where \( T_1 = d_1/c \) is the one-way travel time to the fault. Show that the corresponding time constant \( \tau = 1/a \) is in the four cases:

\[
\tau = \frac{Z_0 C}{2}, \quad \tau = 2Z_0 C, \quad \tau = \frac{2L}{Z_0}, \quad \tau = \frac{L}{2Z_0}
\]

For a resistive fault, show that \( \Gamma_1 = -Z_0 / (2R + Z_0) \), or, \( \Gamma_1 = R / (2R + Z_0) \), for a shunt or series \( R \). Moreover, show that \( \Gamma_1 = (Z_1 - Z_0) / (Z_1 + Z_0) \), where \( Z_1 \) is the parallel (in the shunt-\( R \) case) or series combination of \( R \) with \( Z_0 \) and give an intuitive explanation of this fact. For a series \( C \), show that the voltage wave along the two segments is given as follows, and also derive similar expressions for all the other cases:

\[
V(t, z) = \begin{cases} 
V_0 u(t - z/c) + V_0 \left[1 - e^{-a(t + z/c - 2T_1)} \right] u(t + z/c - 2T_1), & \text{for } 0 < z < d_1 \\
V_0 e^{-a(t - z/c)} u(t - z/c), & \text{for } d_1 < z \leq d_1 + d_2
\end{cases}
\]

Make a plot of \( V_d(t) \) for \( 0 \leq t \leq 5T_1 \), assuming \( a = 1 \) for the \( C \) and \( L \) faults, and \( \Gamma_1 = \mp 1 \) corresponding to a shorted shunt or an opened series fault.

The MATLAB file `TDRmovie.m` generates a movie of the step input as it propagates and gets reflected from the fault. The lengths were \( d_1 = 6, d_2 = 4 \) (in units such that \( c = 1 \)), and the input was \( V_0 = 1 \).

### 12 Coupled Lines

#### 12.1 Coupled Transmission Lines

Coupling between two transmission lines is introduced by their proximity to each other. Coupling effects may be undesirable, such as crosstalk in printed circuits, or they may be desirable, as in directional couplers where the objective is to transfer power from one line to the other.

In Sections 12.1–12.3, we discuss the equations, and their solutions, describing coupled lines and crosstalk [1055–1072]. In Sec. 12.4, we discuss directional couplers, as well as fiber Bragg gratings, based on coupled-mode theory [1073–1094]. Fig. 12.1.1 shows an example of two coupled microstrip lines over a common ground plane, and also shows a generic circuit model for coupled lines.

![Coupled Transmission Lines](image)

**Fig. 12.1.1** Coupled Transmission Lines.

For simplicity, we assume that the lines are lossless. Let \( L_i, C_i, i = 1, 2 \) be the distributed inductances and capacitances per unit length when the lines are isolated from each other. The corresponding propagation velocities and characteristic impedances are: \( v_1 = 1/\sqrt{L_1C_1}, Z_1 = \sqrt{L_1C_1}, i = 1, 2 \). The coupling between the lines is modeled by introducing a mutual inductance and capacitance per unit length, \( L_m, C_m \). Then, the coupled versions of telegrapher’s equations (11.15.1) become:

\[ C_1 \] is related to the capacitance to ground \( C_{1g} \) via \( C_1 = C_{1g} + C_m \), so that the total charge per unit length on line-1 is \( Q_1 = C_1 V_1 - C_m V_2 = C_{1g} (V_1 - V_{d}) + C_m (V_1 - V_{2}), \) where \( V_{d} \) = 0.
12.1. Coupled Transmission Lines

\[
\begin{align*}
\frac{\partial V_1}{\partial z} &= -L_1 \frac{\partial I_1}{\partial t} - L_m \frac{\partial I_2}{\partial t}, \\
\frac{\partial I_1}{\partial z} &= -C_1 \frac{\partial V_1}{\partial t} + C_m \frac{\partial V_2}{\partial t}, \\
\frac{\partial V_2}{\partial z} &= -L_2 \frac{\partial I_2}{\partial t} - L_m \frac{\partial I_1}{\partial t}, \\
\frac{\partial I_2}{\partial z} &= -C_2 \frac{\partial V_2}{\partial t} + C_m \frac{\partial V_1}{\partial t}
\end{align*}
\]

(12.1.1)

When \( L_m = C_m = 0 \), they reduce to the uncoupled equations describing the isolated individual lines. Eqs. (12.1.1) may be written in the 2×2 matrix forms:

\[
\begin{align*}
\frac{\partial V}{\partial z} &= -\begin{bmatrix} L_1 & L_m \\ L_m & L_2 \end{bmatrix} \frac{\partial I}{\partial t}, \\
\frac{\partial I}{\partial z} &= \begin{bmatrix} C_1 & -C_m \\ -C_m & C_2 \end{bmatrix} \frac{\partial V}{\partial t}
\end{align*}
\]

(12.1.2)

where \( V, I \) are the column vectors:

\[
V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}
\]

(12.1.3)

For sinusoidal time dependence \( e^{j\omega t} \), the system (12.1.2) becomes:

\[
\begin{align*}
\frac{dV}{dz} &= -j\omega \begin{bmatrix} L_1 & L_m \\ L_m & L_2 \end{bmatrix} I, \\
\frac{dI}{dz} &= -j\omega \begin{bmatrix} C_1 & -C_m \\ -C_m & C_2 \end{bmatrix} V
\end{align*}
\]

(12.1.4)

It proves convenient to recast these equations in terms of the forward and backward waves that are normalized with respect to the uncoupled impedances \( Z_1, Z_2 \):

\[
\begin{align*}
a_1 &= \frac{V_1 + Z_1 I_1}{2\sqrt{Z_1}}, & b_1 &= \frac{V_1 - Z_1 I_1}{2\sqrt{Z_1}}, \\
\Rightarrow & \quad a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\end{align*}
\]

(12.1.5)

The \( a, b \) waves are similar to the power waves defined in Sec. 14.7. The total average power on the line can be expressed conveniently in terms of these:

\[
P = \frac{1}{2} \text{Re}[V^*I] = \frac{1}{2} \text{Re}[V_1^*I_1] + \frac{1}{2} \text{Re}[V_2^*I_2] = P_1 + P_2
\]

\[
= (|a_1|^2 - |b_1|^2) + (|a_2|^2 - |b_2|^2) = (|a_1|^2 + |a_2|^2) - (|b_1|^2 + |b_2|^2)
\]

\[
= a^*a - b^*b
\]

(12.1.6)

where the \( a \) waves carry power forward, and the \( b \) waves, backward. After some algebra, it can be shown that Eqs. (12.1.4) are equivalent to the system:

\[
\begin{align*}
\frac{d}{dz} \begin{bmatrix} a \\ b \end{bmatrix} &= -j \begin{bmatrix} F & -G \\ G & F \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}
\end{align*}
\]

(12.1.7)

with the matrices \( F, G \) given by:

\[
F = \begin{bmatrix} \beta_1 & \kappa \\ \kappa & \beta_2 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}
\]

