M_{in} = 1 - \frac{|\Gamma_{in}|^2}{1 - |S_{11}\Gamma_G|^2}, \quad M_{out} = 1 - \frac{|\Gamma_{out}|^2}{1 - |S_{11}\Gamma_L|^2}$$ \hspace{1cm} (14.6.17)

These follow from the identity:

$$|1 - \Gamma_1\Gamma_2|^2 - |\Gamma_1 - \Gamma_2|^2 = (1 - |\Gamma_1|^2)(1 - |\Gamma_2|^2)$$  \hspace{1cm} (14.6.18)
14.6. Power Gains

The transducer gain is maximized when the two-port is simultaneously conjugate matched, that is, when $\Gamma_{in} = \Gamma_{G}$ and $\Gamma_{L} = \Gamma_{L}^{\ast}$. Then, $M_{in} = M_{G} = 1$ and the three gains become equal. The common maximum gain achieved by simultaneous matching is called the maximum available gain (MAG):

$$G_{T,max} = G_{a,max} = G_{p,max} = G_{MAG}$$  \hspace{1cm} (14.6.19)

Simultaneous matching is discussed in Sec. 14.8. The necessary and sufficient condition for simultaneous matching is $K \geq 1$, where $K$ is the Rollett stability factor. It can be shown that the MAG can be expressed as:

$$G_{MAG} = \left| \frac{S_{21}}{S_{12}} \right| \left( K - \sqrt{K^2 - 1} \right)$$  \hspace{1cm} (maximum available gain) \hspace{1cm} (14.6.20)

The maximum stable gain (MSG) is the maximum value $G_{MAG}$ can have, which is achievable when $K = 1$:

$$G_{MSG} = \left| \frac{S_{21}}{S_{12}} \right|$$  \hspace{1cm} (maximum stable gain) \hspace{1cm} (14.6.21)

In the unilateral case, the MAG is obtained either by setting $\Gamma_{G} = \Gamma_{G}^{in} = S_{11}^{\ast}$ and $\Gamma_{L} = \Gamma_{L}^{out} = S_{22}^{\ast}$ in Eq. (14.6.14), or by a careful limiting process in Eq. (14.6.20), in which $K \to \infty$ so that both the numerator factor $K - \sqrt{K^2 - 1}$ and the denominator factor $|S_{12}|$ tend to zero. With either method, we find the unilateral MAG:

$$G_{MAG,u} = \left| \frac{S_{21}}{S_{12}} \right|^2 \frac{G_{1}|S_{21}|^2 G_{2}}{G_{1} - |S_{11}|^2 - |S_{22}|^2}$$  \hspace{1cm} (unilateral MAG) \hspace{1cm} (14.6.22)

The maximum unilateral input and output gain factors are:

$$G_{1} = \frac{1}{1 - |S_{11}|^2} , \quad G_{2} = \frac{1}{1 - |S_{22}|^2}$$  \hspace{1cm} (14.6.23)

They are the maxima of the input and output gain factors in Eq. (14.6.14) realized with conjugate matching, that is, with $\Gamma_{G} = S_{11}^{\ast}$ and $\Gamma_{L} = S_{22}^{\ast}$. For any other values of the reflection coefficients (such that $|\Gamma_{G}| < 1$ and $|\Gamma_{L}| < 1$), we have the following inequalities, which follow from the identity (14.6.18):

$$\frac{1 - |\Gamma_{G}|^2}{1 - |S_{11}|^2 \Gamma_{G}^{\ast}} \leq \frac{1 - |\Gamma_{L}|^2}{1 - |S_{22}| \Gamma_{L}^{\ast}} \leq \frac{1 - |\Gamma_{L}|^2}{1 - |S_{22}|^2 \Gamma_{L}^{\ast}} \leq \frac{1 - |\Gamma_{G}|^2}{1 - |S_{11}|^2} \leq 1$$  \hspace{1cm} (14.6.24)

Often two-ports, such as most microwave transistor amplifiers, are approximately unilateral, that is, the measured $S$-parameters satisfy $|S_{12}| \ll |S_{21}|$. To decide whether the two-port should be treated as unilateral, a figure of merit is used, which is essentially the comparison of the maximum unilateral gain to the transducer gain of the actual device under the same matching conditions, that is, $\Gamma_{G} = S_{11}^{\ast}$ and $\Gamma_{L} = S_{22}^{\ast}$.

For these matched values of $\Gamma_{G}, \Gamma_{L}$, the ratio of the bilateral and unilateral transducer gains can be shown to have the form:

$$g_{u} = \frac{G_{T}}{G_{Tu}} = \frac{1}{1 - |U|^2} , \quad U = \frac{S_{12} S_{21} S_{11} S_{22}}{(1 - |S_{11}|^2)(1 - |S_{22}|^2)}$$  \hspace{1cm} (14.6.25)

The quantity $|U|$ is known as the unilateral figure of merit. If the relative gain ratio $g_{u}$ is near unity (typically, within 10 percent of unity), the two-port may be treated as unilateral.

The MATLAB function $sgain$ computes the transducer, available, and operating power gains, given the $S$-parameters and the reflection coefficients $\Gamma_{G}, \Gamma_{L}$. In addition, it computes the unilateral gains, the maximum available gain, and the maximum stable gain. It also computes the unilateral figure of merit ratio (14.6.25). It has usage:

- $G_{T} = sgain(S,G_{G},G_{L})$;  \hspace{1cm} transducer power gain at given $\Gamma_{G}, \Gamma_{L}$
- $G_{a} = sgain(S,G_{G},G_{L},\ast)$;  \hspace{1cm} available power gain at given $\Gamma_{G}$ with $\Gamma_{L} = \Gamma_{L}^{\ast}$
- $G_{p} = sgain(S,G_{G},G_{L},\ast)$;  \hspace{1cm} operating power gain at given $\Gamma_{G}$ with $\Gamma_{L} = \Gamma_{L}^{\ast}$
- $G_{mag} = sgain(S)$;  \hspace{1cm} maximum available gain (MAG)
- $G_{msg} = sgain(S,\ast)$;  \hspace{1cm} maximum stable gain (MSG)
- $G_{u} = sgain(S,\ast)$;  \hspace{1cm} maximum unilateral gain, Eq. (14.6.22)
- $G_{1} = sgain(S,ui)$;  \hspace{1cm} maximum unilateral input gain, Eq. (14.6.23)
- $G_{2} = sgain(S,uo)$;  \hspace{1cm} maximum unilateral output gain, Eq. (14.6.23)
- $g_{u} = sgain(S,\ast)$;  \hspace{1cm} unilateral figure of merit ratio, Eq. (14.6.25)

The MATLAB functions $g_{in}$ and $g_{out}$ compute the input and output reflection coefficients from $S$ and $\Gamma_{G}, \Gamma_{L}$. They have usage:

- $G_{in} = gin(S,G_{G})$;  \hspace{1cm} input reflection coefficient, Eq. (14.4.3)
- $G_{out} = gout(S,G_{G})$;  \hspace{1cm} output reflection coefficient, Eq. (14.4.6)

**Example 14.6.1:** A microwave transistor amplifier uses the Hewlett-Packard AT-41410 NPN bipolar transistor with the following $S$-parameters at 2 GHz [1507]:

$$S_{11} = 0.61 \angle 165^\circ , \quad S_{12} = 3.72 \angle 59^\circ , \quad S_{12} = 0.05 \angle 42^\circ , \quad S_{22} = 0.45 \angle 48^\circ$$

Calculate the input and output reflection coefficients and the various power gains, if the amplifier is connected to a generator and load with impedances $Z_{G} = 10 + 20j$ and $Z_{L} = 30 + 40j$ ohm.

**Solution:** The following MATLAB code will calculate all the required gains:

```matlab
Z0 = 50;  \hspace{1cm} % normalization impedance
ZG = 10+20j; \hspace{1cm} gG = z2g(ZG,Z0); \hspace{1cm} % $\Gamma_{G} = -0.500 + 0.500j \angle 71.53^\circ$
ZL = 30-40j; \hspace{1cm} gL = z2g(ZL,Z0); \hspace{1cm} % $\Gamma_{L} = -0.41 - 0.43j \angle -53.12^\circ$
S = smat([0.61 165 3.72 59 0.05 42 0.45 -48]); \hspace{1cm} % reshape S into matrix
Gin = gin(S,gG);  \hspace{1cm} % $\Gamma_{in} = 0.54 \angle 162.30^\circ$
Gout = gout(S,gG); \hspace{1cm} % $\Gamma_{out} = 0.45 \angle -67.46^\circ$
Gt = sgain(S,gG,gL); \hspace{1cm} % $G_{T} = 4.71$, or, 6.73 dB
Ga = sgain(S,gG,gL,'a'); \hspace{1cm} % $G_{a} = 11.44$, or, 10.58 dB
Gp = sgain(S,gG,gL,'p'); \hspace{1cm} % $G_{p} = 10.51$, or, 10.22 dB
Gu = sgain(S,'u'); \hspace{1cm} % $G_{u} = 27.64$, or, 14.41 dB
G1 = sgain(S,'ui'); \hspace{1cm} % $G_{1} = 1.59$, or, 2.02 dB
G2 = sgain(S,'uo'); \hspace{1cm} % $G_{2} = 1.25$, or, 0.98 dB
```
The amplifier cannot be considered to be unilateral as the unilateral figure of merit ratio $g_u = 1.23$ is fairly large (larger than 10 percent from unity.)

The amplifier is operating at a gain of $G_T = 6.73$ dB, which is far from the maximum value of $G_{MAG} = 16.18$ dB. This is because it is mismatched with the given generator and load impedances.

To realize the optimum gain $G_{MAG}$ the amplifier must 'see' certain optimum generator and load impedances or reflection coefficients. These can be calculated by the MATLAB function smatch and are found to be:

$$\Gamma_G = 0.82 \angle -162.67^\circ \Rightarrow Z_G = g_2z(Z_G, Z_0) = 5.12 - 7.54j \Omega$$
$$\Gamma_L = 0.75 \angle 52.57^\circ \Rightarrow Z_L = g_2z(Z_L, Z_0) = 33.66 + 91.48j \Omega$$

The design of such optimum matching terminations and the function smatch are discussed in Sec. 11.7. The functions $g_2z$ and $z_2g$ were discussed in Sec. 11.7.