(12.1.8)

where \( \beta_1, \beta_2 \) are the uncoupled wavenumbers \( \beta_1 = \omega/v_1 = \omega\sqrt{L_1/C_1} \), \( \beta_2 = \omega\sqrt{L_2/C_2} \) and the coupling parameters \( \kappa, X \) are:

\[
\kappa = \frac{1}{2} \left[ \frac{L_m}{\sqrt{Z_1Z_2}} - C_m\sqrt{Z_1Z_2} \right] = \frac{1}{2} \sqrt{\beta_1\beta_2} \left( \frac{L_m}{\sqrt{L_1L_2}} - \frac{C_m}{\sqrt{C_1C_2}} \right)
\]

(12.1.9)

\[
X = \frac{1}{2} \left[ \frac{L_m}{\sqrt{Z_1Z_2}} + C_m\sqrt{Z_1Z_2} \right] = \frac{1}{2} \sqrt{\beta_1\beta_2} \left( \frac{L_m}{\sqrt{L_1L_2}} + \frac{C_m}{\sqrt{C_1C_2}} \right)
\]

(12.1.10)

A consequence of the structure of the matrices \( F, G \) is that the total power \( P \) defined in (12.1.6) is conserved along \( z \). This follows by writing the power in the following form, where \( I \) is the 2×2 identity matrix:

\[
P = a^*a - b^*b = [a^*, b^*] \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}
\]

(12.1.11)

Using (12.1.7), we find:

\[
\frac{dP}{dz} = j[a^*, b^*] \begin{bmatrix} -F^+ & -G^+ \\ -G & -F \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} F & -G \\ G & F \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0
\]

(12.1.12)

the latter following from the conditions \( F^T = F \) and \( G^T = G \). Eqs. (12.1.6) and (12.1.7) form the basis of coupled-mode theory.

Next, we specialize to the case of two identical lines that have \( L_1 = L_2 = L \) and \( C_1 = C_2 = C \), so that \( \beta_1 = \beta_2 = \omega\sqrt{L/C} \equiv \beta \) and \( Z_1 = Z_2 = \sqrt{L/C} \equiv Z_0 \), and speed \( v_0 = 1/\sqrt{L/C} \). Then, the \( a, b \) waves and the matrices \( F, G \) take the simpler forms:

\[
a = \frac{V + Z_0 I}{2\sqrt{Z_0}}, \quad b = \frac{V - Z_0 I}{2\sqrt{Z_0}} \Rightarrow a = \frac{V + Z_0 I}{2}, \quad b = \frac{V - Z_0 I}{2}
\]

(12.1.10)

\[
F = \begin{bmatrix} \beta & \kappa \\ \kappa & \beta \end{bmatrix}, \quad G = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix}
\]

(12.1.11)

where, for simplicity, we removed the common scale factor \( \sqrt{Z_0} \) from the denominator of \( a, b \). The parameters \( \kappa, X \) are obtained by setting \( Z_1 = Z_2 = Z_0 \) in (12.1.9):

\[
\kappa = \frac{1}{2} \beta \left( \frac{L_m}{L_0} - \frac{C_m}{C_0} \right), \quad X = \frac{1}{2} \beta \left( \frac{L_m}{L_0} + \frac{C_m}{C_0} \right).
\]

(12.1.12)

The matrices \( F, G \) commute with each other. In fact, they are both examples of matrices of the form:

\[
A = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix} = a_0 I + a_1 J, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

(12.1.13)
where \( a_0, a_1 \) are real such that \( |a_0| \neq |a_1| \). Such matrices form a \textit{commutative} subgroup of the group of nonsingular \( 2 \times 2 \) matrices. Their eigenvalues are \( \lambda_{\pm} = a_0 \pm a_1 \) and they can all be diagonalized by a \textit{common} unitary matrix:

\[
Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = [e_+, e_-], \quad e_+ = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad e_- = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

so that we have \( QQ^\dagger = Q^\dagger Q = I \) and \( Ae_\pm = \lambda_{\pm} e_\pm \).

The eigenvectors \( e_\pm \) are referred to as the even and odd modes. To simplify subsequent expressions, we will denote the eigenvalues of \( A \) by \( A_{\pm} = a_0 \pm a_1 \) and the diagonalized matrix by \( \tilde{A} \). Thus,

\[
A = Q\tilde{A}Q^\dagger, \quad \tilde{A} = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix} = \begin{bmatrix} a_0 + a_1 & 0 \\ 0 & a_0 - a_1 \end{bmatrix}
\]

Such matrices, as well as any matrix-valued function thereof, may be diagonalized simultaneously. Three examples of such functions appear in the solution of Eqs. (12.1.7):

\[
\begin{align*}
B &= \sqrt{(F + G)(F - G)} = Q\sqrt{(F + G)(F - G)}Q^\dagger \\
Z &= Z_0\sqrt{(F + G)(F - G)}^{-1} = Z_0Q\sqrt{(F + G)(F - G)}^{-1}Q^\dagger \\
\Gamma &= (Z - Z_0)I(Z + Z_0)I^{-1} = Q(Z - Z_0)I(Z + Z_0)I^{-1}Q^\dagger
\end{align*}
\]

Using the property \( FG = GF \), and differentiating (12.1.7) one more time, we obtain the decoupled second-order equations, with \( B \) as defined in (12.1.16):

\[
\frac{d^2a}{dz^2} = -B^2a, \quad \frac{d^2b}{dz^2} = -B^2b
\]

However, it is better to work with (12.1.7) directly. This system can be decoupled by forming the following linear combinations of the \( a, b \) waves:

\[
\begin{align*}
A &= a - \Gamma b \\
B &= b - \Gamma a
\end{align*}
\]

The \( A, B \) can be written in terms of \( V, I \) and the impedance matrix \( Z \) as follows:

\[
\begin{align*}
A &= (2D)^{-1}(V + ZI) \\
B &= (2D)^{-1}(V - ZI)
\end{align*}
\]

Using (12.1.17), we find that \( A, B \) satisfy the decoupled first-order system:

\[
\frac{d}{dz} \begin{bmatrix} A \\ B \end{bmatrix} = -j \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \Rightarrow \frac{dA}{dz} = -jRA, \quad \frac{dB}{dz} = jRB
\]

with solutions expressed in terms of the matrix exponentials \( e^{\pm jRBz}:

\[
A(z) = e^{-jRbz}A(0), \quad B(z) = e^{jRbz}B(0)
\]

Using (12.1.18), we obtain the solutions for \( V, I \):

\[
\begin{align*}
V(z) &= D[e^{-jRbz}A(0) + e^{jRbz}B(0)] \\
ZI(z) &= D[e^{-jRbz}A(0) - e^{jRbz}B(0)]
\end{align*}
\]

To complete the solution, we assume that both lines are terminated at common generator and load impedances, that is, \( Z_{G1} = Z_{G2} = Z_G \) and \( Z_{I1} = Z_{I2} = Z_I \). The generator voltages \( V_{G1}, V_{G2} \) are assumed to be different. We define the generator voltage vector and source and load matrix reflection coefficients:

\[
\begin{align*}
V_G &= \begin{bmatrix} V_{G1} \\ V_{G2} \end{bmatrix}, \quad \Gamma_G = (Z_{G1} - Z)(Z_{G1} + Z)^{-1} \\
\Gamma_L &= (Z_{I1} - Z)(Z_{I1} + Z)^{-1}
\end{align*}
\]

The terminal conditions for the line are at \( z = 0 \) and \( z = I \):

\[
V_G = V(0) + Z_GI(0), \quad V(I) = Z_I(I)
\]

They may be re-expressed in terms of \( A, B \) with the help of (12.1.18):

\[
A(0) - \Gamma_G B(0) = D^{-1}(Z + Z_GI)^{-1}V_G, \quad B(I) = \Gamma_LA(I)
\]

But from (12.1.19), we have:

\[
e^{jRb}B(0) = B(I) = \Gamma_LA(I) = \Gamma_Le^{-jRb}A(0) = B(0) = \Gamma_Le^{-jRb}A(0)
\]

Inserting this into (12.1.24), we may solve for \( A(0) \) in terms of the generator voltage:

\[
A(0) = D^{-1}[I - \Gamma_G\Gamma_Le^{-jRb}][Z + Z_GI]^{-1}V_G
\]

Using (12.1.26) into (12.1.21), we finally obtain the voltage and current at an arbitrary position \( z \) along the lines:

\[
\begin{align*}
V(z) &= [e^{-jRbz} + \Gamma_Le^{-jRb}A(0)]\left[I - \Gamma_G\Gamma_Le^{-jRb}\right][Z + Z_GI]^{-1}V_G \\
I(z) &= [e^{-jRbz} - \Gamma_Le^{-jRb}B(0)]\left[I - \Gamma_G\Gamma_Le^{-jRb}\right][Z + Z_GI]^{-1}V_G
\end{align*}
\]

These are the coupled-line generalizations of Eqs. (11.9.7). Resolving \( V_G \) and \( V(z) \) into their even and odd modes, that is, expressing them as linear combinations of the eigenvectors \( e_\pm \), we have:

\[
V_G = V_G+e_+ + V_G-e_-, \quad V_G= \frac{V_{G1} \pm V_{G2}}{\sqrt{2}}
\]

\[
V(z) = V_+(z)e_+ + V_-(z)e_-, \quad V_+(z) = \frac{V_1(z) \pm V_2(z)}{\sqrt{2}}
\]

In this basis, the matrices in (12.1.27) are diagonal resulting in the equivalent solution:

\[
\begin{align*}
V(z) &= V_+(z)e_+ + V_-(z)e_- = e^{-jRbz} + \Gamma_Le^{-jRb}A(0) + e^{jRbz}B(0) \\
&= \begin{bmatrix} V_{G1} \pm V_{G2} \\ \frac{1 - \Gamma_G\Gamma_Le^{-jRb}}{Z + Z_G} \end{bmatrix} + \begin{bmatrix} Z_G \frac{1 - \Gamma_G\Gamma_Le^{-jRb}}{Z + Z_G} \\ 1 \frac{1 - \Gamma_G\Gamma_Le^{-jRb}}{Z + Z_G} \end{bmatrix} V_G+e_+
\end{align*}
\]

\(^1\)The matrices \( D, Z, \Gamma_G, \Gamma_L, I, B \) all commute with each other.
where \( \beta_\pm \) are the eigenvalues of \( B, Z_\pm \) the eigenvalues of \( Z\), and \( \Gamma_{G\pm}, \Gamma_{L\pm} \) are:

\[
\Gamma_{G\pm} = \frac{Z_0 - Z_\pm}{Z_0 + Z_\pm}, \quad \Gamma_{L\pm} = \frac{Z_L - Z_\pm}{Z_L + Z_\pm} \tag{12.1.30}
\]

The voltages \( V_1(z), V_2(z) \) are obtained by extracting the top and bottom components of (12.1.29), that is, \( V_{1,2}(z) = \left[V_{\pm}(z) \pm V_-(z)\right] / \sqrt{2} \):

\[
V_1(z) = \frac{e^{-j\beta_+ z} + \Gamma_{L+} e^{-j\beta_- z}}{1 - \Gamma_{G+} \Gamma_{L-}} V_0 + \frac{e^{-j\beta_- z} + \Gamma_{L-} e^{-j\beta_+ z}}{1 - \Gamma_{G-} \Gamma_{L+}} V_-
\]

\[
V_2(z) = \frac{e^{-j\beta_+ z} + \Gamma_{L+} e^{-j\beta_- z}}{1 - \Gamma_{G+} \Gamma_{L-}} V_0 - \frac{e^{-j\beta_- z} + \Gamma_{L-} e^{-j\beta_+ z}}{1 - \Gamma_{G-} \Gamma_{L+}} V_-
\]

where we defined:

\[
V_\pm = \left( \frac{Z_\pm - Z_G}{Z_\pm + Z_G} \right) \frac{V_{G\pm}}{\sqrt{2}} = \frac{1}{4} (1 - \Gamma_{G\pm}) (V_{G1} \pm V_{G2}) \tag{12.1.32}
\]

The parameters \( \beta_\pm, Z_\pm \) are obtained using the rules of Eq. (12.1.15). From Eq. (12.1.12), we find the eigenvalues of the matrices \( F \pm G \):

\[
(F + G) \pm = \beta \pm (\kappa + \chi) = \beta \left( \pm \frac{L_m}{L_0} \right) = \omega \frac{1}{c_0} (L_0 \pm L_m)
\]

\[
(F - G) \pm = \beta \pm (\kappa - \chi) = \beta \left( \pm \frac{C_m}{c_0} \right) = \omega Z_0 (C_0 \mp C_m)
\]

Then, it follows that:

\[
\beta_+ = \sqrt{(F + G)_+ (F - G)_+} = \omega \sqrt{(L_0 + L_m) (C_0 - C_m)}
\]

\[
\beta_- = \sqrt{(F + G)_- (F - G)_-} = \omega \sqrt{(L_0 - L_m) (C_0 + C_m)} \tag{12.1.33}
\]

\[
Z_+ = Z_0 \sqrt{(F + G)_+ (F - G)_+} = \frac{L_0 + L_m}{C_0 - C_m} \tag{12.1.34}
\]

\[
Z_- = Z_0 \sqrt{(F + G)_- (F - G)_-} = \frac{L_0 - L_m}{C_0 + C_m}
\]

Thus, the coupled system acts as two uncoupled lines with wavenumbers and characteristic impedances \( \beta_\pm, Z_\pm \), propagation speeds \( \nu_\pm = 1 / \sqrt{(L_0 \pm L_m) (C_0 \mp C_m)} \), and propagation delays \( T_\pm = 1 / \nu_\pm \). The even mode is energized when \( V_{G2} = V_{G1} \), or, \( V_{G+} \neq 0, V_{G-} = 0 \), and the odd mode, when \( V_{G2} = -V_{G1} \), or, \( V_{G+} = 0, V_{G-} \neq 0 \).