### 14.7 Generalized S-Parameters and Power Waves

The practical usefulness of the $S$-parameters lies in the fact that the definitions (14.1.4) represent forward and backward traveling waves, which can be measured remotely by connecting a network analyzer to the two-port with transmission lines of characteristic impedance equal to the normalization impedance $Z_0$. This was depicted in Fig. 14.1.3.

A generalization of $S$-parameters and wave variables can be given by using in Eq. (14.1.4) two different normalization impedances for the input and output ports.

Anticipating that the two-port will be connected to a generator and load impedances $Z_G$ and $Z_L$, a particularly convenient choice is to use $Z_G$ for the input normalization impedance and $Z_L$ for the output one, leading to the definition of the power waves (as opposed to traveling waves) [1133–1135,1137]:

$$\begin{align*}
a'_1 &= \frac{V_1 + Z_G I_1}{2\sqrt{R_G}} \\
b'_1 &= \frac{V_1 - Z_G I_1}{2\sqrt{R_G}}
\end{align*}$$

$$\begin{align*}
a''_1 &= \frac{V_2 - Z_L I_2}{2\sqrt{R_L}} \\
b''_1 &= \frac{V_2 + Z_L I_2}{2\sqrt{R_L}}
\end{align*}$$

We note that the $b$-waves involve the complex-conjugates of the impedances. The quantities $R_G, R_L$ are the resistive parts of $Z_G, Z_L$ and are assumed to be positive. These definitions reduce to the conventional traveling ones if $Z_G = Z_L = Z_0$.

These "wave" variables can no longer be interpreted as incoming and outgoing waves from the two sides of the two-port. However, as we see below, they have a nice interpretation in terms of power transfer to and from the two-port and simplify the expressions for the power gains. Inverting Eqs. (14.7.1), we have:

$$\begin{align*}
V_1 &= \frac{1}{\sqrt{R_G}} (Z_b^* a'_1 + Z_G b'_1) \\
I_1 &= \frac{1}{\sqrt{R_G}} (a'_1 - b'_1)
\end{align*}$$

$$\begin{align*}
V_2 &= \frac{1}{\sqrt{R_L}} (Z_b^* a''_1 + Z_L b''_1) \\
I_2 &= \frac{1}{\sqrt{R_L}} (a''_1 - b''_1)
\end{align*}$$

The power waves can be related directly to the traveling waves. For example, expressing Eqs. (14.7.1) and (14.1.5) in matrix form, we have for port-1:

$$\begin{bmatrix} a'_1 \\ b'_1 \end{bmatrix} = \frac{1}{2\sqrt{R_G}} \begin{bmatrix} 1 & -Z_G^* \\ 1 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}, \quad \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \frac{1}{\sqrt{Z_0}} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} Z_0 \\ 1 \end{bmatrix} \begin{bmatrix} a'_1 \\ b'_1 \end{bmatrix}$$

It follows that:

$$\begin{bmatrix} a'_1 \\ b'_1 \end{bmatrix} = \frac{1}{2\sqrt{R_G} Z_0} \begin{bmatrix} 1 & -Z_G^* \\ 1 & 1 \end{bmatrix} \begin{bmatrix} Z_0 + Z_G \\ Z_0 - Z_G \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \quad \text{or,}$$

$$\begin{bmatrix} a'_1 \\ b'_1 \end{bmatrix} = \frac{1}{2\sqrt{R_G} Z_0} \begin{bmatrix} Z_0 + Z_G \\ Z_0 - Z_G \end{bmatrix} \begin{bmatrix} Z_G \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$$

The entries of this matrix can be expressed directly in terms of the reflection coefficient $\Gamma_G$. Using the identities of Problem 14.3, we may rewrite Eq. (14.7.3) and its inverse as follows:

$$\begin{align*}
\begin{bmatrix} a'_1 \\ b'_1 \end{bmatrix} &= \frac{1}{\sqrt{1 - |\Gamma_G|^2}} \begin{bmatrix} e^{i\phi_G} & -\Gamma_G e^{i\phi_G} \\ -\Gamma_G e^{-i\phi_G} & e^{-i\phi_G} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \\
\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} &= \frac{1}{\sqrt{1 - |\Gamma_G|^2}} \begin{bmatrix} e^{-j\phi_L} & \Gamma_L e^{j\phi_L} \\ \Gamma_L e^{-j\phi_L} & e^{-j\phi_L} \end{bmatrix} \begin{bmatrix} a'_1 \\ b'_1 \end{bmatrix}
\end{align*}$$

where, noting that the quantity $|1 - \Gamma_G|/(1 - \Gamma_G)$ is a pure phase factor, we defined:

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0}, \quad e^{i\phi_G} = \frac{|1 - \Gamma_G|}{1 - |\Gamma_G|^2}, \quad e^{i\phi_L} = \frac{1 - \Gamma_L}{|1 - \Gamma_L|}$$

Similarly, we have for the power and traveling waves at port-2:

$$\begin{align*}
\begin{bmatrix} a'_2 \\ b'_2 \end{bmatrix} &= \frac{1}{\sqrt{1 - |\Gamma_L|^2}} \begin{bmatrix} e^{i\phi_L} & -\Gamma_L e^{i\phi_L} \\ -\Gamma_L e^{-i\phi_L} & e^{-i\phi_L} \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \\
\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} &= \frac{1}{\sqrt{1 - |\Gamma_L|^2}} \begin{bmatrix} e^{-j\phi_L} & \Gamma_L e^{j\phi_L} \\ \Gamma_L e^{-j\phi_L} & e^{-j\phi_L} \end{bmatrix} \begin{bmatrix} a'_2 \\ b'_2 \end{bmatrix}
\end{align*}$$

where

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}, \quad e^{i\phi_L} = \frac{|1 - \Gamma_L|}{1 - |\Gamma_L|^2}, \quad e^{i\phi_L} = \frac{1 - \Gamma_L}{|1 - \Gamma_L|}$$

The generalized $S$-parameters are the scattering parameters with respect to the power wave variables, that is,
To relate $S'$ to the conventional scattering matrix $S$, we define the following diagonal matrices:

$$
\begin{bmatrix}
    b_1' \\
    b_2'
\end{bmatrix}
= 
\begin{bmatrix}
    S_{11}' \\
    S_{12}' \\
    S_{21}' \\
    S_{22}'
\end{bmatrix}
\begin{bmatrix}
    a_1' \\
    a_2'
\end{bmatrix}
\Rightarrow
b' = S'a'
$$
(14.7.8)

Using these matrices, it follows from Eqs. (14.7.4) and (14.7.6):

$$
a_1' = F_G (a_1 - \Gamma_G b_1) \\
a_2' = F_L (a_2 - \Gamma_L b_2) \\
b_1' = F_G^* (b_1 - \Gamma_G^* a_1) \\
b_2' = F_L^* (b_2 - \Gamma_L^* a_2)
$$
(14.7.10)

Using $b = Sa$, we find

$$
a' = F(a - \Gamma b) = F(I - IS) a \\
b' = F^* (S - I^*) a = F^* (S - I^*)^{-1} F^{-1} a'
$$
(14.7.11)

where $I$ is the $2 \times 2$ unit matrix. Thus, the generalized $S$-matrix is:

$$
S' = F^* (S - I^*)^{-1} F^{-1}
$$
(14.7.12)

We note that $S' = S$ when $Z_G = Z_L = Z_0$, that is, when $F_G = F_L = 0$. The explicit expressions for the matrix elements of $S'$ can be derived as follows:

$$
S'_{11} = \frac{(S_{11} - \Gamma_G^*) (1 - S_{22} F_G) + S_{21} S_{12} F_G}{(1 - S_{11} F_G) (1 - S_{22} F_G) - S_{21} S_{12} F_G F_G} e^{-2j\phi_G}
$$
(14.7.13a)

$$
S'_{22} = \frac{(S_{22} - \Gamma_L^*) (1 - S_{11} F_G) + S_{12} S_{21} F_G}{(1 - S_{11} F_G) (1 - S_{22} F_G) + S_{12} S_{21} F_G F_G} e^{-2j\phi_L}
$$
(14.7.13b)

$$
S'_{21} = \frac{\sqrt{1 - |F_G|^2} S_{21} \sqrt{1 - |F_L|^2}}{(1 - S_{11} F_G) (1 - S_{22} F_G) - S_{21} S_{12} F_G F_G} e^{-j(\phi_G + \phi_L)}
$$

$$
S'_{12} = \frac{\sqrt{1 - |F_L|^2} S_{12} \sqrt{1 - |F_G|^2}}{(1 - S_{11} F_G) (1 - S_{22} F_G) + S_{21} S_{12} F_G F_G} e^{-j(\phi_G + \phi_L)}
$$
(14.7.13b)

The $S'_{11}, S'_{22}$ parameters can be rewritten in terms of the input and output reflection coefficients by using Eq. (14.13.2) and the following factorization identities:

$$
(S_{11} - \Gamma_G^*) (1 - S_{22} F_G) + S_{21} S_{12} F_G = (\Gamma_{in} - \Gamma_G^*) (1 - S_{22} F_G)
$$

$$
(S_{22} - \Gamma_L^*) (1 - S_{11} F_G) + S_{12} S_{21} F_G = (\Gamma_{out} - \Gamma_L^*) (1 - S_{11} F_G)
$$

It then follows from Eq. (14.7.13) that:

$$
S'_{11} = \frac{F_{in} - \Gamma_G^*}{1 - F_{in} F_G} e^{-2j\phi_G}, \quad S'_{22} = \frac{F_{out} - \Gamma_L^*}{1 - F_{out} F_L} e^{-2j\phi_L}
$$
(14.7.14)

Therefore, the mismatch factors (14.6.17) are recognized to be:

$$
M_G = 1 - |S'_{11}|^2, \quad M_L = 1 - |S'_{22}|^2
$$
(14.7.15)