When the coupled lines are immersed in a homogeneous medium, such as two parallel wires in air over a ground plane, then the propagation speeds must be equal to the speed of light within this medium [1065], that is, \( \nu_+ = \nu_- = 1 / \sqrt{\mu c} \). This requires:

\[
(L_0 + L_m) (C_0 - C_m) = \mu c \quad \implies \frac{L_0}{C_0 - C_m} = \frac{\mu c C_0}{\mu c}
\]

\[
(L_0 - L_m) (C_0 + C_m) = \mu c \quad \implies \frac{L_m}{C_0 + C_m} = \frac{\mu c C_m}{\mu c}
\]

Therefore, \( L_m / L_0 = C_m / C_0 \), or, equivalently, \( \kappa = 0 \). On the other hand, in an inhomogeneous medium, such as for the case of the microstrip lines shown in Fig. 12.1.1, the propagation speeds may be different, \( \nu_+ \neq \nu_- \), and hence \( T_+ \neq T_- \).

### 12.2 Crosstalk Between Lines

When only line-1 is energized, that is, \( V_{G1} \neq 0, V_{G2} = 0 \), the coupling between the lines induces a propagating wave in line-2, referred to as crosstalk, which also has some minor influence back on line-1. The near-end and far-end crosstalk are the values of \( V_2(z) \) at \( z = 0 \) and \( z = L \), respectively. Setting \( V_{G2} = 0 \) in (12.1.32), we have from (12.1.31):

\[
V_2(0) = \frac{1}{2} \left[ (1 - \Gamma_{G+}) (1 + \Gamma_{L+} \zeta_*^2) + (1 - \Gamma_{G-}) (1 + \Gamma_{L-} \zeta_*^2) \right] V_0
\]

\[
V_2(L) = \frac{1}{2} \left[ (1 - \Gamma_{G+}) (1 + \Gamma_{L+}) + (1 - \Gamma_{G-}) (1 + \Gamma_{L-}) \right] V_0 \tag{12.2.1}
\]

where we defined \( V = V_{G1} / 2 \) and introduced the \( z \)-transform delay variables \( \zeta_* = e^{j\beta_0 T_*} = e^{j\beta_0 t} \). Assuming purely resistive termination impedances \( Z_G, Z_L \), we may use Eq. (11.15.15) to obtain the corresponding time-domain responses:

\[
V_2(0, t) = \frac{1}{2} \left[ (1 - \Gamma_{G+}) \left( V(t) + \left( 1 + \frac{1}{\Gamma_{G+}} \right) \sum_{m=1}^{\infty} (\Gamma_{G+} \Gamma_{L+})^m V(t - 2mT_+) \right) \right]
\]

\[
\frac{1}{2} \left[ (1 - \Gamma_{G-}) \left( V(t) + \left( 1 + \frac{1}{\Gamma_{G-}} \right) \sum_{m=1}^{\infty} (\Gamma_{G-} \Gamma_{L-})^m V(t - 2mT_-) \right) \right] \tag{12.2.2}
\]

\[
V_2(L, t) = \frac{1}{2} \left[ (1 - \Gamma_{G+}) (1 + \Gamma_{L+}) \sum_{m=0}^{\infty} (\Gamma_{G+} \Gamma_{L+})^m V(t - 2mT_+) - T_+ \right]
\]

\[
\frac{1}{2} \left[ (1 - \Gamma_{G-}) (1 + \Gamma_{L-}) \sum_{m=0}^{\infty} (\Gamma_{G-} \Gamma_{L-})^m V(t - 2mT_-) - T_- \right]
\]

where \( V(t) = V_{G1}(t) / 2 \). Because \( Z_+ \neq Z_0 \), there will be multiple reflections even when the lines are matched to \( Z_0 \) at both ends. Setting \( Z = Z_L = Z_0 \), gives for the reflection coefficients (12.1.30):

\[
\Gamma_{G+} = \Gamma_{L+} = \frac{Z_0 - Z_\pm}{Z_0 + Z_\pm} = -\Gamma_\pm \tag{12.2.3}
\]

In this case, we find for the crosstalk signals:

\[
V_2(0, t) = \frac{1}{2} \left[ (1 + \Gamma_+) \left( V(t) - (1 - \Gamma_+) \sum_{m=1}^{\infty} \Gamma_{2m-1} V(t - 2mT_+) \right) \right]
\]

\[
\frac{1}{2} \left[ (1 + \Gamma_-) \left( V(t) - (1 - \Gamma_-) \sum_{m=1}^{\infty} \Gamma_{2m-1} V(t - 2mT_-) \right) \right] \tag{12.2.4}
\]

\[
V_2(L, t) = \frac{1}{2} \left[ (1 - \Gamma^2) \sum_{m=0}^{\infty} \Gamma_{2m} V(t - 2mT_+) - T_+ \right]
\]

\[
\frac{1}{2} \left[ (1 - \Gamma^2) \sum_{m=0}^{\infty} \Gamma_{2m} V(t - 2mT_-) - T_- \right] \]

\[\]V(t) is the signal that would exist on a matched line-1 in the absence of line-2, \( V = Z_0 V_{G1}/(Z_0 + Z_G) = V_{G1}/2 \), provided \( Z_G = Z_0 \).
Similarly, the near-end and far-end signals on the driven line are found by adding, instead of subtracting, the even- and odd-mode terms:

\[ V_1(0,t) = \frac{1}{2}(1 + \Gamma_+)[V(t) - (1 - \Gamma_+) \sum_{m=1}^{\infty} \Gamma_+^{2m-1}V(t - 2mT_+)] + \frac{1}{2}(1 - \Gamma_-)[V(t) - (1 - \Gamma_-) \sum_{m=1}^{\infty} \Gamma_-^{2m-1}V(t - 2mT_-)] \]

(12.2.5)

\[ V_1(l,t) = \frac{1}{2}(1 - \Gamma_+^{2}) \sum_{m=0}^{\infty} \Gamma_+^{2m}V(t - 2mT_+) - T_+ \]

\[ + \frac{1}{2}(1 - \Gamma_-^{2}) \sum_{m=0}^{\infty} \Gamma_-^{2m}V(t - 2mT_-) - T_- \]

These expressions simplify drastically if we assume weak coupling. It is straightforward to verify that to first-order in the parameters \( L_m/L_0, C_m/C_0 \), or equivalently, to first-order in \( \kappa, \chi \), we have the approximations:

\[ \beta_+ = \beta \pm \Delta \beta = \beta \mp \kappa, \quad Z_+ = Z_0 \pm \Delta Z = Z_0 \mp 0 = Z_0 \frac{\chi}{\beta}, \quad V_+ = V_0 \mp 0 = V_0 \frac{\kappa}{\beta} \]

\[ \Gamma_+ = 0 \pm \Delta \Gamma = \pm \frac{\chi}{2\beta}, \quad T_+ = T \pm \Delta T = T \mp \frac{\kappa}{\beta} \]  

(12.2.6)

where \( T = l/v_0 \). Because the \( \Gamma_+ \)s are already first-order, the multiple reflection terms in the above summations are a second-order effect, and only the lowest terms will contribute, that is, the term \( m = 1 \) for the near-end, and \( m = 0 \) for the far end. Then,

\[ V_2(0,t) = \frac{1}{2}(\Gamma_+ - \Gamma_-)V(t) - \frac{1}{2}[\Gamma_+ V(t - 2T_+) - \Gamma_- V(t - 2T_-)] \]

\[ V_2(l,t) = \frac{1}{2}[V(t - T_+) - V(t - T_-)] \]

Using a Taylor series expansion and (12.2.6), we have to first-order:

\[ V(t - 2T_+) = V(t - 2T + \Delta T) = V(t - 2T) + (\Delta T)\dot{V}(t - 2T), \quad \dot{V} = \frac{dV}{dt} \]

\[ V(t - T_+) = V(t - T + \Delta T) = V(t - T) + (\Delta T)\dot{V}(t - T) \]

Therefore, \( \Gamma_+^{2}V(t - 2T_+) = \Gamma_+^{2}V(t - 2T) + (\Delta T)\dot{V}(t - 2T) \)

\[ \Gamma_+^{2}V(t - T_+) = \Gamma_+^{2}V(t - T) + (\Delta T)\dot{V}(t - T) \]

These can be written in the commonly used form:

\[ V_2(0,t) = K_h [V(t) - V(t - 2T)] \]

(near- and far-end crosstalk)  

(12.2.7)

where \( K_h, K_f \) are known as the backward and forward crosstalk coefficients:

\[ K_h = \frac{X}{2\beta} = \frac{V_0}{4} \left( \frac{L_m}{Z_0} + C_m Z_0 \right), \quad K_f = -\frac{T K}{\beta} = -\frac{V_0 T}{2} \left( \frac{L_m}{Z_0} - C_m Z_0 \right) \]

where we may replace \( l = V_0 T \). The same approximations give for line-1, \( V_1(0,t) = V(t) \) and \( V_1(l,t) = V(t - T) \). Thus, to first-order, line-2 does not act back to disturb line-1.

**Example 12.2.1:** Fig. 12.2.1 shows the signals \( V_1(0,t), V_1(l,t), V_2(0,t), V_2(l,t) \) for a pair of coupled lines matched at both ends. The uncoupled line impedance was \( Z_0 = 50 \Omega \).

![Fig. 12.2.1 Near- and far-end crosstalk signals on lines 1 and 2.](image-url)

For the left graph, we chose \( L_m/L_0 = 0.4, C_m/C_0 = 0.3 \), which results in the even and odd mode parameters (using the exact formulas):

\[ Z_+ = 70.71 \Omega, \quad Z_- = 33.97 \Omega, \quad v_+ = 1.01 v_0, \quad v_- = 1.13 v_0 \]

\[ \Gamma_+ = 0.17, \quad \Gamma_- = -0.19, \quad T_+ = 0.99T, \quad T_- = 0.88T, \quad K_h = 0.175, \quad K_f = 0.05 \]

The right graph corresponds to \( L_m/L_0 = 0.8, C_m/C_0 = 0.7 \), with parameters:

\[ Z_+ = 122.47 \Omega, \quad Z_- = 17.15 \Omega, \quad v_+ = 1.36 v_0, \quad v_- = 1.71 v_0 \]

\[ \Gamma_+ = 0.42, \quad \Gamma_- = 0.49, \quad T_+ = 0.73T, \quad T_- = 0.58T, \quad K_h = 0.375, \quad K_f = 0.05 \]

The generator input to line-1 was a rising step with rise-time \( t_r = T/4 \), that is,

\[ V(t) = \frac{1}{2} V_G(t) = \frac{t}{t_r} [u(t) - u(t - t_r)] + u(t - t_r) \]

The weak-coupling approximations are more closely satisfied for the left case. Eqs. (12.2.7) predict for \( V_2(0,t) \) a trapezoidal pulse of duration \( 2T \) and height \( K_h \), and for \( V_2(l,t) \), a rectangular pulse of width \( t_r \) and height \( K_f t_r / t_r = 0.2 \) starting at \( t = T \):

\[ V_2(l,t) = K_f \frac{dV(t - T)}{dt} = K_f \frac{t_r}{t_r} [u(t) - u(t - T - t_r)] \]

These predictions are approximately correct as can be seen in the figure. The approximation predicts also that \( V_1(0,t) = V(t) \) and \( V_1(l,t) = V(t - T) \), which are not quite true—the effect of line-2 on line-1 cannot be ignored completely.
12.3 Weakly Coupled Lines with Arbitrary Terminations

The interaction between the two lines is seen better in the MATLAB movie xtalkmovie.m, which plots the waves $V_1(z,t)$ and $V_2(z,t)$ as they propagate to and get reflected from their respective loads, and compares them to the uncoupled case $V_0(z,t) = V(t - z/v_0)$.

The waves $V_{1,2}(z,t)$ are computed by the same method as for the movie pulsemovie.m of Example 11.15.1, applied separately to the even and odd modes.

12.3 Weakly Coupled Lines with Arbitrary Terminations

The even-odd mode decomposition can be carried out only in the case of identical lines both of which have the same load and generator impedances. The case of arbitrary terminations has been solved in closed form only for homogeneous media [1062,1065]. It has also been solved for arbitrary media under the weak coupling assumption [1072].

Following [1072], we solve the general equations (12.1.7)-(12.1.9) for weakly coupled lines assuming arbitrary terminating impedances $Z_{Li}, Z_{Gi}$ with reflection coefficients:

$$\Gamma_{Li} = \frac{Z_{L1} - Z_{i}}{Z_{L1} + Z_{i}}, \quad \Gamma_{Gi} = \frac{Z_{G1} - Z_{i}}{Z_{G1} + Z_{i}}, \quad i = 1, 2 \quad (12.3.1)$$

Working with the forward and backward waves, we write Eq. (12.1.7) as the 4×4 matrix equation:

$$\frac{dc}{dz} = -jMc, \quad c = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}, \quad M = \begin{bmatrix} \beta_1 & \kappa & 0 & -X \\ \kappa & \beta_2 & -\chi & 0 \\ 0 & \chi & -\beta_1 & -\kappa \\ X & 0 & -\kappa & -\beta_2 \end{bmatrix} \quad (12.3.2)$$

The weak coupling assumption consists of ignoring the coupling of $a_1, b_1$ on $a_2, b_2$. This amounts to approximating the above linear system by:

$$\frac{dc}{dz} = -jMc, \quad M = \begin{bmatrix} \beta_1 & 0 & 0 & 0 \\ \kappa & \beta_2 & -\chi & 0 \\ 0 & \chi & -\beta_1 & 0 \\ X & 0 & -\kappa & -\beta_2 \end{bmatrix} \quad (12.3.3)$$