The power flow relations (14.2.1) into and out of the two-port are also valid in terms of the power wave variables. Using Eq. (14.7.2), it can be shown that:

$$
P_m = \frac{1}{2} \text{Re}[V_1^* I_1] = \frac{1}{2} |a'_1|^2 - \frac{1}{2} |b'_1|^2
$$
(14.7.16)

$$
P_L = \frac{1}{2} \text{Re}[V_2^* I_2] = \frac{1}{2} |b'_2|^2 - \frac{1}{2} |a'_2|^2
$$
(14.7.17)

In the definitions (14.7.1), the impedances $Z_G, Z_L$ are arbitrary normalization parameters. However, if the two-port is actually connected to a generator $V_G$ with impedance $Z_G$ and a load $Z_L$, then the power waves take particularly simple forms. It follows from Fig. 14.1.4 that $V_G = V_1 + Z_G I_1$ and $V_2 = Z_L I_2$. Therefore, definitions Eq. (14.7.1) give:

$$
a'_1 = \frac{V_1 + Z_G I_1}{2 \sqrt{R_G}} = \frac{V_G}{2 \sqrt{R_G}}
$$
(14.7.18)

$$
a'_2 = \frac{V_2 - Z_L I_2}{2 \sqrt{R_L}} = 0
$$
(14.7.19)

$$
b'_2 = \frac{V_2 + Z_L^* I_2}{2 \sqrt{R_L}} = \frac{Z_L + Z_L^* I_2}{2 \sqrt{R_L}} I_2 = \frac{2 R_L}{2 \sqrt{R_L}} I_2 = \sqrt{R_L} I_2
$$
(14.7.20)

It follows that the available power from the generator and the power delivered to the load are given simply by:

$$
P_{avG} = \frac{|V_G|^2}{8 R_G} = \frac{1}{2} |a'_1|^2
$$
(14.7.21)

$$
P_L = \frac{1}{2} R_L |I_2|^2 = \frac{1}{2} |b'_2|^2
$$
(14.7.22)

Because $a'_2 = 0$, the generalized scattering matrix gives $b'_1 = S'_{11} a'_1$ and $b'_2 = S'_{21} a'_1$.

The power expressions (14.7.10) then become:

$$
P_m = \frac{1}{2} |a'_1|^2 - \frac{1}{2} |b'_1|^2 = (1 - |S'_{11}|^2) \frac{1}{2} |a'_1|^2 = (1 - |S'_{11}|^2) P_{avG}
$$
(14.7.23)

$$
P_L = \frac{1}{2} |b'_2|^2 - \frac{1}{2} |a'_2|^2 = \frac{1}{2} |b'_2|^2 = |S'_{21}|^2 \frac{1}{2} |a'_1|^2 = |S'_{21}|^2 P_{avG}
$$
(14.7.24)

It follows that the transducer and operating power gains are:

$$
G_T = \frac{P_L}{P_{avG}} = |S'_{21}|^2, \quad G_o = \frac{P_L}{P_m} = \frac{|S'_{21}|^2}{1 - |S'_{11}|^2}
$$
(14.7.25)
These also follow from the explicit expressions (14.7.13) and Eqs. (14.6.10) and (14.6.11). We can also express the available power gain in terms of the generalized $S$-parameters, that is, $G_a = S_{21}|^2 / (1 - |S_{22}|^2)$. Thus, we summarize:

$$G_T = |S'_{21}|^2, \quad G_a = \frac{|S'_{21}|^2}{1 - |S_{22}|^2}, \quad G_p = \frac{|S'_{21}|^2}{1 - |S_{11}|^2} \tag{14.7.21}$$

When the load and generator are matched to the network, that is, $G_{in} = G^*_G$ and $G_{out} = G^*_L$, the generalized reflections coefficients vanish, $S_{11} = S_{22} = 0$, making all the gains equal to each other.

### 14.8 Simultaneous Conjugate Matching

We saw that the transducer, available, and operating power gains become equal to the maximum available gain $G_{MAX}$ when both the generator and the load are conjugately matched to the two-port, that is, $G_{in} = G^*_G$ and $G_{out} = G^*_L$. Using Eq. (14.6.8), these conditions read explicitly:

$$
\begin{align*}
\Gamma^*_G &= S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} = S_{11} - \Delta \Gamma_L \\
\Gamma^*_L &= S_{22} + \frac{S_{12} S_{21} \Gamma_G}{1 - S_{22} \Gamma_G} = S_{22} - \Delta \Gamma_G
\end{align*} \tag{14.8.1}
$$

Assuming a bilateral two-port, Eqs. (14.8.1) can be solved in the two unknowns $\Gamma_G, \Gamma_L$ (eliminating one of the unknowns gives a quadratic equation for the other.) The resulting solutions can be expressed in terms of the parameters (14.5.1):

$$
\begin{align*}
\Gamma_G &= \frac{B_1 \pm \sqrt{B_1^2 - 4 C_1^2}}{2 C_1} \quad \text{(simultaneous conjugate match)} \tag{14.8.2} \\
\Gamma_L &= \frac{B_2 \pm \sqrt{B_2^2 - 4 C_2^2}}{2 C_2}
\end{align*}
$$

where the minus signs are used when $B_1 > 0$ and $B_2 > 0$, and the plus signs, otherwise.

A necessary and sufficient condition for these solutions to have magnitudes $|\Gamma_G| < 1$ and $|\Gamma_L| < 1$ is that the Rollett stability factor be greater than unity, $K > 1$. This is satisfied when the two-port is unconditionally stable, which implies that $K > 1$ and $B_1 > 0, B_2 > 0$.

A conjugate match exists also when the two-port is potentially unstable, but with $K > 1$. Necessarily, this means that $B_1 < 0, B_2 < 0$, and also $|\Delta| > 1$. Such cases are rare in practice. For example, most microwave transistors have either $K > 1$ and are stable, or, they are potentially unstable with $K < 1$ and $|\Delta| < 1$.

If the two-port is unilateral, $S_{12} = 0$, then the two equations (14.8.1) decouple, so that the optimum conjugately matched terminations are:

$$\Gamma_G = S^*_G, \quad \Gamma_L = S^*_L \quad \text{(unilateral conjugate match)} \tag{14.8.3}$$

The MATLAB function `smatch` implements Eqs. (14.8.2). It works only if $K > 1$. Its usage is as follows:

$$[g_G, g_L] = \text{smatch}(S);$$

To realize such optimum conjugately matched terminations, matching networks must be used at the input and output of the two-port as shown in Fig. 14.8.1.

The input matching network can be thought as being effectively connected to the impedance $Z_L$ into the actual input terminals. It must transform the generalized reflections coefficients $\Gamma_L$ into the impedance of the connected generator, typically, $Z_G = 50$ ohm.

The output matching network must transform the actual load impedance, here $Z_D$, into the optimum load impedance $Z_L = Z^*_L$.

Fig. 14.8.1 Input and output matching networks.

The matching networks may be realized in several possible ways, as discussed in Chap. 13. Stub matching, quarter-wavelength matching, or lumped $L$-section or $Π$-section networks may be used. In designing the matching networks, it proves convenient to first design the reverse network as mentioned in Sec. 13.13.

Fig. 14.8.2 shows the procedure for designing the output matching network using a reversed stub matching transformer or a reversed quarter-wave transformer with a parallel stub. In both cases the reversed network is designed to transform the load impedance $Z^*_L$ into $Z_D$.

**Example 14.8.1:** A microwave transistor amplifier uses the Hewlett-Packard AT-41410 NPN bipolar transistor having $S$-parameters at 2 GHz [1507]:

$$
\begin{align*}
S_{11} &= 0.61 \angle 165^o, \quad S_{21} = 3.72 \angle 59^o, \quad S_{12} = 0.05 \angle 42^o, \quad S_{22} = 0.45 \angle -48^o
\end{align*}
$$

Determine the optimum conjugately matched source and load terminations, and design appropriate input and output matching networks.

**Solution:** This is the continuation of Example 14.6.1. The transistor is stable with $K = 1.1752$ and $|\Delta| = 0.1086$. The function `smatch` gives:

$$
(\Gamma_G, \Gamma_L) = \text{smatch}(S) \Rightarrow \Gamma_G = 0.8179 \angle -162.6697^o, \quad \Gamma_L = 0.7495 \angle 52.5658^o
$$

The corresponding source, load, input, and output impedances are (with $Z_0 = 50$):

$$
Z_G = Z^*_G = 5.1241 - 7.5417j \ \Omega, \quad Z_L = Z^*_L = 33.6758 + 91.4816j \ \Omega
$$
In Fig. 14.8.4, the input stub must transform $Z_0$ to $Z_0$. It can be designed with the help of the function stub1, which gives the two solutions:

$$dl = \text{stub1}(Z_\text{in}/Z_0, 'po') = \begin{bmatrix} 0.3038 & 0.4271 \\ 0.1962 & 0.0247 \end{bmatrix}$$

We choose the lower one, which has the shortest lengths. Thus, the stub length is $d = 0.1962\lambda$ and the segment length $l = 0.0247\lambda$. Both segments can be realized with microstrips of characteristic impedance $Z_0 = 50$ ohm. Similarly, the output matching network can be designed by:

$$dl = \text{stub1}(Z_\text{out}/Z_0, 'po') = \begin{bmatrix} 0.3162 & 0.1194 \\ 0.1838 & 0.2346 \end{bmatrix}$$

Again, we choose the lower solutions, $d = 0.1838\lambda$ and $l = 0.2346\lambda$. The solutions using shorted shunt stubs are:

$$\text{stub1}(Z_\text{in}/Z_0) = \begin{bmatrix} 0.0538 & 0.4271 \\ 0.4462 & 0.0247 \end{bmatrix}, \quad \text{stub1}(Z_\text{out}/Z_0) = \begin{bmatrix} 0.0662 & 0.1194 \\ 0.4338 & 0.2346 \end{bmatrix}$$

Using microstrip lines with alumina substrate ($\varepsilon_r = 9.8$), we obtain the following values for the width-to-height ratio, effective permittivity, and wavelength:

$$u = \frac{w}{h} = \text{mstrip}\varepsilon_r, Z_0 = 0.9711$$
$$c_{\text{eff}} = \text{mstrip}\varepsilon_r, u = 6.5630$$
$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_{\text{eff}}}} = 5.8552 \text{ cm}$$

where $\lambda_0 = 15$ cm is the free-space wavelength at 2 GHz. It follows that the actual segment lengths are $d = 1.1486$ cm, $l = 0.1447$ cm for the input network, and $d = 1.0763$ cm, $l = 1.3734$ cm for the output network.