Its solution is given by $c(z) = e^{-j\beta_1 z}c(0)$, where the transition matrix $e^{-j\beta_1 z}$ can be expressed in closed form as follows:

$$e^{-j\beta_1 z} = \begin{bmatrix} e^{-j\beta_1 z} & 0 & 0 & 0 \\ \kappa e^{-j\beta_1 z} \chi & e^{-j\beta_2 z} & -\chi e^{-j\beta_1 z} \chi & 0 \\ \kappa e^{-j\beta_1 z} \chi & 0 & e^{-j\beta_2 z} & 0 \\ X e^{-j\beta_1 z} & 0 & 0 & e^{-j\beta_2 z} \end{bmatrix}, \quad \kappa = \frac{\kappa}{\beta_1 - \beta_2} \quad (12.3.4)$$

The transition matrix $e^{-j\beta_1 z}$ may be written in terms of the z-domain delay variables $\zeta_i = e^{\beta_i z}, i = 1, 2$, where $T_i$ are the one-way travel times along the lines, that is, $T_i = 1/v_i$. Then, we find:

$$\begin{bmatrix} a_1(t) \\ a_2(t) \\ b_1(t) \\ b_2(t) \end{bmatrix} = \begin{bmatrix} \zeta_1^{-1} & 0 & 0 & 0 \\ \kappa(\zeta_1^{-1} - \zeta_2^{-1}) & \zeta_2^{-1} & \chi(\zeta_1 - \zeta_2^{-1}) & 0 \\ 0 & 0 & \zeta_1 & 0 \\ \kappa(\zeta_1^{-1} - \zeta_2) & 0 & \kappa(\zeta_1 - \zeta_2) & \zeta_2 \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \\ b_1(0) \\ b_2(0) \end{bmatrix} \quad (12.3.5)$$

These must be appended by the appropriate terminating conditions. Assuming that only line-1 is driven, we have:

$$V_1(0) + ZGI_1(0) = V_{G1}, \quad V_1(l) = ZL1I_1(l)$$
$$V_2(0) + ZGL2(0) = 0, \quad V_2(l) = ZL2I_2(l)$$

which can be written in terms of the $a, b$ waves:

$$a_1(0) - \Gamma_{G1}b_1(0) = U_1, \quad b_1(l) = \Gamma_{L1}a_1(l)$$
$$a_2(0) - \Gamma_{G2}b_2(0) = 0, \quad b_2(l) = \Gamma_{L2}a_2(l), \quad U_1 = \frac{2}{\sqrt{\zeta_1}} \frac{1 - \Gamma_{G1}}{1 - \Gamma_{G1}\Gamma_{L1}\zeta_1^{-2}} V_{G1} \quad (12.3.6)$$

Eqs. (12.3.3) and (12.3.4) provide a set of eight equations in eight unknowns. Once these are solved, the near- and far-end voltages may be determined. For line-1, we find:

$$V_1(0) = \frac{\sqrt{\zeta_1}}{2} [a_1(0) + b_1(0)] = \frac{1 + \Gamma_{L1}\zeta_1^{-2}}{1 - \Gamma_{G1}\Gamma_{L1}\zeta_1^{-2}} V$$
$$V_1(l) = \frac{\sqrt{\zeta_1}}{2} [a_1(l) + b_1(l)] = \frac{\zeta_1^{-1} + (1 + \Gamma_{G1})}{1 - \Gamma_{G1}\Gamma_{L1}\zeta_1^{-2}} V$$

where $V = (1 - \Gamma_{G1})V_{G1}/2 = Z_1V_{G1}/(Z_1 + Z_{G1})$. For line-2, we have:

$$V_2(0) = \frac{\zeta_2^{-1}}{2} \Gamma_{L2} \zeta_2^{-1} + \chi(1 - \zeta_1^{-1}\zeta_2^{-1}) (1 + \Gamma_{L1}\Gamma_{L2} \zeta_1^{-1}\zeta_2^{-1}) V_{20}$$
$$V_2(l) = \frac{\zeta_2^{-1}}{2} \Gamma_{L2} \zeta_2^{-1} + \chi(1 - \zeta_1^{-1}\zeta_2^{-1}) (1 + \Gamma_{G2} \zeta_2^{-1}) V_{2l}$$

where $V_{20} = (1 + \Gamma_{G2})V = (1 + \Gamma_{G2})V_{G1}/2$ and $V_{2l} = (1 + \Gamma_{L2})V$, and we defined $\kappa, \chi$ by:

$$\kappa = \frac{\zeta_2^{-1}}{\zeta_1}, \quad \chi = \frac{\zeta_2^{-1}}{\zeta_1} = \frac{\omega \zeta_1}{\beta_1 - \beta_2} = \frac{1}{Z_1^2} \left( \frac{L_m}{Z_1} - C_m Z_1 \right) \quad (12.3.7)$$

In the case of identical lines with $Z_1 = Z_2 = Z_0$ and $\beta_1 = \beta_2 = \beta = \omega/v_0$, we must take the limit:

$$\lim_{\beta \to \beta_1} e^{-j\beta t} = \frac{d}{d\beta} e^{-j\beta t} = -je^{-j\beta t}$$

Then, we obtain:

$$\kappa(\zeta_1^{-1} - \zeta_2^{-1}) - j\omega K_Re^{-j\beta t} = -j\omega \frac{1}{2} \left( \frac{L_m}{Z_0} - C_m Z_0 \right) e^{-j\beta t} \quad (12.3.8)$$

Thus, we define $K_f, K_b$ were defined in (12.2.8). Setting $\zeta_1 = \zeta_2 = \zeta = e^{j\beta t} = e^{j\omega t}$, we obtain the crosstalk signals:
12.4. Coupled-Mode Theory

In its simplest form, coupled-mode or coupled-wave theory provides a paradigm for the interaction between two waves and the exchange of energy from one of the other to the propagator. Reviews and earlier literature may be found in Refs. [1073–1094], see also [784–803] for the relationship to fiber Bragg gratings and distributed feedback lasers.

There are several mechanical and electrical analogs of coupled-mode theory, such as a pair of coupled pendula, or two masses at the ends of two springs with a third spring connecting the two, or two LC circuits with a coupling capacitor between them. In these examples, the exchange of energy is taking place over time instead of over space.

Coupled-wave theory is inherently directional. If two forward-moving waves are strongly coupled, then their interactions with the corresponding backward waves may be ignored. Similarly, if a forward- and a backward-moving wave are strongly coupled, then their interactions with the corresponding oppositely moving waves may be ignored. Fig. 12.4.1 depicts these two cases of co-directional and contra-directional coupling.

![Fig. 12.4.1 Directional Couplers.](image)

Eqs. (12.1.7) form the basis of coupled-mode theory. In the co-directional case, if we assume that there are only forward waves at \( z = 0 \), that is, \( a(0) \neq 0 \) and \( b(0) = 0 \), then it may shown that the effect of the backward waves on the forward ones becomes a second-order effect in the coupling constants, and therefore, may be ignored. To see this, we solve the second of Eqs. (12.1.7) for \( b \) in terms of \( a \), assuming zero initial conditions, and substitute it in the first:

\[
\mathbf{b}(z) = -j \int_0^z e^{\delta(z'-z)} G a(z') \, dz' \Rightarrow \frac{da}{dz} = -j F a + \int_0^z G e^{\beta(z'-z)} G a(z') \, dz'
\]

The second term is second-order in \( G \), or in the coupling constant \( \chi \). Ignoring this term, we obtain the standard equations describing a co-directional coupler:

\[
\frac{da}{dz} = -j F a \Rightarrow \frac{d}{dz} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -j \begin{bmatrix} \beta_1 - \kappa \\ \beta_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (12.4.1)
\]

For the contra-directional case, a similar argument that assumes the initial conditions \( a_2(0) = b_1(0) = 0 \) gives the following approximation that couples the \( a_1 \) and \( b_2 \) waves:

\[
\frac{d}{dz} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix} = -j \begin{bmatrix} \beta_1 \kappa \beta_2 \\ -\beta_2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix} \quad (12.4.2)
\]

The conserved powers are in the two cases:

\[
P = |a_1|^2 + |b_2|^2, \quad P = |a_1|^2 - |b_2|^2 \quad (12.4.3)
\]

The solution of Eq. (12.4.1) is obtained with the help of the transition matrix \( e^{-jFz} \):

\[
e^{-jFz} = e^{-jFz} \begin{bmatrix} \cos \sigma z - j \frac{\delta}{\sigma} \sin \sigma z & -j \frac{\kappa}{\sigma} \sin \sigma z \\ -j \frac{\beta_1}{\sigma} \sin \sigma z & \cos \sigma z + j \frac{\delta}{\sigma} \sin \sigma z \end{bmatrix} \quad (12.4.4)
\]

where

\[
\beta = \frac{\beta_1 + \beta_2}{2}, \quad \delta = \frac{\beta_1 - \beta_2}{2}, \quad \sigma = \sqrt{\delta^2 + \kappa^2} \quad (12.4.5)
\]

Thus, the solution of (12.4.1) is:

\[
\begin{bmatrix} a_1(z) \\ a_2(z) \end{bmatrix} = e^{-jFz} \begin{bmatrix} \cos \sigma z - j \frac{\delta}{\sigma} \sin \sigma z & -j \frac{\kappa}{\sigma} \sin \sigma z \\ -j \frac{\beta_1}{\sigma} \sin \sigma z & \cos \sigma z + j \frac{\delta}{\sigma} \sin \sigma z \end{bmatrix} \begin{bmatrix} a_1(0) \\ a_2(0) \end{bmatrix} \quad (12.4.6)
\]

Starting with initial conditions \( a_1(0) = 1 \) and \( a_2(0) = 0 \), the total initial power will be \( P = |a_1(0)|^2 + |a_2(0)|^2 = 1 \). As the waves propagate along the \( z \)-direction, power is exchanged between lines 1 and 2 according to:

\[
P_1(z) = |a_1(z)|^2 = \cos^2 \sigma z + \frac{\delta^2}{\sigma^2} \sin^2 \sigma z + \frac{\kappa^2}{\sigma^2} \sin^2 \sigma z = 1 - P_2(z) \quad (12.4.7)
\]

Fig. 12.4.2 shows the two cases for which \( \delta/\kappa = 0 \) and \( \delta/\kappa = 0.5 \). In both cases, maximum exchange of power occurs periodically at distances that are odd multiples of \( z = \pi/2 \sigma \). Complete power exchange occurs only in the case \( \delta = 0 \), or equivalently, when \( \beta_1 = \beta_2 \). In this case, we have \( \sigma = \kappa \) and \( P_1(z) = \cos^2 \kappa z, P_2(z) = \sin^2 \kappa z \).
12.5 Fiber Bragg Gratings

As an example of contra-directional coupling, we consider the case of a fiber Bragg grating (FBG), that is, a fiber with a segment that has a periodically varying refractive index, as shown in Fig. 12.5.1.

\[
\begin{bmatrix}
A(z)
B(z)
\end{bmatrix} = \begin{bmatrix}
\alpha & \beta \\
-\beta^* & -\alpha^*
\end{bmatrix}
\begin{bmatrix}
a(z)
\end{bmatrix}
\]

Where \(K = 2\pi/\Lambda\) is the Bloch wavenumber, \(\Lambda\) is the period, and \(a(z), b(z)\) represent the forward and backward waves. The following transformation removes the phase factor \(e^{-jKz}\) from the coupling constant:

\[
\begin{bmatrix}
A(z)
B(z)
\end{bmatrix} = \begin{bmatrix}
e^{jKz/2} & 0 \\
0 & e^{-jKz/2}
\end{bmatrix}
\begin{bmatrix}
A(z)
B(z)
\end{bmatrix}
\]

where \(\delta = \beta - K/2\) is referred to as a detuning parameter. The conserved power is given by \(P(z) = |a(z)|^2 - |b(z)|^2\). The fields at \(z = 0\) are related to those at \(z = l\) by:

\[
\begin{bmatrix}
A(0)
B(0)
\end{bmatrix} = e^{jFl} \begin{bmatrix}
A(l)
B(l)
\end{bmatrix}, \quad \text{with} \quad \Gamma = \begin{bmatrix}
\delta & K \\
-K^* & -\delta
\end{bmatrix}
\]

(12.5.4)

The transfer matrix \(e^{jFl}\) is given by:

\[
\begin{bmatrix}
\cos \sigma l + j \frac{\delta}{\sigma} \sin \sigma l & j \frac{\kappa}{\sigma} \sin \sigma l \\
-j \kappa^* \sin \sigma l & \cos \sigma l - j \frac{\delta}{\sigma} \sin \sigma l
\end{bmatrix} = \begin{bmatrix}
U_{11} & U_{12} \\
U_{21} & U_{22}
\end{bmatrix}
\]

(12.5.5)

where \(\sigma = \sqrt{\delta^2 - |\kappa|^2}\). If \(|\delta| < |\kappa|\), then \(\sigma\) becomes imaginary. In this case, it is more convenient to express the transfer matrix in terms of the quantity \(\sigma y = \sqrt{\delta^2 - |\kappa|^2}\).
Fiber Bragg Gratings

12. Coupled Lines

where we replaced $U_{12}^* = T / U$ and $U_{11} = 1 / T$. Assuming a quarter-wavelength spacing $d = \lambda_B / 4 = \Lambda / 2$, we have $\beta d = (\delta + \pi / \Lambda) d = \delta d + \pi / 2$. Replacing $\phi = e^{i \beta d} e^{i \pi / 2} = j e^{i \beta d}$, we obtain:

$$\Gamma_{\text{comp}} = \frac{\Gamma (T^* e^{i \phi} - T e^{-i \phi})}{T^* e^{i \phi} - |\Gamma|^2 T e^{-i \phi}} \quad (12.5.13)$$

At $\delta = 0$, we have $T = T^* = 1 / \cosh |k|$, and therefore, $\Gamma_{\text{comp}} = 0$. Fig. 12.5.4 depicts the reflectance, $|\Gamma_{\text{comp}}|^2$, and transmittance, $1 - |\Gamma_{\text{comp}}|^2$, for the case $k l = 2$.