In the quarter-wavelength method shown in Fig. 14.8.5, we use the function qwt2 to carry out the design of the required impedances of the microstrip segments. We have for the input and output networks:

$$[Z_1, Z_2] = \text{qwt2}(Z_\text{in}, Z_0) = [28.4817, -11.0232] \Omega$$
$$[Z_1, Z_2] = \text{qwt2}(Z_\text{out}, Z_0) = [118.7832, 103.8782] \Omega$$

For the input case, we find $Z_2 = -11.0232 \Omega$, which means that we should use either a 3λ/8-shorted stub or a λ/8-opened one. We choose the latter. Similarly, for the output case, we have $Z_2 = 103.8782 \Omega$, and we choose a 3λ/8-opened stub. The parameters of each microstrip segment are:

- $Z_1 = 28.4817 \Omega$, $u = 2.5832$, $c_{\text{eff}} = 7.2325$, $\lambda = 5.578$ cm, $\lambda/4 = 1.394$ cm
- $Z_2 = 11.0232 \Omega$, $u = 8.9424$, $c_{\text{eff}} = 8.2974$, $\lambda = 5.207$ cm, $\lambda/8 = 0.651$ cm
- $Z_1 = 118.7832 \Omega$, $u = 0.0656$, $c_{\text{eff}} = 5.8799$, $\lambda = 6.186$ cm, $\lambda/4 = 1.547$ cm
- $Z_2 = 103.8782 \Omega$, $u = 0.1169$, $c_{\text{eff}} = 7.9503$, $\lambda = 6.149$ cm, $3\lambda/8 = 2.306$ cm
Simultaneous Conjugate Matching

Finally, the designs using $L$-sections shown in Fig. 14.8.6, can be carried out with the help of the function $lmatch$. We have the dual solutions for the input and output networks:

$$\begin{bmatrix} X_1 \newline X_2 \end{bmatrix} = lmatch(Z_0, Z_{in}, 'n') = \begin{bmatrix} 16.8955 & -22.7058 \\ -16.8955 & 7.6223 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \newline X_2 \end{bmatrix} = lmatch(Z_{out}, Z_0, 'n') = \begin{bmatrix} 57.9268 & -107.7472 \\ 502.4796 & 7.6223 \end{bmatrix}$$

According to the usage of $lmatch$, the output network transforms $Z_0$ into $Z_{out}^*$, but that is equal to $Z_L$ as required.

Choosing the first rows as the solutions in both cases, the shunt part $X_1$ will be inductive and the series part $X_2$, capacitive. At 2 GHz, we find the element values:

$$\begin{align*}
L_1 &= \frac{X_1}{\omega} = 1.3445 \text{ nH}, \\
C_1 &= -\frac{1}{\omega X_2} = 3.5047 \text{ pF} \\
L_2 &= \frac{X_1}{\omega} = 4.6097 \text{ nH}, \\
C_2 &= -\frac{1}{\omega X_2} = 0.7386 \text{ pF}
\end{align*}$$

The output network, but not the input one, also admits a reversed $L$-section solution:

$$\begin{bmatrix} X_1 \newline X_2 \end{bmatrix} = lmatch(Z_{out}, Z_0, 'r') = \begin{bmatrix} 71.8148 & 68.0353 \\ -71.8148 & 114.9280 \end{bmatrix}$$

The essential MATLAB code used to generate the above results was as follows:

```matlab
Z0 = 50; f = 2; w = 2*pi*f; la0 = 30/f; er = 9.8; % f in GHz
S = smat([0.61 165 3.72 59 0.05 42 0.45 -48]); % S-matrix
[gG, gL] = smatch(S); % simultaneous conjugate match
smith;
p(0G, '.'); plot(conj(S(1,1)), 'o');
p(0L, '.'); plot(conj(S(2,2)), 'o');
ZG = g2z(gG, Z0); Zin = conj(ZG);
ZL = g2z(gL, Z0); Zout = conj(ZL);
d1 = stub1(Zin/Z0, 'po');
d1 = stub1(Zout/Z0, 'po'); % single-stub design
u = mstripu(er, Z0);
e = mstripu(u, er, Z0);
la = la0/sqrt(eff);
[Z1, Z2] = qwt2(Zin, Z0); % quarter-wavelength with \lambda/8 stub
[X12 = lmatch(Zout, Z0, 'r');]
[X12 = lmatch(Zout, Z0, 'r');]

One could replace the stubs with balanced stubs, as discussed in Sec. 13.9, or use $II$- or $T$-sections instead of $L$-sections.

14.9 Power Gain Circles

For a stable two-port, the maximum transducer gain is achieved at single pair of points $\Gamma_G, \Gamma_L$. When the gain $G$ is required to be less than $G_{\text{MAG}}$, there will be many possible pairs $\Gamma_G, \Gamma_L$ at which the gain $G$ is realized. The locus of such points $\Gamma_G$ and $\Gamma_L$ on the $\Gamma$-plane is typically a circle of the form:

$$|\Gamma - c| = r$$

where $c, r$ are the center and radius of the circle and depend on the desired value of the gain $G$.

In practice, several types of such circles are used, such as unilateral, operating, and available power gain circles, as well as constant noise figure circles, constant SWR circles, and others.

The gain circles allow one to select appropriate values for $\Gamma_G, \Gamma_L$ that, in addition to providing the desired gain, also satisfy other requirements, such as striking a balance between minimizing the noise figure and maximizing the gain.

The MATLAB function $sgcirc$ calculates the stability circles as well as the operating, available, and unilateral gain circles. Its complete usage is:

```matlab
[c, r] = sgcirc(S, 's'); % source stability circle
[c, r] = sgcirc(S, 'l'); % load stability circle
[c, r] = sgcirc(S, 'p', G); % operating power gain circle
[c, r] = sgcirc(S, 'a', G); % available power gain circle
```


14.10 Unilateral Gain Circles

We consider only the unconditionally stable unilateral case, which has \(|S_{11}| < 1\) and \(|S_{22}| < 1\). The dependence of the transducer power gain on \(G_T\) and \(G_L\) decouples and the value of the gain may be adjusted by separately choosing \(G_T\) and \(G_L\). We have from Eq. (14.6.14):

\[
G_T = \frac{1}{|1 - S_{11}|^2} \frac{1 - |S_{22}|^2}{|1 - S_{22}G_L|^2} = G_S |S_{21}|^2 G_L
\]  

The input and output gain factors \(G_S, G_L\) satisfy the inequalities (14.6.24). Concentrating on the output gain factor, the corresponding gain circle is obtained as the locus of points \(G_L\) that will lead to a fixed value, say \(G_L = G\), which necessarily must be less than the maximum \(G_2\) given in Eq. (14.6.23), that is,

\[
\frac{1 - |S_{22}|^2}{|1 - S_{22}G_L|^2} = G_L \leq G_2 = \frac{1}{1 - |S_{22}|^2}
\]  

Normalizing the gain \(G\) to its maximum value \(g = G/G_2 = G (1 - |S_{22}|^2)\), we may rewrite (14.10.2) in the form:

\[
\frac{1 - |S_{22}|^2}{|1 - S_{22}G_L|^2} = g \leq 1
\]  

This equation can easily be rearranged into the equation of a circle \(|I_L - c| = r\), with center and radius given by:

\[
c = \frac{gS_{22}^*}{1 - (1-g)|S_{22}|^2}, \quad r = \sqrt{\frac{1 - g(1 - |S_{22}|^2)}{1 - (1-g)|S_{22}|^2}}
\]  

When \(g = 1\) or \(G = G_2\), the gain circle collapses onto a single point, that is, the optimum point \(I_L = S_{22}^2\). Similarly, we find for the constant gain circles of the input gain factor:

\[
c = \frac{gS_{11}^*}{1 - (1-g)|S_{11}|^2}, \quad r = \sqrt{\frac{1 - g(1 - |S_{11}|^2)}{1 - (1-g)|S_{11}|^2}}
\]  

where here, \(g = G/G_1 = G (1 - |S_{11}|^2)\) and the circles are \(|I_L - c| = r\).

Both sets of \(c, r\) satisfy the conditions \(|c| < 1\) and \(|c + r < 1\), the latter implying that the circles lie entirely within the unit circle \(|r| < 1\), that is, within the Smith chart.

Example 14.10.1: A unilateral microwave transistor has S-parameters:

\[
S_{11} = 0.8 < 120^\circ, \quad S_{21} = 4 < 60^\circ, \quad S_{12} = 0, \quad S_{22} = 0.2 < 30^\circ
\]

The unilateral MAG and the maximum input and output gains are obtained as follows:

\[
G_{MAG, \text{in}} = \text{sgain}(S, \text{‘u’}) = 16.66 \text{ dB}
\]

\[
G_1 = \text{sgain}(S, \text{‘ui’}) = 4.44 \text{ dB}
\]

\[
G_2 = \text{sgain}(S, \text{‘uo’}) = 0.18 \text{ dB}
\]

Most of the gain is accounted for by the factor \(|S_{22}|^2\), which is 12.04 dB. The constant input gain circles for \(G_2 = 1, 2, 3\) dB are shown in Fig. 14.10.1. Their centers lie along the ray to \(S_{22}^2\). For example, the center and radius of the 3-dB case were computed by

\[
[c_1, r_1] = \text{sgcirc}(S, \text{‘ui’}, 1) \Rightarrow c_1 = 0.701 \angle -120^\circ, \quad r_1 = 0.233
\]

Fig. 14.10.1 Unilateral input gain circles.

Because the output does not provide much gain, we may choose the optimum value \(I_L = S_{22}^2 = 0.2 < 30^\circ\). Then, with any point \(I_L\) along the 3-dB input gain circle the total transducer gain will be in dB:

\[
G_T = G_G + |S_{22}|^2 + G_L = 3 + 12.04 + 0.18 = 15.22 \text{ dB}
\]

Points along the 3-dB circle are parametrized as \(I_L = c_1 + r_1 e^{i \phi}\), where \(\phi\) is any angle. Choosing \(\phi = \arg(S_{22}) - \pi\) will correspond to the point on the circle that lies closest to the origin, that is, \(I_L = 0.468 < 120^\circ\), as shown in Fig. 14.10.1. The corresponding generator and load impedances will be:

\[
Z_G = 69.21 + 14.42 j \Omega, \quad Z_L = 23.15 - 24.02 j \Omega
\]

The MATLAB code used to generate these circles was:

```matlab
S = smat([0.8, 120, 4, 60, 0, 0, 0.2, -30]);
[c1, r1] = sgcirc(S, ‘ui’, 1);
[c2, r2] = sgcirc(S, ‘ui’, 2);
[c3, r3] = sgcirc(S, ‘ui’, 3);
smith; smithcir(c1, r1); smithcir(c2, r2); smithcir(c3, r3);
c = exp(-j*angle(S(1,1))); line([0,real(c)], [0,imag(c)]);
```
14.11 Operating and Available Power Gain Circles

Because the transducer power gain $G_T$ depends on two independent parameters—the source and load reflection coefficients—it is difficult to find the simultaneous locus of points for $\Gamma_G$, $\Gamma_L$ that will result in a given value for the gain.

If the generator is matched, $\Gamma_{in} = \Gamma_G^c$, then the transducer gain becomes equal to the operating gain $G_T = G_p$ and depends only on the load reflection coefficient $\Gamma_L$. The locus of points $\Gamma_L$ that result in fixed values of $G_p$ are the operating power gain circles. Similarly, the available power gain circles are obtained by matching the load end, $\Gamma_L = \Gamma_{out}$, and varying $\Gamma_G$ to achieve fixed values of the available power gain.

Using Eqs. (14.6.11) and (14.5.8), the conditions for achieving a constant value, say $G$, for the operating or the available power gains are:

\[
G_p = \frac{1}{1 - |\Gamma_{in}|^2} |S_{21}|^2 \left( \frac{1 - |\Gamma_{in}|^2}{|1 - S_{22}\Gamma_{in}|^2} \right) = G, \quad \Gamma_G^c = \Gamma_{in} = \frac{S_{11} - \Delta \Gamma_L}{1 - S_{22} \Gamma_{in}}
\]

\[
G_a = \frac{1 - |\Gamma_G|^2}{|1 - S_{11}\Gamma_G|^2} |S_{21}|^2 \left( \frac{1}{1 - |\Gamma_{out}|^2} \right) = G, \quad \Gamma_L^a = \Gamma_{out} = \frac{S_{22} - \Delta \Gamma_G}{1 - S_{11} \Gamma_G}
\]

We consider the operating gain first. Defining the normalized gain $g = G/|S_{21}|^2$, substituting $\Gamma_{in}$, and using the definitions (14.5.1), we obtain the condition:

\[
g = \frac{1 - |\Gamma_{in}|^2}{|1 - S_{22}\Gamma_{in}|^2 - |S_{11} - \Delta \Gamma_L|^2} = \frac{1 - |\Gamma_{in}|^2}{|S_{22} - \Delta S_{11}\Gamma_L - (S_{22} - \Delta S_{11})\Gamma_L^a + 1 - |S_{11}|^2}
\]

This can be rearranged into the form:

\[
|\Gamma_L|^2 = \frac{gC_2}{1 + gD_2} \Gamma_L - \frac{gC_2^*}{1 + gD_2} \Gamma_L^a = \frac{1 - g(1 - |S_{11}|^2)}{1 + gD_2}
\]

and then into the circle form:

\[
|\Gamma_L|^2 - \frac{gC_2}{1 + gD_2} \Gamma_L = \frac{1 - g(1 - |S_{11}|^2)}{1 + gD_2}
\]

Using the identities (14.5.2) and $1 - |S_{11}|^2 = 2K|S_{12}S_{21}| + D_2$, which follows from (14.5.1), the right-hand side of the above circle form can be written as:

\[
g^2|C_2|^2 + \frac{1 - g(1 - |S_{11}|^2)}{1 + gD_2} = \frac{g^2|S_{12}S_{21}|^2 - 2gK|S_{12}S_{21}| + 1}{(1 + gD_2)^2}
\]

Thus, the operating power gain circle will be $|\Gamma_L - c|^2 = r^2$ with center and radius:

\[
c = \frac{gC_2^*}{1 + gD_2}, \quad r = \frac{\sqrt{g^2|S_{12}S_{21}|^2 - 2gK|S_{12}S_{21}| + 1}}{1 + gD_2}
\]

The points $\Gamma_L$ on this circle result into the value $G_p = G$ for the operating gain. Such points can be parametrized as $\Gamma_L = c + r e^{i\phi}$, where $0 \leq \phi \leq 2\pi$. As $\Gamma_L$ traces this circle, the conjugately matched source coefficient $\Gamma_G^c = \Gamma_{in}$ will also trace a circle because $\Gamma_{in}$ is related to $\Gamma_L$ by the bilinear transformation (14.5.8).

In a similar fashion, we find the available power gain circles to be $|\Gamma_G - c|^2 = r^2$, where $g = G/|S_{21}|^2$ and:

\[
c = \frac{gC_2^*}{1 + gD_1}, \quad r = \frac{\sqrt{g^2|S_{12}S_{21}|^2 - 2gK|S_{12}S_{21}| + 1}}{1 + gD_1}
\]
14.11. Operating and Available Power Gain Circles

For an unconditionally stable two-port, the gain \( G \) must be 0 \( \leq G \leq G_{\text{MAG}} \), with \( G_{\text{MAG}} \) given by Eq. (14.6.20). It can be shown easily that the quantities under the square roots in the definitions of the radii \( r \) in Eqs. (14.11.3) and (14.11.4) are non-negative. The gain circles lie inside the unit circle for all such values of \( G \). The radii \( r \) vanish when \( G = G_{\text{MAG}} \), that is, the circles collapse into single points corresponding to the simultaneous conjugate matched solutions of Eq. (14.8.2).

The MATLAB function \texttt{sgcirc} calculates the center and radii \( c, r \) of the operating and available power gain circles. It has usage, where \( G \) must be entered in dB:

\[
\begin{align*}
[c, r] &= \texttt{sgcirc}(S, 'p', G); \quad \text{operating power gain circle} \\
[c, r] &= \texttt{sgcirc}(S, 'a', G); \quad \text{available power gain circle}
\end{align*}
\]

Example 14.11.1: A microwave transistor amplifier uses the Hewlett-Packard AT-41410 NPN bipolar transistor with the following S-parameters at 2 GHz [1507]:

\[
S_{11} = 0.61 \angle 165^\circ, \quad S_{21} = 3.72 \angle 59^\circ, \quad S_{12} = 0.05 \angle 24^\circ, \quad S_{22} = 0.45 \angle -48^\circ
\]

Calculate \( G_{\text{MAG}} \) and plot the operating and available power gain circles for \( G = 13, 14, 15 \) dB. Then, design source and load matching circuits for the case \( G = 15 \) dB by choosing the reflection coefficient that has the smallest magnitude.

Solution: The MAG was calculated in Example 14.6.1, \( G_{\text{MAG}} = 16.18 \) dB. The gain circles and the corresponding load and source stability circles are shown in Fig. 14.11.1. The operating gain and load stability circles were computed and plotted by the MATLAB statements:

\[
\begin{align*}
[c1, r1] &= \texttt{sgcirc}(S, 'p', 13); \quad % c_1 = 0.4443 \angle -52.56^\circ, r_1 = 0.5212 \\
[c2, r2] &= \texttt{sgcirc}(S, 'p', 14); \quad % c_2 = 0.5297 \angle -52.56^\circ, r_2 = 0.4205 \\
[c3, r3] &= \texttt{sgcirc}(S, 'p', 15); \quad % c_3 = 0.6253 \angle -52.56^\circ, r_3 = 0.2968 \\
[cL, rL] &= \texttt{sgcirc}(S, 'l'); \quad % c_L = 2.9600 \angle -52.56^\circ, r_L = 0.9753
\end{align*}
\]

\[
\begin{align*}
\text{smith; smithcir(c1, r1, 1.7);} & \quad \text{% display portion of circle with } |r_L| \leq 1.7 \\
\text{smithcir(c1, r1); smithcir(c2, r2); smithcir(c3, r3);}
\end{align*}
\]

The gain circles lie entirely within the unit circle, for example, we have \( r_3 + |c_3| = 0.9221 < 1 \), and their centers lie along the ray of \( c_L \). As \( G \) traces the 15-dB circle, the corresponding \( I_G = I_{\text{out}} \) traces its own circle, also lying within the unit circle. The following MATLAB code computes and adds that circle to the above Smith chart plots:

\[
\begin{align*}
\phi &= \texttt{linspace}(0, 2\pi, \text{61}); \quad % \text{equally spaced angles at } 1^\circ \text{ intervals} \\
\text{gGmag} &= \texttt{conj(gin(S, gG));} \quad % \text{points on } 15\text{-dB operating gain circle} \\
\text{gGmag} &= \texttt{plot(gammaG);}
\end{align*}
\]

In particular, the point \( I_L \) on the 15-dB circle that lies closest to the origin is \( I_L = c_3 - r_3 e^{j\angle(c_3)} \). The corresponding matched load will be \( I_G = I_{\text{out}} = 0.6805 \angle -163.88^\circ \). These and the corresponding source and load impedances were computed by the MATLAB statements:

\[
\begin{align*}
\text{gL} &= c_3 - r_3 e^{j\angle(c_3)}; \quad % \text{points on } 15\text{-dB available gain circle} \\
\text{ZG} &= \texttt{g2z(gG);} \quad % \text{circle of conjugate matched source points}
\end{align*}
\]

The source and load impedances normalized to \( Z_0 = 50 \) ohm are:

\[
\begin{align*}
Z_G = \frac{Z_G}{Z_0} &= 0.1938 - 0.1363j, \quad Z_L = \frac{Z_L}{Z_0} = 1.2590 + 0.7361j
\end{align*}
\]

The matching circuits can be designed in a variety of ways as in Example 14.8.1. Using open shunt stubs, we can determine the stub and line segment lengths with the help of the function \texttt{stub1}:

\[
\begin{align*}
dl &= \texttt{stub1}(z_L, 'po'); \quad % \text{plot entire source stability circle} \\
dl &= \texttt{stub1}(z_L, 'po'); \quad % \text{plot entire source stability circle}
\end{align*}
\]

In both cases, we may choose the lower solutions as they have shorter total length \( d + l \). The available power gain circles can be determined in a similar fashion with the help of the MATLAB statements:

\[
\begin{align*}
\text{[c1, r1]} &= \texttt{sgcirc}(S, 'a', 13); \quad % c_1 = 0.5384 \angle -162.67^\circ, r_1 = 0.4373 \\
\text{[c2, r2]} &= \texttt{sgcirc}(S, 'a', 14); \quad % c_2 = 0.6227 \angle -162.67^\circ, r_2 = 0.3422 \\
\text{[c3, r3]} &= \texttt{sgcirc}(S, 'a', 15); \quad % c_3 = 0.7111 \angle -162.67^\circ, r_3 = 0.2337 \\
\text{[cG, rG]} &= \texttt{sgcirc}(S, 's'); \quad % c_G = 1.5748 \angle -162.67^\circ, r_G = 0.5162
\end{align*}
\]

\[
\begin{align*}
\text{smith; smithcir(cG, rG);} & \quad % \text{plot entire source stability circle} \\
\text{smithcir(c1, r1); smithcir(c2, r2); smithcir(c3, r3);}
\end{align*}
\]

Again, the circles lie entirely within the unit circle. As \( G \) traces the 15-dB circle, the corresponding matched load \( I_L = I_{\text{out}} \) traces its own circle on the \( \Gamma \)-plane. It can be plotted with:

\[
\begin{align*}
\text{phi} &= \texttt{linspace}(0, 2\pi, \text{61}); \quad % \text{equally spaced angles at } 1^\circ \text{ intervals} \\
\text{gGmag} &= \texttt{conj(gin(S, gGmag));} \quad % \text{points on } 15\text{-dB available gain circle} \\
\text{gGmag} &= \texttt{plot(gammaG);}
\end{align*}
\]
14.11. Operating and Available Power Gain Circles

In particular, the point \( I_G = c_1 - r_2 e^{j\pi c_1} = 0.4774 \angle -126.67 \)° lies closest to the origin. The corresponding matched load will have \( I_L = \Gamma_{\text{out}} = 0.5728 \angle 50.76 \)°. The resulting normalized impedances are:

\[
Z_G = \frac{Z_G}{\eta} = 0.3609 - 0.1329 j, \quad Z_L = \frac{Z_L}{\eta} = 1.1135 + 1.4704 j
\]

and the corresponding stub matching networks will have lengths:

\[
\text{stubl}(z_G^*, 'po') = \begin{bmatrix} 0.3684 \\
0.1316 \\
0.3905 \\
0.0613 \\
0.1030 \end{bmatrix}, \quad \text{stubl}(z_L^*, 'po') = \begin{bmatrix} 0.3488 \\
0.1512 \\
0.2560 \end{bmatrix}
\]

The lower solutions have the shortest lengths. For both the operating and available gain cases, the stub matching circuits will be similar to those in Fig. 14.8.4.

When the two-port is potentially unstable (but with \( |S_{11}| < 1 \) and \( |S_{22}| < 1 \)) the stability circles intersect with the unit-circle, as shown in Fig. 14.5.2. In this case, the operating and available power gain circles also intersect the unit-circle and at the same points as the stability circles.

We demonstrate this in the specific case of \( K < 1, |S_{11}| < 1, |S_{22}| < 1 \), but with \( D_2 > 0 \), an example of which is shown in Fig. 14.11.2. The intersection of an operating gain circle with the unit-circle is obtained by setting \( |I_L| = 1 \) in the circle equation \(|I_L - c| = r\). Writing \( I_L = c \) and \( c = c e^{j\theta_c} \), we have:

\[
r^2 = |I_L - c|^2 = 1 - 2|c| \cos(\theta_c) + |c|^2 \Rightarrow \cos(\theta_c) = \frac{1 + |c|^2 - r^2}{2|c|}
\]

Similarly, the intersection of the load stability circle with the unit-circle leads to the relationship:

\[
r^2 = |I_L - c_L|^2 = 1 - 2|c_L| \cos(\theta_c) + |c_L|^2 \Rightarrow \cos(\theta_c) = \frac{1 + |c_L|^2 - r^2}{2|c_L|}
\]

Because \( c = gC_2^* / (1 + gD_2) \), \( c_L = C_2^* / D_2 \), and \( D_2 > 0 \), it follows that the phase angles of \( c \) and \( c_L \) will be equal, \( \theta_c = \theta_c \). Therefore, in order for the load stability circle and the gain circle to intersect at the same \( I_L = c \), the following condition must be satisfied:

\[
\cos(\theta_c) = \frac{1 + |c|^2 - r^2}{2|c|} = \frac{1 + |c_L|^2 - r^2}{2|c_L|}
\]

Using the identities \( 1 - |S_{11}|^2 = B_2 - D_2 \) and \( 1 - |S_{11}|^2 = (|c_L|^2 - r^2)D_2 \), which follow from Eqs. (14.5.1) and (14.5.6), we obtain:

\[
\frac{1 + |c_L|^2 - r^2}{2|c_L|} = \frac{1 + (B_2 - D_2)/D_2}{2|C_2|/|D_2|} = \frac{B_2}{2|C_2|}
\]

where we used \( D_2 > 0 \). Similarly, Eq. (14.11.2) can be written in the form:

\[
r^2 = |c|^2 + \frac{1 - g(1 - |S_{11}|^2)}{1 + gD_2} \Rightarrow |c|^2 = \frac{g(1 - |S_{11}|^2) - 1}{1 + gD_2} = \frac{g(B_2 - D_2) - 1}{1 + gD_2}
\]

Therefore, we have:

\[
\frac{1 + |c|^2 - r^2}{2|c|} = \frac{1 + (g(B_2 - D_2) - 1) / (1 + gD_2)}{2|C_2|/|1 + gD_2|} = \frac{B_2}{2|C_2|}
\]

Thus, Eq. (14.11.5) is satisfied. This condition has two solutions for \( \theta_L \) that correspond to the two points of intersection with the unit-circle. When \( D_2 > 0 \), we have \( \arg c = \arg C_2 = -\arg C_2 \). Therefore, the two solutions for \( I_G = e^{j\theta_c} \) will be:

\[
\Gamma_L = e^{j\theta_L}, \quad \theta_L = -\arg(C_2) \pm \cos \left( \frac{B_2}{2|C_2|} \right)
\]

Similarly, the points of intersection of the unit-circle and the available gain circles and source stability circle are:

\[
\Gamma_G = e^{j\theta_G}, \quad \theta_G = -\arg(C_1) \pm \cos \left( \frac{B_1}{2|C_1|} \right)
\]

Actually, these expressions work also when \( D_2 < 0 \) and \( D_1 < 0 \).

Example 14.11.2: The microwave transistor Hewlett-Packard AT-41410 NPN is potentially unstable at 1 GHz with the following S-parameters [1507]:

\[
S_{11} = 0.6 \angle -163^\circ, \quad S_{21} = 7.12 \angle 86^\circ, \quad S_{12} = 0.039 \angle 35^\circ, \quad S_{22} = 0.50 \angle -38^\circ
\]

Calculate \( G_{\text{MSG}} \) and plot the operating and available power gain circles for \( G = 20, 21, 22 \) dB. Then, design source and load matching circuits for the 22-dB case by choosing the reflection coefficients that have the smallest magnitudes.

Solution: The MSG computed from Eq. (14.6.21) is \( G_{\text{MSG}} = 22.61 \) dB. Fig. 14.11.2 depicts the operating and available power gain circles as well as the load and source stability circles. The stability parameters are: \( K = 0.7667, \rho = 0.8643, |\Delta| = 0.1893, D_1 = 0.3242, D_2 = 0.2142 \). The computations and plots are done with the following MATLAB code:¹

\[
S = \text{smat}([0.60, -163, 7.12, 86, 0.039, 35, 0.50, -38]); \quad \%	ext{S-parameters}
\]

\[
[K, mu, D1, D2, B1, B2, C1, C2, D1, D2] = \text{sparm}(S); \quad \%	ext{stability parameters}
\]

\[
G_{\text{MSG}} = \text{db}(\text{sgain}(S,'msg')) ; \quad \%	ext{operating power gain circles:}
\]

\[
c1 = 0.6418 \angle 50.86^\circ, \quad r1 = 0.4768
\]

\[
c2 = 0.7502 \angle 50.86^\circ, \quad r2 = 0.4221
\]

\[
c3 = 0.8666 \angle 50.86^\circ, \quad r3 = 0.3893
\]

\[
|\Delta| = 1.0008 \angle 216\delta 0^\circ, \quad r_0 = 1.2965
\]

\[
|\Delta| = 1.7456 \angle 171.69^\circ, \quad r_0 = 0.8566
\]

\[
\text{smithgraph}(cl, r1, 1.5); \quad \%	ext{plot Smith charts}
\]

\[
\text{smithgraph}(c2, r2, 1.5); \quad \%	ext{plot gain circles}
\]

\[
\text{plot}(gG, 'r'); \quad \%	ext{matching plot}
\]

¹The function db converts absolute scales to dB. The function ab converts from dB to absolute units.
Every device is a source of internally generated noise. The noise entering the device and the internal noise must be added to obtain the total input system noise. If the device is an amplifier, the total system noise power will be amplified at the output by the gain of the device. If the output load is matched, this gain will be the available gain.

The internally generated noise is quantified in practice either by the effective noise temperature $T_e$, or by the noise figure $F$ of the device. The internal noise power is given by $P_n = kT_eB$, where $k$ is the Boltzmann constant and $B$ the bandwidth in Hz. These concepts are discussed further in Sec. 16.8. The relationship between $T_e$ and $F$ is defined in terms of a standard reference temperature $T_0 = 290$ K (degrees Kelvin):

$$F = 1 + \frac{T_e}{T_0}$$  \hspace{1cm} (14.12.1)

The noise figure is usually quoted in dB, $F_{\text{dB}} = 10 \log_{10} F$. Because the available gain of a two-port depends on the source impedance $Z_G$, or the source reflection coefficient $\Gamma_G$,

$$F = F_{\text{min}} + \frac{R_n}{R_G|Z_{\text{Opt}}|^2} |Z_G - Z_{\text{Opt}}|^2$$  \hspace{1cm} (14.12.2)

where $R_G = \text{Re}(Z_G)$ and $R_n$ is an equivalent noise resistance. We note that $F = F_{\text{min}}$ when $Z_G = Z_{\text{Opt}}$. Defining the normalized noise resistance $r_n = R_n/Z_0$, where $Z_0 = 50$ ohm, we may write Eq. (14.12.2) in terms of the corresponding source reflection coefficients:

$$F = F_{\text{min}} + 4r_n \frac{|\Gamma_G - \Gamma_{\text{Opt}}|^2}{|1 + \Gamma_{\text{Opt}}|^2(1 - |\Gamma_G|^2)}$$  \hspace{1cm} (14.12.3)

The parameters $F_{\text{min}}, r_n,$ and $\Gamma_{\text{Opt}}$ characterize the noise properties of the two-port and are usually known.

In designing low-noise microwave amplifiers, one would want to achieve the minimum noise figure and the maximum gain. Unfortunately, the optimum source reflection coefficient $\Gamma_{\text{Opt}}$ does not necessarily correspond to the maximum available gain.

The noise figure circles and the available gain circles are useful tools that allow one to obtain a balance between low noise and high gain designs. The noise figure circles are the locus of points $\Gamma_G$ that correspond to fixed values of $F$. They are obtained by rewriting Eq. (14.12.3) as the equation of a circle $|\Gamma_G - c|^2 = r^2$. We write Eq. (14.12.3) in the form:

$$\frac{|\Gamma_G - \Gamma_{\text{Opt}}|^2}{1 - |\Gamma_{\text{Opt}}|^2} = N,$$  \hspace{1cm} (14.12.4)

where $N = (F - F_{\text{min}})[1 + \Gamma_{\text{Opt}}]^2/4r_n$ which can be rearranged into the circle equation:

$$|\Gamma_G - \frac{\Gamma_{\text{Opt}}}{N+1}|^2 = \frac{N^2 + N(1 - |\Gamma_{\text{Opt}}|^2)}{(N+1)^2}$$

Thus, the center and radius of the noise figure circle are:

$$c = \frac{\Gamma_{\text{Opt}}}{N+1}, \hspace{1cm} r = \frac{\sqrt{N^2 + N(1 - |\Gamma_{\text{Opt}}|^2)}}{N+1}$$  \hspace{1cm} (14.12.5)
The MATLAB function `nfccirc` implements Eq. (14.12.5). Its inputs are the noise parameters \( F_{\text{min}}, r_n, \Gamma_{\text{Gopt}} \), and the desired value of \( F \) in dB, and its outputs are \( c, r \):

\[ [c, r] = nfccirc(F, F_{\text{min}}, r_n, g_{\text{Gopt}}); \]

The function `nfig` implements Eq. (14.12.3). Its inputs are \( F_{\text{min}}, r_n, \Gamma_{\text{Gopt}} \), and a vector of values of \( \Gamma_G \), and its output is the corresponding vector of values of \( F \):

\[ F = nfig(F_{\text{min}}, r_n, \Gamma_{\text{Gopt}}); \]

**Example 14.12.1:** The microwave transistor of Example 14.11.1 has the following noise parameters at 2 GHz [1507]:

\[ F_{\text{min}} = 1.6 \text{ dB}, \quad r_n = 0.16, \quad \Gamma_{\text{Gopt}} = 0.26 \angle 172^\circ. \]

Determine the matched load \( \Gamma_L \) corresponding to \( \Gamma_{\text{Gopt}} \) and calculate the available gain. The larger the noise figure \( F \), the larger the radius of its circle. As \( F \) increases, so does \( \Gamma_G \). Points around this circle can be parametrized as \( \Gamma_G = \frac{S_{22} - \Delta \Gamma_{\text{Gopt}}}{1 - S_{11} \Gamma_{\text{Gopt}}} \) = 0.4927 \angle 52.50^\circ.

The value of the available gain at \( \Gamma_{\text{Gopt}} \) is \( G_{\text{Aopt}} = 13.66 \text{ dB}. \) This is to be compared with the MAG of 16.18 dB determined in Example 14.11.1. To increase the available gain, we must also increase the noise figure. Fig. 14.12.1 shows the locations of the optimum reflection coefficients, as well as several noise figure circles.

The MATLAB code for generating this graph was:

```matlab
S = smat([0.61, 165, 3.72, 59, 0.05, 42, 0.45, -48]);
Fmin = 1.6; r_n = 0.16; gGopt = p2c(0.26, 172);

Gmag = db(sgain(S, 'mag'));
Gaopt = db(sgain(S, gGopt, 'a'))
glopt = conj(gout(S, gGopt));
[c1, r1] = nfccirc(1.7, Fmin, r_n, gGopt);
[c2, r2] = nfccirc(1.8, Fmin, r_n, gGopt);
[c3, r3] = nfccirc(1.9, Fmin, r_n, gGopt);
[c4, r4] = nfccirc(2.0, Fmin, r_n, gGopt);

smith; plot([gGopt, gLopt], '.');
smithcir(c1, r1); smithcir(c2, r2); smithcir(c3, r3); smithcir(c4, r4);
```

Thus, we see that increasing the gain comes at the price of increasing the noise figure. As \( F_G \) traces the \( F_G = 1.8 \text{ dB} \) circle, the available gain \( G_a \) varies as shown in Fig. 14.12.2. Points around this circle can be parametrized as \( \Gamma_G = c_2 + r_2 e^{i\phi}, \) with \( 0 \leq \phi \leq 2\pi \). Fig. 14.12.2 plots \( G_a \) versus the angle \( \phi \). We note that the gain varies between the limits \( 12.22 \leq G_a \leq 14.81 \text{ dB} \).

**Solution:** The conjugate matched load corresponding to \( \Gamma_{\text{Gopt}} \) is:

\[ \Gamma_{\text{Lopt}} = \Gamma_{\text{out}}^* = \left[ \frac{S_{22} - \Delta \Gamma_{\text{Gopt}}}{1 - S_{11} \Gamma_{\text{Gopt}}} \right]^* = 0.4927 \angle 52.50^\circ. \]

The MATLAB code for generating this graph was:

```matlab
S = smat([0.61, 165, 3.72, 59, 0.05, 42, 0.45, -48]);
Fmin = 1.6; r_n = 0.16; gGopt = p2c(0.26, 172);

c = 0.61; r = 0.16; gG = smatch(S);
F = nfig(Fmin, r_n, gopt, gG);

gG = smatch(S);
F = nfig(Fmin, r_n, gopt, gG);

smith; plot([gGopt, gLopt], '.');
smithcir(c1, r1); smithcir(c2, r2); smithcir(c3, r3); smithcir(c4, r4);
```

The larger the noise figure \( F \), the larger the radius of its circle. As \( F \) increases, so does the available gain. But as the gain increases, the radius of its circle decreases. Thus, for a fixed value of \( F \), there will be a maximum value of the available gain corresponding to that gain circle that has the smallest radius and is tangent to the noise figure circle.

In the extreme case of the maximum available gain, the available gain circle collapses to a point—the simultaneous conjugate matched point \( \Gamma_G = 0.8179 \angle -162.67^\circ \) with a corresponding noise figure of \( F = 4.28 \text{ dB} \). These results can be calculated by the MATLAB statements:

```matlab
S = smat([0.61, 165, 3.72, 59, 0.05, 42, 0.45, -48]);
Fmin = 1.6; r_n = 0.16; gGopt = p2c(0.26, 172);

c = 0.61; r = 0.16; gG = smatch(S);
F = nfig(Fmin, r_n, gopt, gG);

smith; plot([gGopt, gLopt], '.');
smithcir(c1, r1); smithcir(c2, r2); smithcir(c3, r3); smithcir(c4, r4);
```

The maximum value, \( G_a = 14.81 \text{ dB} \), is reached when \( F_G = 0.4478 \angle -169.73^\circ \), with a resulting matched load \( \Gamma_L = \Gamma_{\text{out}} = 0.5574 \angle 52.50^\circ \). The two points \( \Gamma_G, \Gamma_L \), as well as the
14.12. Noise Figure Circles

\[ F = 1.8 \text{ dB} \]

\[ G_a = 14.81 \text{ dB} \]

![Noise Figure Circle](image)

Fig. 14.12.3 Maximum available gain for given noise figure.

\[ G_a = 14.81 \text{ dB gain circle, which is tangential to the 1.8-db noise figure circle, are shown in Fig. 14.12.3.} \]

The following MATLAB code performs these calculations and plots:

```matlab
phi = linspace(0,2*pi,72); % angle in 1/20 increments
gG = c2 + r2*exp(j*phi); % ΓG around the c2,r2 circle
G = db(gain(S,gc,g,'c')); % available gain in dB
plot(phi*180/pi, G); % maximum available gain
[Ga,i] = max(G); % ΓG for maximum gain
gammaG = gc(i); % matched load f_L
[ca,ra] = sgcir(S,'a',Ga); % available gain circle
smith; smithcir(c2,r2); smithcir(ca,ra);
plot({'gammaG, gammaL', 'r'});
```

The maximum gain and the point of tangency with the noise figure circle are determined by direct search, that is, evaluating the gain around the 1.8-db noise figure circle and finding where it reaches a maximum.

The input and output stub matching networks can be designed with the help of the function `stub1`. The normalized source and load impedances are:

\[ Z_G = \frac{1 + \Gamma_G}{1 - \Gamma_G} = 0.3840 - 0.0767j, \quad Z_L = \frac{1 + \Gamma_L}{1 - \Gamma_L} = 1.0904 + 1.3993j \]

The stub matching networks have lengths:

\[ \text{stub1}(x_G, 'po') = \begin{bmatrix} 0.3749 & 0.3977 \\ 0.1251 & 0.0738 \end{bmatrix}, \quad \text{stub1}(x_L, 'po') = \begin{bmatrix} 0.3519 & 0.0991 \\ 0.1481 & 0.2250 \end{bmatrix} \]

The lower solutions have shorter total lengths \( d + l \). The implementation of the matching networks with microstrip lines will be similar to that in Fig. 14.8.4.

If the two-port is potentially unstable, one must be checked that the resulting solutions for \( \Gamma_G, \Gamma_L \) both lie in their respective stability regions. Problems 14.6 and 14.7 illustrate the design of such potentially unstable low noise microwave amplifiers.

14.13 Problems

14.1 Using the relationships (14.4.3) and (14.4.6), derive the following identities:

\[ (Z_{11} + Z_G) (Z_{22} + Z_L) - Z_{12} Z_{21} = (Z_{12} + Z_G) (Z_{21} + Z_L) \]

\[ (1 - S_{11} \Gamma_G) (1 - S_{22} \Gamma_L) - S_{12} S_{21} \Gamma_G \Gamma_L = (1 - S_{22} \Gamma_L) (1 - \Gamma_G \Gamma_L) - (1 - S_{11} \Gamma_L) (1 - \Gamma_G \Gamma_L) \]

Using Eqs. (14.4.4) and (14.4.5), show that:

\[ \frac{Z_{21}}{Z_{22} + Z_L} = \frac{S_{21}}{1 - S_{22} \Gamma_L} \frac{1 - \Gamma_L}{1 - \Gamma_G}, \quad \frac{Z_{21}}{Z_{11} + Z_G} = \frac{S_{21}}{1 - S_{11} \Gamma_G} \frac{1 - \Gamma_G}{1 - \Gamma_L} \]

Finally, for the real part \( R_L = \text{Re}(Z_L) \), show that:

\[ Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \Rightarrow R_L = Z_0 \frac{1 - |\Gamma_L|^2}{1 - |\Gamma_L|^2} \]

14.2 Computer Experiment. The Hewlett-Packard ATF-10136 GaAs FET transistor has the following S-parameters at 4 GHz and 8 GHz [1507]:

\[ S_{11} = 0.54 \angle 120^\circ, \quad S_{12} = 3.60 \angle 61^\circ, \quad S_{12} = 0.137 \angle 31^\circ, \quad S_{22} = 0.22 \angle 49^\circ \]

\[ S_{11} = 0.60 \angle 87^\circ, \quad S_{21} = 0.29 \angle 32^\circ, \quad S_{12} = 0.21 \angle 36^\circ, \quad S_{22} = 0.32 \angle 48^\circ \]

Determine the stability parameters, stability circles, and stability regions at the two frequencies.

14.3 Derive the following relationships, where \( R_G = \text{Re}(Z_G) \):

\[ \frac{Z_{21}}{2 \sqrt{R_G Z_L}} = \frac{1}{\sqrt{1 - |\Gamma_G|^2}} \frac{|1 - \Gamma_L|}{1 - \Gamma_G}, \quad \frac{Z_{21}}{2 \sqrt{R_G Z_L}} = - \frac{\Gamma_G}{\sqrt{1 - |\Gamma_G|^2}} \frac{|1 - \Gamma_G|}{1 - \Gamma_G} \]

14.4 Derive Eqs. (14.7.13) relating the generalized S-parameters to the conventional S-parameters.

14.5 Derive the expression Eq. (14.6.20) for the maximum available gain \( G_{max} \) and show that it is the maximum of all three gains, that is, transducer, available, and operating gains.

14.6 Computer Experiment. The microwave transistor of Example 14.11.2 has the following noise parameters at a frequency of 1 GHz [1507]: \( F_{min} = 1.3 \text{ dB}, \quad F_n = 0.16, \quad F_{Gopt} = 0.06 \angle 49^\circ \).

Determine the matched load \( \Gamma_{Gopt} \) corresponding to \( \Gamma_{Gopt} \) and calculate the available gain.

Then, plot the noise figure circles for \( F = 1.4, 1.5, 1.6 \text{ dB} \).

For the 1.5-db noise figure circle, determine the values of \( \Gamma_G, \Gamma_L \) that correspond to the maximum possible available gain.

Design microstrip stub matching circuits for the computed values of \( \Gamma_G, \Gamma_L \).
14.13. Problems

14.7 Computer Experiment. The Hewlett-Packard ATF-36163 pseudomorphic high electron mobility transistor (PHEMT) has the following S- and noise parameters at 6 GHz [1507]:

\[
\begin{align*}
S_{11} &= 0.75 \angle -131^\circ, \quad S_{21} = 3.95 \angle 55^\circ, \quad S_{12} = 0.13 \angle -12^\circ, \quad S_{22} = 0.27 \angle -116^\circ \\
\Gamma_{\text{min}} &= 0.66 \text{ dB}, \quad r_n = 0.15, \quad \Gamma_{\text{Gopt}} = 0.55 \angle 88^\circ
\end{align*}
\]

Plot the \( F = 0.7, 0.8, 0.9 \) dB noise figure circles. On the 0.7-dB circle, determine the source reflection coefficient \( \Gamma_G \) that corresponds to maximum available gain, and then determine the corresponding matched load coefficient \( \Gamma_L \).

Design microstrip stub matching circuits for the computed values of \( \Gamma_G, \Gamma_L \).

14.8 Computer Experiment. In this experiment, you will carry out two low-noise microwave amplifier designs, including the corresponding input and output matching networks. The first design fixes the noise figure and finds the maximum gain that can be used. The second design fixes the desired gain and finds the minimum noise figure that may be achieved.

The Hewlett-Packard Agilent ATF-34143 PHEMT transistor is suitable for low-noise amplifiers in cellular/PCS base stations, low-earth-orbit and multipoint microwave distribution systems, and other low-noise applications.

At 2 GHz, its S-parameters and noise-figure data are as follows, for biasing conditions of \( V_{DS} = 4 \) V and \( I_{DS} = 40 \) mA:

\[
\begin{align*}
S_{11} &= 0.700 \angle -150^\circ, \quad S_{12} = 0.081 \angle 19^\circ \\
S_{21} &= 6.002 \angle 73^\circ, \quad S_{22} = 0.210 \angle -150^\circ \\
\Gamma_{\text{min}} &= 0.22 \text{ dB}, \quad r_n = 0.09, \quad \Gamma_{\text{Gopt}} = 0.66 \angle 67^\circ
\end{align*}
\]

a. At 2 GHz, the transistor is potentially unstable. Calculate the stability parameters \( K, \mu, \Delta, D_1, D_2 \). Calculate the MSG in dB.

Draw a basic Smith chart and place on it the source and load stability circles (display only a small portion of each circle outside the Smith chart.) Then, determine the parts of the Smith chart that correspond to the source and load stability regions.

b. For the given optimum reflection coefficient \( \Gamma_{\text{Gopt}} \), calculate the corresponding load reflection coefficient \( \Gamma_{\text{Lopt}} \) assuming a matched load.

Place the two points \( \Gamma_{\text{Gopt}}, \Gamma_{\text{Lopt}} \) on the above Smith chart and determine whether they lie in their respective stability regions.

c. Calculate the available gain \( G_{a\text{opt}} \) in dB that corresponds to \( \Gamma_{\text{Gopt}} \).

Add the corresponding available gain circle to the above Smith chart. (Note that the source stability circle and the available gain circles intersect the Smith chart at the same points.)

d. Add to your Smith chart the noise figure circles corresponding to the noise figure values of \( F = 0.25, 0.30, 0.35 \) dB.

For the case \( F = 0.35 \) dB, calculate and plot the available gain \( G_a \) in dB as \( \Gamma_G \) traces the noise-figure circle. Determine the maximum value of \( G_a \) and the corresponding value of \( \Gamma_G \).

Place on your Smith chart the available gain circle corresponding to this maximum \( G_a \). Place also the corresponding point \( \Gamma_G \), which should be the point of tangency between the gain and noise figure circles.

calculate and place on the Smith chart the corresponding load reflection coefficient \( \Gamma_L = \Gamma_{\text{Lopt}} \). Verify that the two points \( \Gamma_G, \Gamma_L \) lie in their respective stability regions.

In addition, for comparison purposes, place on your Smith chart the available gain circles corresponding to the values \( G_a = 15 \) and \( 16 \) dB.

e. The points \( \Gamma_G \) and \( \Gamma_L \) determined in the previous question achieve the maximum gain for the given noise figure of \( F = 0.35 \) dB.

Design input and output stub matching networks that match the amplifier to a 50-ohm generator and a 50-ohm load. Use “parallel/open” microstrip stubs having 50-ohm characteristic impedance and alumina substrate of relative permittivity of \( \epsilon_r = 9.8 \).

Determine the stub lengths \( d, l \) in in units of \( \lambda \), the wavelength inside the microstrip lines. Choose always the solution with the shortest total length \( d + l \).

Determine the effective permittivity \( \epsilon_{\text{eff}} \) of the stubs, the stub wavelength \( \lambda \) in cm, and the width/height ratio, \( w/h \). Then, determine the stub lengths \( d, l \) in cm.

Finally, make a schematic of your final design that shows both the input and output matching networks (as in Fig.10.8.3.)

f. The above design sets \( F = 0.35 \) dB and finds the maximum achievable gain. Carry out an alternative design as follows. Start with a desired available gain of \( G_a = 16 \) dB and draw the corresponding available gain circle on your Smith chart.

As \( \Gamma_G \) traces the portion of this circle that lies inside the Smith chart, compute the corresponding noise figure \( F \). (Points on the circle can be parametrized by \( \Gamma_G = c + re^{i\theta} \), but you must keep only those that have \( |\Gamma_G| < 1 \).)

Find the minimum among these values of \( F \) in dB and calculate the corresponding value of \( \Gamma_G \). Calculate the corresponding matched \( \Gamma_L \).

Add to your Smith chart the corresponding noise figure circle and place on it the points \( \Gamma_G \) and \( \Gamma_L \).

g. Design the appropriate stub matching networks as in part 14.8.