Quarter-wave phase-shifted FBGs are similar to the Fabry-Perot resonators discussed in Sec. 6.5. Improved designs having narrow and flat transmission bands can be obtained by cascading several quarter-wave FBGs with different lengths [784–804]. Some applications of FBGs in DWDM systems were pointed out in Sec. 6.7.

12.6 Diffuse Reflection and Transmission

Another example of contra-directional coupling is the two-flux model of Schuster and Kubelka-Munk describing the absorption and multiple scattering of light propagating in a turbid medium [1095–1111].

The model has a large number of applications, such as radiative transfer in stellar atmospheres, reflectance spectroscopy, reflection and transmission properties of papers, paints, skin tissue, dental materials, and the sea.

The model assumes a simplified parallel-plane geometry, as shown in Fig. 12.6.1. Let $I_+ (z)$ be the forward and backward radiation intensities per unit frequency interval at location $z$ within the material. The model is described by the two coefficients $k, s$ of absorption and scattering per unit length. For simplicity, we assume that $k, s$ are independent of $z$.

Within a layer $dz$, the forward intensity $I_+$ will be diminished by an amount of $I_+ k dz$ due to absorption and an amount of $I_+ s d z$ due to scattering, and it will be increased by an amount of $I_+ s d z$ arising from the backward-moving intensity that is getting scattered...
The reflectance and transmittance corresponding to a black, non-reflecting, background are obtained by setting \( R_g = 0 \) in Eq. (12.6.4):

\[
R_0 = \frac{-U_{21}}{U_{22}} = \frac{s \sinh \beta l}{\beta \cosh \beta l + \alpha \sinh \beta l}
\]

\[
T_0 = \frac{1}{U_{22}} = \frac{\beta}{\beta \cosh \beta l + \alpha \sinh \beta l}
\]

The reflectance of an infinitely-thick medium is obtained in the limit \( l \to \infty \):

\[
R_\infty = \frac{s}{s + k + \sqrt{k(k + 2s)}} \Rightarrow \frac{k}{s} = \frac{(R_\infty - 1)^2}{2R_\infty}
\]

For the special case of an absorbing but non-scattering medium (\( k \neq 0, s = 0 \)), we have \( \alpha = \beta = k \) and the transfer matrix (12.6.3) and Eq. (12.6.4) simplify into:

\[
U = e^{-fI} \begin{bmatrix} e^{-kl} & 0 \\ 0 & e^{kl} \end{bmatrix}, \quad R = e^{-2kl} R_g, \quad T = e^{-kl}
\]

These are in accordance with our expectations for exponential attenuation with distance. The intensities are related by \( I_0(s) = I_+(l) = e^{-kl} I_-(0) \) and \( I_0(-s) = I_-(-l) = e^{-kl} I_+(0) \). Thus, the reflectance corresponds to traversing a forward and a reverse path of length \( l \), and the transmittance only a forward path.

Perhaps, the most surprising prediction of this model (first pointed out by Schuster) is that, in the case of a non-absorbing but scattering medium (\( k = 0, s \neq 0 \)), the transmittance is not attenuating exponentially, but rather, inversely with distance. Indeed, setting \( \alpha = s \) and taking the limit \( \beta \to 0 \) with \( \beta \to 0 \), we find:

\[
U = e^{-fI} \begin{bmatrix} 1 - sl & sl \\ -sl & 1 + sl \end{bmatrix}, \quad R = \frac{sl + (1 - sl) R_g}{1 + sl - sR_g}, \quad T = \frac{1}{1 + sl - sR_g}
\]

In particular, for the case of a non-reflecting background, we have:

\[
R_0 = \frac{sl}{1 + sl}, \quad T_0 = \frac{1}{1 + sl}
\]

### 12.7 Problems

12.1 Show that the coupled telegrapher’s equations (12.1.4) can be written in the form (12.1.7).

12.2 Consider the practical case in which two lines are coupled only over a middle portion of length \( l \), with their beginning and ending segments being uncoupled, as shown below:

Assuming weakly coupled lines, how should Eqs. (12.3.6) and (12.3.9) be modified in this case? \( \text{[Hint: Replace the segments to the left of the reference plane A and to the right of plane B by their Thévenin equivalents.]} \)
12.7. Problems

12.3 Derive the transition matrix $e^{-j\hat{M}z}$ of weakly coupled lines described by Eq. (12.3.2).

12.4 Verify explicitly that Eq. (12.4.6) is the solution of the coupled-mode equations (12.4.1).

12.5 Computer Experiment—Fiber Bragg Gratings. Reproduce the results and graphs of Figures 12.5.2 and 12.5.3.

13 Impedance Matching

13.1 Conjugate and Reflectionless Matching

The Thévenin equivalent circuits depicted in Figs. 11.11.1 and 11.11.3 also allow us to answer the question of maximum power transfer. Given a generator and a length-$d$ transmission line, maximum transfer of power from the generator to the load takes place when the load is conjugate matched to the generator, that is,

$$Z_L = Z_{th}^*$$

(conjugate match) \hspace{1cm} (13.1.1)

The proof of this result is postponed until Sec. 16.4. Writing $Z_{th} = R_{th} + jX_{th}$ and $Z_L = R_L + jX_L$, the condition is equivalent to $R_L = R_{th}$ and $X_L = -X_{th}$. In this case, half of the generated power is delivered to the load and half is dissipated in the generator’s Thévenin resistance. From the Thévenin circuit shown in Fig. 11.11.1, we find for the current through the load:

$$I_L = \frac{V_{th}}{Z_{th} + Z_L} = \frac{V_{th}}{R_{th} + R_L + j(X_{th} + X_L)} = \frac{V_{th}}{2R_{th}}$$

Thus, the total reactance of the circuit is canceled. It follows then that the power delivered by the Thévenin generator and the powers dissipated in the generator’s Thévenin resistance and the load will be:

$$P_{tot} = \frac{1}{2} \text{Re}(V_{th}^* I_L) = \frac{|V_{th}|^2}{4R_{th}}$$
$$P_{th} = \frac{1}{2} R_{th} |I_L|^2 = \frac{|V_{th}|^2}{8R_{th}} = \frac{1}{2} P_{tot}$$
$$P_L = \frac{1}{2} R_L |I_L|^2 = \frac{|V_{th}|^2}{8R_{th}} = \frac{1}{2} P_{tot}$$

Assuming a lossless line (real-valued $Z_0$ and $\beta$), the conjugate match condition can also be written in terms of the reflection coefficients corresponding to $Z_L$ and $Z_{th}$:

$$\Gamma_L = \Gamma_{th}^* = \Gamma_G e^{-j\beta d}\]$$

(conjugate match) \hspace{1cm} (13.1.3)

Moving the phase exponential to the left, we note that the conjugate match condition can be written in terms of the same quantities at the input side of the transmission line: