Electromagnetic Waves

and

Antennas

Exercise book

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Chapter1

Maxwell's Equations

1.1 Exercise

Prove the vector algebra identities:

a) $\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{B}\left(\mathbf{A}\cdot\mathbf{C}\right)-\mathbf{C}\left(\mathbf{A}\cdot\mathbf{B}\right)$

It is possible to write the vectors in the form:

$$\begin{cases}\mathbf{A}=A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}}\\ \mathbf{B}=B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}\\ \mathbf{C}=C_x\hat{\mathbf{x}}+C_y\hat{\mathbf{y}}+C_z\hat{\mathbf{z}}\end{cases} \tag{1.1.1}$$

and to use the follow relationship:

$$\begin{aligned}\mathbf{U}\times\mathbf{V}&=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ U_x & U_y & U_z\\ V_x & V_y & V_z\end{vmatrix}=\\ &=\hat{\mathbf{x}}\left(U_yV_z-U_zV_y\right)-\hat{\mathbf{y}}\left(U_xV_z-U_zV_x\right)+\hat{\mathbf{z}}\left(U_xV_y-U_yV_x\right)\end{aligned} \tag{1.1.2}$$

Now we can prove the algebra identities with simply mathematical substitutions:

$$\begin{aligned}&\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\mathbf{A}\times\left(\hat{\mathbf{x}}\left(B_yC_z-B_zC_y\right)-\hat{\mathbf{y}}\left(B_xC_z-B_zC_x\right)+\hat{\mathbf{z}}\left(B_xC_y-B_yC_x\right)\right)=\\ &=\hat{\mathbf{x}}\left(A_y\left(B_xC_y-B_yC_x\right)+A_z\left(B_xC_z-B_zC_x\right)\right)\\ &\quad-\hat{\mathbf{y}}\left(A_x\left(B_xC_y-B_yC_x\right)-A_z\left(B_yC_z-B_zC_y\right)\right)\\ &\quad+\hat{\mathbf{z}}\left(-A_x\left(B_xC_z-B_zC_x\right)-A_y\left(B_yC_z-B_zC_y\right)\right)\end{aligned} \tag{1.1.3}$$

Expanding the terms in (1.1.3), we have:

$$\begin{aligned}&\mathbf{A}\times\left(\mathbf{B}\times\mathbf{C}\right)=\\ &+\hat{\mathbf{x}}\left(A_yB_xC_y-A_yB_yC_x+A_zB_xC_z-A_zB_zC_x\right)\\ &+\hat{\mathbf{y}}\left(A_xB_yC_x-A_xB_xC_y+A_zB_yC_z-A_zB_zC_y\right)\\ &+\hat{\mathbf{z}}\left(A_xB_zC_x-A_xB_xC_z-A_yB_yC_z+A_yB_zC_y\right)\end{aligned} \tag{1.1.4}$$

Let us write eq. (1.1.4) in matrix form, separating the terms with the minus sign and the terms with the plus sign:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\begin{bmatrix}0 & B_yA_xC_x & B_zA_xC_x\\ B_xA_yC_y & 0 & B_zA_yC_y\\ B_xA_zC_z & B_yA_zC_z & 0\end{bmatrix}-\begin{bmatrix}0 & C_yA_xB_x & C_zA_xB_x\\ C_xA_yB_y & 0 & C_zA_yB_y\\ C_xA_zB_z & C_yA_zB_z & 0\end{bmatrix} \quad (1.1.5)$$

Note that the elements of the diagonal of each matrix are zero. Each term can be filled with the product of the three component with the same subscript $(a_{ii} = A_iB_iC_i)$:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\begin{bmatrix}B_xA_xC_x & B_yA_xC_x & B_zA_xC_x\\ B_xA_yC_y & B_yA_yC_y & B_zA_yC_y\\ B_xA_zC_z & B_yA_zC_z & B_zA_zC_z\end{bmatrix}-\begin{bmatrix}C_xA_xB_x & C_yA_xB_x & C_zA_xB_x\\ C_xA_yB_y & C_yA_yB_y & C_zA_yB_y\\ C_xA_zB_z & C_yA_zB_z & C_zA_zB_z\end{bmatrix}=$$

$$\begin{aligned}&=+A_xC_x\left(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}\right)+A_yC_y\left(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}\right)+A_zC_z\left(B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}}\right)-\\ &\quad -A_xB_x\left(C_x\hat{\mathbf{x}}+C_y\hat{\mathbf{y}}+C_z\hat{\mathbf{z}}\right)-A_yB_y\left(C_x\hat{\mathbf{x}}+C_y\hat{\mathbf{y}}+C_z\hat{\mathbf{z}}\right)-A_zB_z\left(C_x\hat{\mathbf{x}}+C_y\hat{\mathbf{y}}+C_z\hat{\mathbf{z}}\right)= \quad (1.1.6)\\ &=\mathbf{B}\left(A_xC_x+A_yC_y+A_zC_z\right)-\mathbf{C}\left(A_xB_x+A_yB_y+A_zB_z\right)=\\ &=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})\end{aligned}$$

b) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

Using relationships (1.1.1) and (1.1.2), we can write:

$$\begin{aligned}&\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{A}\cdot\left(\hat{\mathbf{x}}\left(B_yC_z-B_zC_y\right)-\hat{\mathbf{y}}\left(B_xC_z-B_zC_x\right)+\hat{\mathbf{z}}\left(B_xC_y-B_yC_x\right)\right)=\\ &\left(A_xB_yC_z-A_xB_zC_y\right)-\left(A_yB_xC_z-A_yB_zC_x\right)+\left(A_zB_xC_y-A_zB_yC_x\right)= \quad (1.1.7)\\ &\left(A_xB_yC_z+A_yB_zC_x+A_zB_xC_y\right)-\left(A_xB_zC_y+A_yB_xC_z+A_zB_yC_x\right)\end{aligned}$$

$$\begin{aligned}&\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{B}\cdot\left(\hat{\mathbf{x}}\left(C_yA_z-C_zA_y\right)-\hat{\mathbf{y}}\left(C_xA_z-C_zA_x\right)+\hat{\mathbf{z}}\left(A_xC_y-A_yC_x\right)\right)=\\ &\left(B_xC_yA_z-B_xC_zA_y\right)-\left(B_yC_xA_z-B_yC_zA_x\right)+\left(B_zA_xC_y-B_zA_yC_x\right)=\\ &\left(B_xC_yA_z+B_yC_zA_x+B_zC_xA_y\right)-\left(B_xC_zA_y+B_yC_xA_z+B_zC_yA_x\right)\underset{\text{order them}}{\overset{\uparrow}{=}} \quad (1.1.8)\\ &\left(A_xB_yC_z+A_yB_zC_x+A_zB_xC_y\right)-\left(A_xB_zC_y+A_yB_xC_z+A_zB_yC_x\right)\end{aligned}$$

$$\begin{aligned}&\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{C}\cdot\left(\hat{\mathbf{x}}\left(A_yB_z-A_zB_y\right)-\hat{\mathbf{y}}\left(A_xB_z-A_zB_x\right)+\hat{\mathbf{z}}\left(A_xB_y-A_yB_x\right)\right)=\\ &\left(C_xA_yB_z-C_xA_zB_y\right)-\left(C_yA_xB_z-C_yA_zB_x\right)+\left(C_zA_xB_y-C_zA_yB_x\right)=\\ &\left(C_xA_yB_z+C_yA_zB_x+C_zA_xB_y\right)-\left(C_xA_zB_y+C_yA_xB_z+C_zA_yB_x\right)\underset{\text{order them}}{\overset{\uparrow}{=}} \quad (1.1.9)\\ &\left(A_xB_yC_z+A_yB_zC_x+A_zB_xC_y\right)-\left(A_xB_zC_y+A_yB_xC_z+A_zB_yC_x\right)\end{aligned}$$

If we compare the last row of each expression, we note that they are identical so the algebra identity is verified.

c) $|\mathbf{A}\times\mathbf{B}|^2 + |\mathbf{A}\cdot\mathbf{B}|^2 = |\mathbf{A}|^2\,|\mathbf{B}|^2$

Using relationships (1.1.1) and (1.1.2), we can write:

$$|\mathbf{A}\times\mathbf{B}|^2 + |\mathbf{A}\cdot\mathbf{B}|^2 = \left|\hat{\mathbf{x}}\left(A_yB_z - A_zB_y\right) - \hat{\mathbf{y}}\left(A_xB_z - A_zB_x\right) + \hat{\mathbf{z}}\left(A_xB_y - A_yB_x\right)\right|^2 + \\ +\left(A_xB_x + A_yB_y + A_zB_z\right)^2 =$$

$$\left(\sqrt{\left(A_yB_z - A_zB_y\right)^2 + \left(A_xB_z - A_zB_x\right)^2 + \left(A_xB_y - A_yB_x\right)^2}\right)^2 + \left(A_xB_x + A_yB_y + A_zB_z\right)^2 =$$

$$\left(A_yB_z - A_zB_y\right)^2 + \left(A_xB_z - A_zB_x\right)^2 + \left(A_xB_y - A_yB_x\right)^2 + \left(A_xB_x + A_yB_y + A_zB_z\right)^2 =$$

$$A_y^2B_z^2 + A_z^2B_y^2 - 2A_yB_zA_zB_y + A_x^2B_z^2 + A_z^2B_x^2 - 2A_xB_zA_zB_x + \\ A_x^2B_y^2 + A_y^2B_x^2 - 2A_xB_yA_yB_x + \left(A_xB_x + A_yB_y + A_zB_z\right)^2 =$$

$$A_y^2B_z^2 + A_z^2B_y^2 - 2A_yB_zA_zB_y + A_x^2B_z^2 + A_z^2B_x^2 - 2A_xB_zA_zB_x + \\ A_x^2B_y^2 + A_y^2B_x^2 - 2A_xB_yA_yB_x + A_x^2B_x^2 + A_y^2B_y^2 + A_z^2B_z^2 + \\ 2A_xB_yA_yB_x + 2A_xB_zA_zB_x + 2A_xB_yA_yB_x \underset{\substack{\uparrow \\ \text{cancel the opposites}}}{=}$$

$$A_y^2B_z^2 + A_z^2B_y^2 + A_x^2B_z^2 + A_z^2B_x^2 + A_x^2B_y^2 + A_y^2B_x^2 + A_x^2B_x^2 + A_y^2B_y^2 + A_z^2B_z^2 =$$

$$\left(A_x^2 + A_y^2 + A_z^2\right)\left(B_x^2 + B_y^2 + B_z^2\right) = |\mathbf{A}|^2\,|\mathbf{B}|^2$$

d) $\mathbf{A} = \hat{\mathbf{n}}\times\mathbf{A}\times\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\mathbf{A})\hat{\mathbf{n}}$

Does it make a difference whether $\hat{\mathbf{n}}\times\mathbf{A}\times\hat{\mathbf{n}}$ is taken to mean $(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}$ or $\hat{\mathbf{n}}\times(\mathbf{A}\times\hat{\mathbf{n}})$?

The unit vector $\hat{\mathbf{n}}$can be expressed as follow:

$$\begin{cases} \hat{\mathbf{n}} = n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}} \\ |\hat{\mathbf{n}}| = \sqrt{n_x^2 + n_y^2 + n_z^2} = 1 \end{cases} \qquad (1.1.10)$$

Let us begin considering the first case:

$$(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}=\left[\hat{\mathbf{x}}\left(n_yA_z-n_zA_y\right)-\hat{\mathbf{y}}\left(n_xA_z-n_zA_x\right)+\hat{\mathbf{z}}\left(n_xA_y-n_yA_x\right)\right]\times\hat{\mathbf{n}}=$$

$$\begin{aligned}&+\hat{\mathbf{x}}\left[\left(n_zA_x-n_xA_z\right)n_z-\left(n_xA_y-n_yA_x\right)n_y\right]+\\&-\hat{\mathbf{y}}\left[\left(n_yA_z-n_zA_y\right)n_z-\left(n_xA_y-n_yA_x\right)n_x\right]+\\&+\hat{\mathbf{z}}\left[\left(n_yA_z-n_zA_y\right)n_y-\left(n_zA_x-n_xA_z\right)n_x\right]=\end{aligned}\tag{1.1.11}$$

$$\begin{aligned}&+\hat{\mathbf{x}}\left[n_z^2A_x-n_xn_zA_z-n_xn_yA_y+n_y^2A_x\right]+\\&-\hat{\mathbf{y}}\left[n_yn_zA_z-n_z^2A_y-n_x^2A_y+n_yn_xA_x\right]+\\&+\hat{\mathbf{z}}\left[n_y^2A_z-n_zn_yA_y-n_zn_xA_x+n_x^2A_z\right]\end{aligned}$$

And now consider the second case:

$$\hat{\mathbf{n}}\times(\mathbf{A}\times\hat{\mathbf{n}})=\hat{\mathbf{n}}\times\left[\hat{\mathbf{x}}\left(A_yn_z-A_zn_y\right)-\hat{\mathbf{y}}\left(A_xn_z-A_zn_x\right)+\hat{\mathbf{z}}\left(A_xn_y-A_yn_x\right)\right]=$$

$$\begin{aligned}&+\hat{\mathbf{x}}\left[n_y\left(A_xn_y-A_yn_x\right)-n_z\left(A_zn_x-A_xn_z\right)\right]+\\&-\hat{\mathbf{y}}\left[n_x\left(A_xn_y-A_yn_x\right)-n_z\left(A_yn_z-A_zn_y\right)\right]+\\&+\hat{\mathbf{z}}\left[n_x\left(A_zn_x-A_xn_z\right)-n_y\left(A_yn_z-A_zn_y\right)\right]=\end{aligned}\tag{1.1.12}$$

$$\begin{aligned}&+\hat{\mathbf{x}}\left[A_xn_y^2-A_yn_yn_x-A_zn_zn_x+A_xn_z^2\right]+\\&-\hat{\mathbf{y}}\left[A_xn_xn_y-A_yn_x^2-A_yn_z^2+A_zn_zn_y\right]+\\&+\hat{\mathbf{z}}\left[A_zn_x^2-A_xn_xn_z-A_yn_yn_z+A_zn_y^2\right]\end{aligned}$$

It is very easy to show that $(\hat{\mathbf{n}}\times\mathbf{A})\times\hat{\mathbf{n}}=\hat{\mathbf{n}}\times(\mathbf{A}\times\hat{\mathbf{n}})$.

The second term of the identity can be written as:

$$
\begin{aligned}
&(\hat{\mathbf{n}}\cdot\mathbf{A})\hat{\mathbf{n}} = \left(n_x A_x + n_y A_y + n_z A_z\right)\left(n_x\hat{\mathbf{x}} + n_y\hat{\mathbf{y}} + n_z\hat{\mathbf{z}}\right) = \\
&+\hat{\mathbf{x}}\left[n_x\left(n_x A_x + n_y A_y + n_z A_z\right)\right] + \\
&+\hat{\mathbf{y}}\left[n_y\left(n_x A_x + n_y A_y + n_z A_z\right)\right] + \\
&+\hat{\mathbf{z}}\left[n_z\left(n_x A_x + n_y A_y + n_z A_z\right)\right] = \\
&+\hat{\mathbf{x}}\left[n_x^2 A_x + n_x n_y A_y + n_x n_z A_z\right] + \\
&+\hat{\mathbf{y}}\left[n_y n_x A_x + n_y^2 A_y + n_y n_z A_z\right] + \\
&+\hat{\mathbf{z}}\left[n_z n_x A_x + n_z n_y A_y + n_z^2 A_z\right]
\end{aligned}
\tag{1.1.13}
$$

Adding the two results, we obtain:

$$
\begin{aligned}
&\hat{\mathbf{n}}\times\mathbf{A}\times\hat{\mathbf{n}} + (\hat{\mathbf{n}}\cdot\mathbf{A})\hat{\mathbf{n}} = \\
&+\hat{\mathbf{x}}\left[A_x n_y^2 - A_y n_y n_x - A_z n_z n_x + A_x n_z^2\right] + \\
&-\hat{\mathbf{y}}\left[A_x n_x n_y - A_y n_x^2 - A_y n_z^2 + A_z n_z n_y\right] + \\
&+\hat{\mathbf{z}}\left[A_z n_x^2 - A_x n_x n_z - A_y n_y n_z + A_z n_y^2\right] + \\
&+\hat{\mathbf{x}}\left[n_x^2 A_x + n_x n_y A_y + n_x n_z A_z\right] + \\
&+\hat{\mathbf{y}}\left[n_y n_x A_x + n_y^2 A_y + n_y n_z A_z\right] + \\
&+\hat{\mathbf{z}}\left[n_z n_x A_x + n_z n_y A_y + n_z^2 A_z\right] \underset{\uparrow}{=}
\end{aligned}
$$

change signs in parentheses at first $\hat{y}$ and add

$$
\begin{aligned}
&+\hat{\mathbf{x}}\left[A_x n_y^2 - A_y n_y n_x - A_z n_z n_x + A_x n_z^2 + n_x^2 A_x + n_x n_y A_y + n_x n_z A_z\right] + \\
&+\hat{\mathbf{y}}\left[A_y n_x^2 + A_y n_z^2 - A_x n_x n_y - A_z n_z n_y + n_y n_x A_x + n_y^2 A_y + n_y n_z A_z\right] + \\
&+\hat{\mathbf{z}}\left[A_z n_x^2 - A_x n_x n_z - A_y n_y n_z + A_z n_y^2 + n_z n_x A_x + n_z n_y A_y + n_z^2 A_z\right] = \\
&+\hat{\mathbf{x}}A_x\left[n_y^2 + n_z^2 + n_x^2\right] + +\hat{\mathbf{y}}A_y\left[n_x^2 + n_z^2 + n_y^2\right] + +\hat{\mathbf{z}}A_z\left[n_x^2 + n_y^2 + n_z^2\right] = \mathbf{A}
\end{aligned}
\tag{1.1.14}
$$

1.2 Exercise

Prove the vector analysis identities:

1. $\nabla \times (\nabla \phi) = 0$
2. $\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi$ (Green's first identity)
3. $\nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \nabla^2 \psi - \psi \nabla^2 \phi$ (Green's second identity)
4. $\nabla \cdot (\phi \mathbf{A}) = (\nabla \phi) \cdot \mathbf{A} + \phi \nabla \cdot \mathbf{A}$
5. $\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi \nabla \times \mathbf{A}$
6. $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
7. $\nabla \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
8. $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

First of all we have to express the operator ∇ in general orthogonal coordinates in four common applications. All vector components are presented with respect to the normalized base $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$:

$$\begin{cases} \nabla \phi = \dfrac{\hat{\mathbf{e}}_1}{h_1}\dfrac{\partial \phi}{\partial q_1} + \dfrac{\hat{\mathbf{e}}_2}{h_2}\dfrac{\partial \phi}{\partial q_2} + \dfrac{\hat{\mathbf{e}}_3}{h_3}\dfrac{\partial \phi}{\partial q_3} \\ \nabla^2 \phi = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial q_1}\left(\dfrac{h_2 h_3}{h_1}\dfrac{\partial \phi}{\partial q_1}\right) + \dfrac{\partial}{\partial q_2}\left(\dfrac{h_1 h_3}{h_2}\dfrac{\partial \phi}{\partial q_2}\right) + \dfrac{\partial}{\partial q_3}\left(\dfrac{h_1 h_2}{h_3}\dfrac{\partial \phi}{\partial q_3}\right)\right] \\ \nabla \cdot \mathbf{F} = \dfrac{1}{h_1 h_2 h_3}\left[\dfrac{\partial}{\partial q_1}(F_1 h_2 h_3) + \dfrac{\partial}{\partial q_2}(F_2 h_1 h_3) + \dfrac{\partial}{\partial q_3}(F_3 h_1 h_2)\right] \\ \nabla \times \mathbf{F} = \dfrac{1}{h_1 h_2 h_3}\begin{Vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{Vmatrix} = \\ \quad + \dfrac{\hat{\mathbf{e}}_1}{h_2 h_3}\left[\dfrac{\partial}{\partial q_2}(h_3 F_3) - \dfrac{\partial}{\partial q_3}(h_2 F_2)\right] + \dfrac{\hat{\mathbf{e}}_2}{h_1 h_3}\left[\dfrac{\partial}{\partial q_3}(h_1 F_1) - \dfrac{\partial}{\partial q_1}(h_3 F_3)\right] + \\ \quad + \dfrac{\hat{\mathbf{e}}_3}{h_1 h_2}\left[\dfrac{\partial}{\partial q_1}(h_2 F_2) - \dfrac{\partial}{\partial q_2}(h_1 F_1)\right] \end{cases} \tag{1.2.1}$$

where (h_1, h_2, h_3) are the metric coefficients. For common geometries they are defined as follow:

$$\begin{cases} h_1 = 1, \quad h_2 = 1, \quad h_3 = 1 & \text{(rectangular coordinates)} \\ h_1 = 1, \quad h_2 = r, \quad h_3 = 1 & \text{(cylindrical coordinates)} \\ h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\vartheta & \text{(spherical coordinates)} \end{cases} \tag{1.2.2}$$

For simplicity, the proves are done using rectangular coordinates $(h_1 = 1,\ h_2 = 1,\ h_3 = 1)$:

- Identity n° 1

$$\nabla\times(\nabla\phi) = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ \left(\frac{\partial\phi}{\partial q_1}\right) & \left(\frac{\partial\phi}{\partial q_2}\right) & \left(\frac{\partial\phi}{\partial q_3}\right) \end{vmatrix} =$$

$$= \begin{bmatrix} \hat{\mathbf{e}}_1\left(\frac{\partial}{\partial q_2}\frac{\partial\phi}{\partial q_3} - \frac{\partial}{\partial q_3}\frac{\partial\phi}{\partial q_2}\right) - \hat{\mathbf{e}}_2\left(\frac{\partial}{\partial q_1}\frac{\partial\phi}{\partial q_3} - \frac{\partial}{\partial q_3}\frac{\partial\phi}{\partial q_1}\right) + \\ +\hat{\mathbf{e}}_3\left(\frac{\partial}{\partial q_1}\frac{\partial\phi}{\partial q_2} - \frac{\partial}{\partial q_2}\frac{\partial\phi}{\partial q_1}\right) \end{bmatrix} = 0$$

For the property of linearity of the derivate operator $\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}\phi = \frac{\partial}{\partial q_j}\frac{\partial}{\partial q_i}\phi$, so each term in the parentheses vanishes and also the result.

- Identity n° 2

$$\nabla\cdot(\phi\nabla\psi) = \nabla\cdot\left(\hat{\mathbf{e}}_1\phi\frac{\partial\psi}{\partial q_1} + \hat{\mathbf{e}}_2\phi\frac{\partial\psi}{\partial q_2} + \hat{\mathbf{e}}_3\phi\frac{\partial\psi}{\partial q_3}\right) =$$

$$= \frac{\partial}{\partial q_1}\left(\phi\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\phi\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\phi\frac{\partial\psi}{\partial q_3}\right) =$$

$$= \left(\frac{\partial\phi}{\partial q_1}\frac{\partial\psi}{\partial q_1} + \phi\frac{\partial^2\psi}{\partial q_1}\right) + \left(\frac{\partial\phi}{\partial q_2}\frac{\partial\psi}{\partial q_2} + \phi\frac{\partial^2\psi}{\partial q_2}\right) + \left(\frac{\partial\phi}{\partial q_3}\frac{\partial\psi}{\partial q_3} + \phi\frac{\partial^2\psi}{\partial q_3}\right) =$$

$$= \phi\left(\frac{\partial^2\psi}{\partial q_1} + \frac{\partial^2\psi}{\partial q^2} + \frac{\partial^2\psi}{\partial q^3}\right) + \left(\frac{\partial\phi}{\partial q_1}\frac{\partial\psi}{\partial q_1} + \frac{\partial\phi}{\partial q_2}\frac{\partial\psi}{\partial q_2} + \frac{\partial\phi}{\partial q_3}\frac{\partial\psi}{\partial q_3}\right) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

- Identity n° 3

First of all we expand the sum inside parentheses:

$$\begin{cases} \phi\nabla\psi = \phi\hat{\mathbf{e}}_1\frac{\partial\psi}{\partial q_1} + \phi\hat{\mathbf{e}}_2\frac{\partial\psi}{\partial q_2} + \phi\hat{\mathbf{e}}_3\frac{\partial\psi}{\partial q_3} \\ \psi\nabla\phi = \psi\hat{\mathbf{e}}_1\frac{\partial\phi}{\partial q_1} + \psi\hat{\mathbf{e}}_2\frac{\partial\phi}{\partial q_2} + \psi\hat{\mathbf{e}}_3\frac{\partial\phi}{\partial q_3} \end{cases}$$

so

$$\left(\phi\nabla\psi-\psi\nabla\phi\right)=\hat{\mathbf{e}}_1\left(\phi\frac{\partial\psi}{\partial q_1}-\psi\frac{\partial\phi}{\partial q_1}\right)+\hat{\mathbf{e}}_2\left(\phi\frac{\partial\psi}{\partial q_2}-\psi\frac{\partial\phi}{\partial q_2}\right)+\hat{\mathbf{e}}_3\left(\phi\frac{\partial\psi}{\partial q_3}-\psi\frac{\partial\phi}{\partial q_3}\right)$$

Now we can apply the dot product:

$$\nabla\cdot\left(\phi\nabla\psi-\psi\nabla\phi\right)=\left[\frac{\partial}{\partial q_1}\left(\phi\frac{\partial\psi}{\partial q_1}-\psi\frac{\partial\phi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\phi\frac{\partial\psi}{\partial q_2}-\psi\frac{\partial\phi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\phi\frac{\partial\psi}{\partial q_3}-\psi\frac{\partial\phi}{\partial q_3}\right)\right]=$$

$$=+\left(\frac{\partial\phi}{\partial q_1}\frac{\partial\psi}{\partial q_1}+\phi\frac{\partial^2\psi}{\partial q_1}-\frac{\partial\psi}{\partial q_1}\frac{\partial\phi}{\partial q_1}-\psi\frac{\partial^2\phi}{\partial q_1}\right)+\left(\frac{\partial\phi}{\partial q_2}\frac{\partial\psi}{\partial q_2}+\phi\frac{\partial^2\psi}{\partial q_2}-\frac{\partial\psi}{\partial q_2}\frac{\partial\phi}{\partial q_2}-\psi\frac{\partial^2\phi}{\partial q_2}\right)+$$

$$+\left(\frac{\partial\phi}{\partial q_3}\frac{\partial\psi}{\partial q_3}+\phi\frac{\partial^2\psi}{\partial q_3}-\frac{\partial\psi}{\partial q_3}\frac{\partial\phi}{\partial q_3}-\psi\frac{\partial^2\phi}{\partial q_3}\right)\underset{\substack{\uparrow\\ \text{cancel opposite terms in parentheses}}}{=}$$

$$=\phi\left(\frac{\partial^2\psi}{\partial q_1}+\frac{\partial^2\psi}{\partial q_2}+\frac{\partial^2\psi}{\partial q_3}\right)-\psi\left(\frac{\partial^2\phi}{\partial q_1}+\frac{\partial^2\phi}{\partial q_2}+\frac{\partial^2\phi}{\partial q_3}\right)=\phi\nabla^2\psi-\psi\nabla^2\phi$$

- Identity n°4

$$\nabla\cdot\left(\phi\mathbf{A}\right)=\nabla\cdot\left(\phi A_1\hat{\mathbf{e}}_1+\phi A_2\hat{\mathbf{e}}_2+\phi A_3\hat{\mathbf{e}}_3\right)=\left[\frac{\partial}{\partial q_1}\left(\phi A_1\right)+\frac{\partial}{\partial q_2}\left(\phi A_2\right)+\frac{\partial}{\partial q_3}\left(\phi A_3\right)\right]=$$

$$=\left[\left(\phi\frac{\partial A_1}{\partial q_1}+A_1\frac{\partial\phi}{\partial q_1}\right)+\left(\phi\frac{\partial A_2}{\partial q_2}+A_2\frac{\partial\phi}{\partial q_2}\right)+\left(\phi\frac{\partial A_3}{\partial q_3}+A_3\frac{\partial\phi}{\partial q_3}\right)\right]=$$

$$=\left(A_1\frac{\partial\phi}{\partial q_1}+A_2\frac{\partial\phi}{\partial q_2}+A_3\frac{\partial\phi}{\partial q_3}\right)+\phi\left(\frac{\partial A_1}{\partial q_1}+\frac{\partial A_2}{\partial q_2}+\frac{\partial A_3}{\partial q_3}\right)=\left(\nabla\phi\right)\cdot\mathbf{A}+\phi\nabla\cdot\mathbf{A}$$

- Identity n° 5

$$\nabla\times\left(\phi\mathbf{A}\right)=\begin{vmatrix}\hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3\\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3}\\ \phi A_1 & \phi A_2 & \phi A_3\end{vmatrix}=$$

$$+\hat{\mathbf{e}}_1\left[\frac{\partial}{\partial q_2}(\phi A_3)-\frac{\partial}{\partial q_3}(\phi A_2)\right]+\hat{\mathbf{e}}_2\left[\frac{\partial}{\partial q_3}(\phi A_1)-\frac{\partial}{\partial q_1}(\phi A_3)\right]+\hat{\mathbf{e}}_3\left[\frac{\partial}{\partial q_1}(\phi A_2)-\frac{\partial}{\partial q_2}(\phi A_1)\right]=$$

$$=\hat{\mathbf{e}}_1\left[\left(\frac{\partial\phi}{\partial q_2}A_3+\frac{\partial A_3}{\partial q_2}\phi\right)-\left(\frac{\partial\phi}{\partial q_3}A_2+\frac{\partial A_2}{\partial q^3}\phi\right)\right]+\hat{\mathbf{e}}_2\left[\left(\frac{\partial\phi}{\partial q_3}A_1+\frac{\partial A_1}{\partial q_3}\phi\right)-\left(\frac{\partial\phi}{\partial q_1}A_3+\frac{\partial A_3}{\partial q_1}\phi\right)\right]+$$
$$+\hat{\mathbf{e}}_3\left[\left(\frac{\partial\phi}{\partial q_1}A_2+\frac{\partial A_2}{\partial q_1}\phi\right)-\left(\frac{\partial\phi}{\partial q_2}A_1+\frac{\partial A_1}{\partial q^2}\phi\right)\right]=$$

$$=\hat{\mathbf{e}}_1\left[\frac{\partial\phi}{\partial q_2}A_3-\frac{\partial\phi}{\partial q_3}A_2\right]+\hat{\mathbf{e}}_2\left[\frac{\partial\phi}{\partial q_3}A_1-\frac{\partial\phi}{\partial q_1}A_3\right]+\hat{\mathbf{e}}_3\left[\frac{\partial\phi}{\partial q_1}A_2-\frac{\partial\phi}{\partial q_2}A_1\right]+$$
$$+\hat{\mathbf{e}}_1\left[\frac{\partial A_3}{\partial q_2}\phi-\frac{\partial A_2}{\partial q_3}\phi\right]+\hat{\mathbf{e}}_2\left[\frac{\partial A_1}{\partial q_3}\phi-\frac{\partial A_3}{\partial q_1}\phi\right]+\hat{\mathbf{e}}_3\left[\frac{\partial A_2}{\partial q_1}\phi-\frac{\partial A_1}{\partial q_2}\phi\right]=(\nabla\phi)\times\mathbf{A}+\phi\nabla\times\mathbf{A}$$

- Identity n° 6

$$\nabla\cdot(\nabla\times\mathbf{A})=\nabla\cdot\left[\hat{\mathbf{e}}_1\left(\frac{\partial A_3}{\partial q_2}-\frac{\partial A_2}{\partial q_3}\right)-\hat{\mathbf{e}}_2\left(\frac{\partial A_3}{\partial q_1}-\frac{\partial A_1}{\partial q_3}\right)+\hat{\mathbf{e}}_3\left(\frac{\partial A_2}{\partial q_1}-\frac{\partial A_1}{\partial q_2}\right)\right]=$$
$$=\left[\frac{\partial}{\partial q_1}\left(\frac{\partial A_3}{\partial q_2}-\frac{\partial A_2}{\partial q_3}\right)-\frac{\partial}{\partial q_2}\left(\frac{\partial A_3}{\partial q_1}-\frac{\partial A_1}{\partial q_3}\right)+\frac{\partial}{\partial q_3}\left(\frac{\partial A_2}{\partial q_1}-\frac{\partial A_1}{\partial q_2}\right)\right]=$$
$$=\left[\frac{\partial}{\partial q_1}\frac{\partial A_3}{\partial q_2}-\frac{\partial}{\partial q_1}\frac{\partial A_2}{\partial q_3}-\frac{\partial}{\partial q_2}\frac{\partial A_3}{\partial q_1}-\frac{\partial}{\partial q^2}\frac{\partial A_1}{\partial q_3}+\frac{\partial}{\partial q_3}\frac{\partial A_2}{\partial q_1}-\frac{\partial}{\partial q_3}\frac{\partial A_1}{\partial q_2}\right]=0$$

For the linearity of the derivate operator $\frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j}\phi=\frac{\partial}{\partial q_j}\frac{\partial}{\partial q_i}\phi$, so the term in brackets is null.

- Identity n°7

To evaluate the expression $\nabla\cdot\mathbf{A}\times\mathbf{B}$, we have to calculate first the cross product and then the divergence of vector $\mathbf{A}\times\mathbf{B}$. This choise is obligated by the fact that if first we calculated the divergence of the vector $\mathbf{A}$, the results would be a scalar. Cross product with the vector $\mathbf{B}$ would be impossible. So we have:

$$\nabla\cdot\left(\mathbf{A}\times\mathbf{B}\right)=\nabla\cdot\left[\hat{\mathbf{e}}_{\mathbf{1}}\left(A_2B_3-A_3B_2\right)-\hat{\mathbf{e}}_{\mathbf{2}}\left(A_1B_3-A_3B_1\right)+\hat{\mathbf{e}}_{\mathbf{3}}\left(A_1B_2-A_2B_1\right)\right]=$$

$$=\frac{\partial}{\partial q_1}\left(A_2B_3-A_3B_2\right)-\frac{\partial}{\partial q_2}\left(A_1B_3-A_3B_1\right)+\frac{\partial}{\partial q_3}\left(A_1B_2-A_2B_1\right)=$$

$$=\frac{\partial}{\partial q_1}\left(A_2B_3\right)-\frac{\partial}{\partial q_1}\left(A_3B_2\right)-\frac{\partial}{\partial q_2}\left(A_1B_3\right)+\frac{\partial}{\partial q_2}\left(A_3B_1\right)+\frac{\partial}{\partial q_3}\left(A_1B_2\right)-\frac{\partial}{\partial q_3}\left(A_2B_1\right)=$$

$$=\left(B_3\frac{\partial A_2}{\partial q_1}+A_2\frac{\partial B_3}{\partial q_1}\right)-\left(B_2\frac{\partial A_3}{\partial q_1}+A_3\frac{\partial B_2}{\partial q_1}\right)-\left(B_3\frac{\partial A_1}{\partial q_2}+A_1\frac{\partial B_3}{\partial q_2}\right)+$$

$$+\left(B_1\frac{\partial A_3}{\partial q_2}+A_3\frac{\partial B_1}{\partial q_2}\right)+\left(B_2\frac{\partial A_1}{\partial q_3}+A_1\frac{\partial B_2}{\partial q_3}\right)-\left(A_2\frac{\partial B_1}{\partial q_3}+B_1\frac{\partial A_2}{\partial q_3}\right)=$$

$$=B_1\left(\frac{\partial A_3}{\partial q_2}-\frac{\partial A_2}{\partial q_3}\right)+B_2\left(\frac{\partial A_1}{\partial q_3}-\frac{\partial A_3}{\partial q_1}\right)+B_3\left(\frac{\partial A_2}{\partial q_1}-\frac{\partial A_1}{\partial q_2}\right)+$$

$$+A_1\left(\frac{\partial B_2}{\partial q_3}-\frac{\partial B_3}{\partial q_2}\right)+A_2\left(\frac{\partial B_3}{\partial q_1}-\frac{\partial B_1}{\partial q_3}\right)+A_3\left(\frac{\partial B_1}{\partial q_2}-\frac{\partial B_2}{\partial q_1}\right)=$$

$$=\mathbf{B}\cdot\left(\nabla\times\mathbf{A}\right)-\mathbf{A}\cdot\left(\nabla\times\mathbf{B}\right)$$

- Identity n° 8

$$\nabla\times\left(\nabla\times\mathbf{A}\right)=\nabla\times\left(\hat{\mathbf{e}}_{\mathbf{1}}\left(\frac{\partial A_3}{\partial q_2}-\frac{\partial A_2}{\partial q_3}\right)-\hat{\mathbf{e}}_2\left(\frac{\partial A_3}{\partial q_1}-\frac{\partial A_1}{\partial q_3}\right)+\hat{\mathbf{e}}_3\left(\frac{\partial A_2}{\partial q_1}-\frac{\partial A_1}{\partial q_2}\right)\right)=$$

$$=\hat{\mathbf{e}}_{\mathbf{1}}\left(\frac{\partial}{\partial q_2}\left(\frac{\partial A_2}{\partial q_1}-\frac{\partial A_1}{\partial q_2}\right)-\frac{\partial}{\partial q_3}\left(\frac{\partial A_1}{\partial q_3}-\frac{\partial A_3}{\partial q_1}\right)\right)+$$

$$-\hat{\mathbf{e}}_2\left(\frac{\partial}{\partial q_1}\left(\frac{\partial A_2}{\partial q_1}-\frac{\partial A_1}{\partial q_2}\right)-\frac{\partial}{\partial q_3}\left(\frac{\partial A_3}{\partial q_2}-\frac{\partial A_2}{\partial q_3}\right)\right)+$$

$$+\hat{\mathbf{e}}_3\left(\frac{\partial}{\partial q_1}\left(\frac{\partial A_1}{\partial q_3}-\frac{\partial A_3}{\partial q_1}\right)-\frac{\partial}{\partial q_2}\left(\frac{\partial A_3}{\partial q_2}-\frac{\partial A_2}{\partial q_3}\right)\right)=$$

$$=\hat{\mathbf{e}}_{\mathbf{1}}\left(\frac{\partial}{\partial q_2}\frac{\partial A_2}{\partial q_1}-\frac{\partial^2 A_1}{\partial q_2}-\frac{\partial^2 A_1}{\partial q_3}+\frac{\partial}{\partial q_3}\frac{\partial A_3}{\partial q_1}\right)+$$

$$-\hat{\mathbf{e}}_2\left(\frac{\partial^2 A_2}{\partial q_1}-\frac{\partial}{\partial q_1}\frac{\partial A_1}{\partial q_2}-\frac{\partial}{\partial q_3}\frac{\partial A_3}{\partial q_2}+\frac{\partial^2 A_2}{\partial q_3}\right)+$$

$$+\hat{\mathbf{e}}_3\left(\frac{\partial}{\partial q_1}\frac{\partial A_1}{\partial q_3}-\frac{\partial^2 A_3}{\partial q_1}-\frac{\partial^2 A_3}{\partial q_2}+\frac{\partial}{\partial q_2}\frac{\partial A_2}{\partial q_3}\right)=$$

$$=\hat{\mathbf{e}}_1\left(\frac{\partial}{\partial q_2}\frac{\partial A_2}{\partial q_1}+\frac{\partial}{\partial q_3}\frac{\partial A_3}{\partial q_1}\right)-\hat{\mathbf{e}}_2\left(-\frac{\partial}{\partial q_1}\frac{\partial A_1}{\partial q_2}-\frac{\partial}{\partial q_3}\frac{\partial A_3}{\partial q_2}\right)+\hat{\mathbf{e}}_3\left(\frac{\partial}{\partial q_1}\frac{\partial A_1}{\partial q_3}+\frac{\partial}{\partial q_2}\frac{\partial A_2}{\partial q_3}\right)+$$

$$+\hat{\mathbf{e}}_1\left(-\frac{\partial^2 A_1}{\partial q_2}-\frac{\partial^2 A_1}{\partial q_3}\right)-\hat{\mathbf{e}}_2\left(\frac{\partial^2 A_2}{\partial q_1}+\frac{\partial^2 A_2}{\partial q_3}\right)+\hat{\mathbf{e}}_3\left(-\frac{\partial^2 A_3}{\partial q_1}-\frac{\partial^2 A_3}{\partial q_2}\right)=$$

$$=\hat{\mathbf{e}}_1\left[\frac{\partial}{\partial q_1}\left(\frac{\partial A_2}{\partial q_2}+\frac{\partial A_3}{\partial q_3}\right)\right]+\hat{\mathbf{e}}_2\left[\frac{\partial}{\partial q_2}\left(\frac{\partial A_1}{\partial q_1}+\frac{\partial A_3}{\partial q_3}\right)\right]+\hat{\mathbf{e}}_3\left[\frac{\partial}{\partial q_3}\left(\frac{\partial A_1}{\partial q_1}+\frac{\partial A_2}{\partial q_2}\right)\right]-$$

$$-\hat{\mathbf{e}}_1\left(\frac{\partial^2 A_1}{\partial q_2}+\frac{\partial^2 A_1}{\partial q_3}\right)-\hat{\mathbf{e}}_2\left(\frac{\partial^2 A_2}{\partial q_1}+\frac{\partial^2 A_2}{\partial q_3}\right)-\hat{\mathbf{e}}_3\left(+\frac{\partial^2 A_3}{\partial q_1}+\frac{\partial^2 A_3}{\partial q_2}\right)=\nabla\left(\nabla\cdot\mathbf{A}\right)-\nabla^2\mathbf{A}$$

1.3 Exercise

Consider the infinitesimal volume element $\Delta x \Delta y \Delta z$ shown below, such that its upper half lies in medium ε_1 and its lower half in medium ε_2. The axes are oriented such that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$.

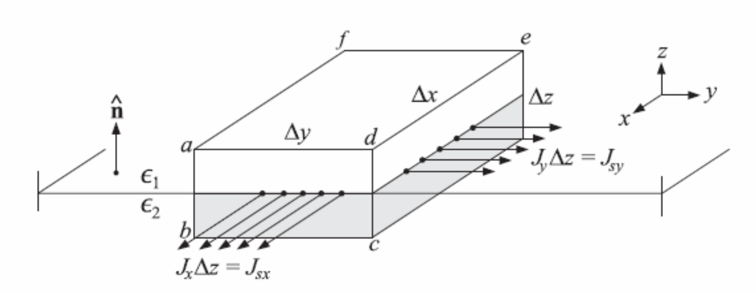

Fig. 1.3.1: Infinitesimal volume element between two medium.

1. Applying the integrated form of Ampère's law to the infinitesimal face *abcd*, show that

$$H_{2y} - H_{1y} = J_x \Delta z + \frac{\partial D_x}{\partial t}\Delta z \qquad (1.3.1)$$

2. In the limit $\Delta z \to 0$, the second term in the right–hand side may be assumed to go to zero, whereas the first term will be non–zero and may be set equal to the surface current density, that is, $J_{sx} \equiv \lim_{\Delta z \to 0}(J_x \Delta z)$. Show that this leads to the boundary condition $H_{1y} - H_{2y} = -J_{sx}$. Similarly, shows that $H_{1x} - H_{2x} = J_{sy}$, and that these two boundary conditions can be combined vectorially into:

$$\hat{\mathbf{n}} \times (\mathbf{H_1} - \mathbf{H_2}) = \mathbf{J_s} \qquad (1.3.2)$$

3. Apply the integrated form of Gauss's law to the same volume element and show the boundary condition $D_{1z} - D_{2z} = \rho_s = \lim_{\Delta z \to 0}(\rho \Delta z)$.

Solution

- Question n° 1

In its historically original form, Ampère's circuital law relates the magnetic field to its electric current source. The law can be written in two forms, the integral form and the differential form. The forms are equivalent, and related by the Kelvin–Stokes theorem. The identity demonstrated by Stokes is the follow:

$$\iint_S (\nabla \times \mathbf{F})\, dS = \oint_{c(S)} \mathbf{F} \cdot d\underline{\ell} \qquad (1.3.3)$$

So applying (1.3.3) to the second Maxwell's equation, we obtain the Ampere's law in integral form with few simply steps:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Integrate terms of the identity over an opened surface S:

$$\iint_S (\nabla \times \mathbf{H}) \cdot \hat{\mathbf{n}} dS = \iint_S \mathbf{J} \cdot \hat{\mathbf{n}} dS + \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot \hat{\mathbf{n}} dS$$

and apply the Stokes theorem:

$$\oint_{c(S)} \mathbf{H} \cdot d\underline{\ell} = \iint_S \mathbf{J} \cdot \hat{\mathbf{n}} dS + \iint_S \frac{\partial \mathbf{D}}{\partial t} \cdot \hat{\mathbf{n}} dS \tag{1.3.4}$$

where $\underline{\ell}$ is the infinitesimal vector, tangent to the curved line c that bounds the surface S.

Now we can consider the infinitesimal face *abcd,* that has area $S = \Delta z \Delta y$ and perimeter $p = 2\Delta z + 2\Delta y$. The left–hand side of (1.3.4) can be decomposed into a sum of four integral expression, one for each infinitesimal side of the rectangular *abcd*, and we have to define the sense of integration. Choose an counterclockwise path so that, using the right–hand rule, the normal is $\hat{\mathbf{x}}$. Note that the z–parallel sides have the first half in the medium 1 and the second in medium 2. So the integral on that part of the path needs to be decomposed into two integral with different arguments. For simplicity, denote the points of contact between mediums along the segments $\overline{ab}$ and $\overline{cd}$ with O_1 and O_2, respectively.

On the contrary, to solve the right–side of (1.3.4) we have to identify the correct component of $\mathbf{J}$ and $\mathbf{D}$ that flows through the face *abcd*, i.e. the component J_x and D_x.

So we obtain:

$$-\int_a^{O_1} \mathbf{H}_1 \cdot d\hat{\mathbf{z}} - \int_{O_1}^{b} \mathbf{H}_2 \cdot d\hat{\mathbf{z}} + \int_b^c \mathbf{H}_2 \cdot d\hat{\mathbf{y}} + \int_c^{O_2} \mathbf{H}_2 \cdot d\hat{\mathbf{z}} + \int_{O_2}^{d} \mathbf{H}_1 \cdot d\hat{\mathbf{z}} - \int_d^a \mathbf{H}_1 \cdot d\hat{\mathbf{y}} =$$
$$= J_x \Delta z \Delta y + \frac{\partial D_x}{\partial t} \Delta z \Delta y$$

H_1 and H_2 are constant inside each medium, so the line integrals can be written as:

$$-\cancel{H_{1z}\frac{\Delta z}{2}} - \cancel{H_{2z}\frac{\Delta z}{2}} + H_{2y}\Delta y + \cancel{H_{2z}\frac{\Delta z}{2}} + \cancel{H_{1z}\frac{\Delta z}{2}} - H_{1y}\Delta y = J_x \Delta z \Delta y + \frac{\partial D_x}{\partial t} \Delta z \Delta y$$

i.e.

$$H_{2y} \cancel{\Delta y} - H_{1y} \cancel{\Delta y} = J_x \Delta z \cancel{\Delta y} + \frac{\partial D_x}{\partial t} \Delta z \cancel{\Delta y}$$

$$H_{2y} - H_{1y} = J_x \Delta z + \frac{\partial D_x}{\partial t} \Delta z$$

- Question n° 2

In the limit $\Delta z \to 0$, the eq. (1.3.1) is reduced in $H_{1y} - H_{2y} = -J_{sx}$ and similarly $H_{1x} - H_{2x} = J_{sy}$. In order to obtain eq. (1.3.2), we can subtract vectorially these two boundary conditions:

$$\hat{\mathbf{y}}\left(H_{1x} - H_{2x}\right) - \hat{\mathbf{x}}\left(H_{1y} - H_{2y}\right) = J_{sx}\hat{\mathbf{x}} + J_{sy}\hat{\mathbf{y}}$$

$$\hat{\mathbf{n}} \times \left(\mathbf{H}_1 - \mathbf{H}_2\right) = \mathbf{J}_s$$

where $\hat{\mathbf{n}} = \hat{\mathbf{z}}$.

- Question n° 3

Gauss's law relates the electric field to its electric charge sources. Like Ampère's circuital law, it can be written in two forms, the integral form and the differential form. The forms are equivalent, and related by the divergence theorem:

$$\iiint_V (\nabla \cdot \mathbf{F})\, dV = \oiint_{S(V)} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \tag{1.3.5}$$

So applying (1.3.5) to the third Maxwell's equation, we obtain the Gauss's law in integral form with few simply steps:

$$\nabla \cdot \mathbf{D} = \rho$$

Integrate terms of the identity over a volume V:

$$\iiint_V (\nabla \cdot \mathbf{D})\, dV = \iiint_V \rho\, dV$$

and apply the divergence theorem:

$$\oiint_{S(V)} \mathbf{D} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \rho\, dV = Q_{in} \tag{1.3.6}$$

where $\hat{\mathbf{n}}$ is the outgoing unit vector normal to the closed surface S that bounds the volume V.

Now consider the volume $V = \Delta x \Delta y \Delta z$. The left–hand side of (1.3.6) can be decomposed into two integrals with arguments $\mathbf{D_1}$ and $\mathbf{D_2}$, respectively in the medium 1 and medium 2. The right–hand side of (1.3.6) is a simple volume integral. So we have:

$$\iint_{S_1} (\mathbf{D}_1 \cdot \hat{\mathbf{n}})\, dS_1 + \iint_{S_2} (\mathbf{D}_2 \cdot \hat{\mathbf{n}})\, dS_2 = \iiint_V \rho\, dV = \rho \Delta x \Delta y \Delta z \tag{1.3.7}$$

where S_1 and S_2 are portions of S in the medium 1 and medium 2, respectively and ρ is considered constant inside the volume V.

The terms on the right–hand side of eq. (1.3.7) can be decomposed into several surface integrals, one for each side of parallelepiped $\Delta x \Delta y \Delta z$:

$$D_{1z}\Delta x\Delta y + \cancel{D_{1y}\frac{\Delta z}{2}}\Delta x + \cancel{D_{1x}\frac{\Delta z}{2}}\Delta y - \cancel{D_{1y}\frac{\Delta z}{2}}\Delta x - \cancel{D_{1x}\frac{\Delta z}{2}}\Delta y -$$

$$-D_{2z}\Delta x\Delta y + \cancel{D_{2y}\frac{\Delta z}{2}}\Delta x + \cancel{D_{2x}\frac{\Delta z}{2}}\Delta y - \cancel{D_{2y}\frac{\Delta z}{2}}\Delta x - \cancel{D_{2x}\frac{\Delta z}{2}}\Delta y = \rho\Delta x\Delta y\Delta z$$

i.e.

$$D_{1z} - D_{2z} = \rho\Delta z$$

In the limit $\Delta z \to 0$, the amount $\rho\Delta z$ collapses in ρ_s which is the surface electric charge density.

1.4 Exercise

Show that time average of the product of two harmonic quantities $A(\mathrm{t}) = \mathrm{Re}\left[\mathrm{A}e^{j\omega t}\right]$ and $B(\mathrm{t}) = \mathrm{Re}\left[\mathrm{B}e^{j\omega t}\right]$ with phasors A, B is given by:

$$\overline{A(\mathrm{t})B(\mathrm{t})} = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} A(\mathrm{t})B(\mathrm{t})\mathrm{dt} = \frac{1}{2}\mathrm{Re}\left[\mathrm{AB}^*\right] \tag{1.4.1}$$

where $\mathrm{T} = 2\pi/\omega$ is one period. Then show that the time–averaged values of the cross and dot products of two time–harmonic vector quantities $\boldsymbol{A}(\mathrm{t}) = \mathrm{Re}\left[\mathbf{A}e^{j\omega t}\right]$ and $\boldsymbol{B}(\mathrm{t}) = \mathrm{Re}\left[\mathbf{B}e^{j\omega t}\right]$ can be expressed in terms of the corresponding phasors as follows:

$$\overline{\boldsymbol{A}(\mathrm{t})\times\boldsymbol{B}(\mathrm{t})} = \frac{1}{2}\mathrm{Re}\left[\mathbf{A}\times\mathbf{B}^*\right] \tag{1.4.2}$$

$$\overline{\boldsymbol{A}(\mathrm{t})\cdot\boldsymbol{B}(\mathrm{t})} = \frac{1}{2}\mathrm{Re}\left[\mathbf{A}\cdot\mathbf{B}^*\right] \tag{1.4.3}$$

Solution

First of all, we express the harmonic quantities $A(\mathrm{t})$ and $B(\mathrm{t})$ in their extended form:

$$\begin{cases} A(\mathrm{t}) = \mathrm{A}\cos\left(\omega \mathrm{t} + \varphi_1\right) \\ B(\mathrm{t}) = \mathrm{B}\cos\left(\omega \mathrm{t} + \varphi_2\right) \end{cases} \tag{1.4.4}$$

and substitute eq. (1.4.4) into eq. (1.4.1):

$$\overline{A(\mathrm{t})B(\mathrm{t})} = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} A(\mathrm{t})B(\mathrm{t})\mathrm{dt} = \frac{1}{\mathrm{T}}\int_0^{\mathrm{T}} \mathrm{AB}\cos\left(\omega\mathrm{t}+\varphi_1\right)\cos\left(\omega\mathrm{t}+\varphi_2\right)\mathrm{dt} \tag{1.4.5}$$

Now we have to use Euler's formula:

$$\begin{cases} e^{jx} = \cos x + j\sin x \\ e^{-jx} = \cos x - j\sin x \end{cases} \Leftrightarrow \begin{cases} \cos x = \dfrac{e^{jx} + e^{-jx}}{2} \\ \sin x = \dfrac{e^{jx} - e^{-jx}}{2j} \end{cases} \tag{1.4.6}$$

Substitute eq. (1.4.6) into eq. (1.4.5) and we obtain:

$$\overline{A(t)B(t)} = \frac{1}{T}\int_0^T AB\left(\frac{e^{j\omega t}e^{\varphi_1} + e^{-j\omega t}e^{-\varphi_1}}{2}\right)\left(\frac{e^{j\omega t}e^{\varphi_2} + e^{-j\omega t}e^{-\varphi_2}}{2}\right)dt =$$

$$= \frac{AB}{2T}\int_0^T \frac{\left(e^{j\omega t}e^{\varphi_1} + e^{-j\omega t}e^{-\varphi_1}\right)\left(e^{j\omega t}e^{\varphi_2} + e^{-j\omega t}e^{-\varphi_2}\right)}{2}dt =$$

$$= \frac{AB}{2T}\int_0^T \frac{\cancel{e^{2j\omega t}e^{(\varphi_1+\varphi_2)}} + e^{(\varphi_1-\varphi_2)} + e^{-(\varphi_1-\varphi_2)} + \cancel{e^{-2j\omega t}e^{-(\varphi_1+\varphi_2)}}}{2}dt =$$

$$= \frac{AB}{2T}\int_0^T \cos(\varphi_1-\varphi_2)\,dt = \frac{AB\cos(\varphi_1-\varphi_2)}{2\cancel{T}}\cancel{T} = \frac{1}{2}AB\cos(\varphi_1-\varphi_2) = \frac{1}{2}\mathrm{Re}\left[AB^*\right]$$

Operating in similar way, we can demonstrate the time–averaged values of the cross and dot products of two time–harmonic vector quantities.

- Cross Product

$$\overline{\boldsymbol{A}(t)\times\boldsymbol{B}(t)} = \frac{1}{T}\int_0^T \left(\boldsymbol{A}(t)\times\boldsymbol{B}(t)\right)dt = \frac{1}{T}\int_0^T \mathrm{Re}[Ae^{j\omega t}]\mathbf{a}\times\mathrm{Re}[Be^{j\omega t}]\mathbf{b}\,dt = \frac{\mathbf{a}\times\mathbf{b}}{T}\int_0^T \mathrm{Re}[Ae^{j\omega t}]\mathrm{Re}[Be^{j\omega t}]dt$$

The result of integral is note by previous exercise, so:

$$\overline{\boldsymbol{A}(t)\times\boldsymbol{B}(t)} = \frac{\mathbf{a}\times\mathbf{b}}{2}\mathrm{Re}[AB^*] = \frac{1}{2}\mathrm{Re}[\mathbf{a}A\times\mathbf{b}B^*] = \frac{1}{2}\mathrm{Re}[\mathbf{A}\times\mathbf{B}^*]$$

- Dot Product

$$\overline{\boldsymbol{A}(t)\cdot\boldsymbol{B}(t)} = \frac{1}{T}\int_0^T \left(\boldsymbol{A}(t)\cdot\boldsymbol{B}(t)\right)dt = \frac{1}{T}\int_0^T \mathrm{Re}[Ae^{j\omega t}]\mathbf{a}\cdot\mathrm{Re}[Be^{j\omega t}]\mathbf{b}\,dt = \frac{\mathbf{a}\cdot\mathbf{b}}{T}\int_0^T \mathrm{Re}[Ae^{j\omega t}]\mathrm{Re}[Be^{j\omega t}]dt =$$

$$= \frac{\mathbf{a}\cdot\mathbf{b}}{2}\mathrm{Re}[AB^*] = \frac{1}{2}\mathrm{Re}[\mathbf{a}A\cdot\mathbf{b}B^*] = \frac{1}{2}\mathrm{Re}[\mathbf{A}\cdot\mathbf{B}^*]$$

1.5 Exercise

Assuming that $\mathbf{B} = \mu\mathbf{H}$:

1. Show that Maxwell's equations

$$\boxed{\begin{aligned}&\nabla \times \mathbf{E} = -j\omega\mathbf{B}\\ &\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D}\\ &\nabla \cdot \mathbf{D} = \rho\\ &\nabla \cdot \mathbf{B} = 0\end{aligned}}$$

   imply the following complex–value version of Poynting's theorem:

$$\nabla \cdot \left(\mathbf{E} \times \mathbf{H}^*\right) = -j\omega\mu\mathbf{H} \cdot \mathbf{H}^* - \mathbf{E} \cdot \mathbf{J}_{\text{tot}}^* \qquad (1.5.1)$$

   where $\mathbf{J}_{\text{tot}} = \mathbf{J} + j\omega\mathbf{D}$.

2. Extracting the real–parts of both sides of eq. (1.5.1) and integrating over a volume V bounded by closed surface, show the time–averaged form of energy conservation:

$$-\oiint_{S(V)} \frac{1}{2}\text{Re}[\mathbf{E} \times \mathbf{H}^*] \cdot \hat{\mathbf{n}}dS = \iiint_V \frac{1}{2}\text{Re}\left[\mathbf{E} \cdot \mathbf{J}_{\text{tot}}^*\right]dV \qquad (1.5.2)$$

   which states that the net time–averaged power floating into a volume is dissipated into heat.

3. For a lossless dielectric, show that the integrals in (1.5.2) are zero and provide an interpretation.

Solution

- Question n° 1

Using the identity $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$ and Maxwell's equations, we have:

$$\nabla \cdot \left(\mathbf{E} \times \mathbf{H}^*\right) = \mathbf{H}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot \left(\nabla \times \mathbf{H}^*\right) = \mathbf{H}^* \cdot (-j\omega\mathbf{B}) - \mathbf{E} \cdot \left(\mathbf{J}^* - j\omega\mathbf{D}^*\right) =$$
$$= -j\omega\mu\mathbf{H} \cdot \mathbf{H}^* - \mathbf{E} \cdot \mathbf{J}_{\text{tot}}^*$$

- Question n° 2

Integrate over a volume V the right–hand side of eq. (1.5.1) and apply the divergence's theorem:

$$\iiint_V \left(\nabla \cdot \left(\mathbf{E} \times \mathbf{H}^*\right)\right)dV = \oiint_{S(V)} \left(\mathbf{E} \times \mathbf{H}^*\right)\hat{\mathbf{n}}dS$$

and now calculate the time–averaged value:

$$\frac{1}{T}\int_0^T\left[\oiint_{S(V)}\left(\mathbf{E}\times\mathbf{H}^*\right)\hat{\mathbf{n}}dS\right]dt$$

Invert the order of integrals:

$$\oiint_{S(V)}\left[\frac{1}{T}\int_0^T\left(\mathbf{E}\times\mathbf{H}^*\right)dt\right]\hat{\mathbf{n}}dS=\oiint_{S(V)}\frac{1}{2}\text{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\hat{\mathbf{n}}dS \qquad (1.5.3)$$

In similar way on left–hand side, we obtain:

$$\frac{1}{T}\int_0^T\left[\iiint_V\left(-j\omega\mu\mathbf{H}\cdot\mathbf{H}^*-\mathbf{E}\cdot\mathbf{J}_{tot}^*\right)dV\right]dt=$$
$$\iiint_V\left[\frac{1}{T}\int_0^T\left(-j\omega\mu\mathbf{H}\cdot\mathbf{H}^*-\mathbf{E}\cdot\mathbf{J}_{tot}^*\right)dt\right]dV= \qquad (1.5.4)$$
$$\iiint_V\left[\frac{1}{2}\text{Re}\left[-j\omega\mu\mathbf{H}\cdot\mathbf{H}^*\right]-\frac{1}{2}\text{Re}\left[\mathbf{E}\cdot\mathbf{J}_{tot}^*\right]\right]dV$$

The real part of $j\omega\mu\mathbf{H}\cdot\mathbf{H}^*$ is zero because the product $\mathbf{H}\cdot\mathbf{H}^*=|\mathbf{H}|^2$ is real and so the quantity $j\omega\mu\mathbf{H}\cdot\mathbf{H}^*$ is imaginary. Only the term associated with the heat survives and we can write:

$$-\oiint_{S(V)}\frac{1}{2}\text{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]\hat{\mathbf{n}}dS=\iiint_V\frac{1}{2}\text{Re}\left[\mathbf{E}\cdot\mathbf{J}_{tot}^*\right]dV \qquad (1.5.5)$$

The minus sign is been associated with the left–hand side because it identifies the quantity of energy that goes in the volume V –while the Poynting's vector is defined outgoing from V– and the right–hand side represents the energy dissipated as heat.

- Question n° 3

Inside a lossless dielectric, the current density $\mathbf{J}$ is zero while the displacement current $\mathbf{D}$ is simply equal to $\varepsilon\mathbf{E}$. So:

$$\iiint_V\frac{1}{2}\text{Re}\left[\mathbf{E}\cdot(-j\omega\varepsilon\mathbf{E}^*)\right]dV=0 \qquad (1.5.6)$$

being the real part of $j\omega\varepsilon\mathbf{E}\cdot\mathbf{E}^*$ zero. Moreover zero for the right–hand side of (1.5.5), that represents the quantity of energy ingoing the volume bounded by the surface S, implies that not all the energy remains inside the volume. Exactly in steady state the quantity of energy ingoing is equal to the outgoing one. It is correct because electromagnetic wave pass through the dielectric.

1.6 Exercise

Assuming that $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$,

1. Show that Maxwell's equations imply the following relationships:

$$\rho E_x + \left(\mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}\right)_x = \nabla\cdot\left(\varepsilon E_x\mathbf{E} - \hat{\mathbf{x}}\frac{1}{2}\varepsilon E^2\right) \tag{1.6.1}$$

$$\left(\mathbf{J}\times\mathbf{B}\right)_x + \left(\frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}\right)_x = \nabla\cdot\left(\mu H_x\mathbf{H} - \hat{\mathbf{x}}\frac{1}{2}\mu H^2\right) \tag{1.6.2}$$

where the subscript x means the x–component.

2. From eq. (1.6.1) and (1.6.2), derive the following relationship that represent momentum conservation:

$$f_x + \frac{\partial G_x}{\partial t} = \nabla\cdot T_x \tag{1.6.3}$$

where f_x, G_x are the x–components of the vectors $\mathbf{f} = \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}$ and $\mathbf{G} = \mathbf{D}\times\mathbf{B}$, and T_x is defined to be the vector(equal to Maxwell's stress tensor acting on the unit vector $\hat{\mathbf{x}}$):

$$\mathbf{T}_x = \varepsilon E_x\mathbf{E} + \mu H_x\mathbf{H} - \hat{\mathbf{x}}\frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right)$$

3. Write similar equations of y, z components. The quantity G_x is interpreted as the field momentum (in the x–direction) per unit of volume, that is, the momentum density.

Solution

- Question n° 1

Let us begin with eq. (1.6.2) because it is easy to note from the left–hand side that it is the cross product of the second Maxwell's equation (i.e. $\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$) with the vector $\mathbf{B}$ and then we extract the x–component. So we have to demonstrate the right–hand side of eq. (1.6.2). We can write:

$$\left(\nabla\times\mathbf{H}\right)\times\mathbf{B} = \left[\hat{\mathbf{x}}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\right]\times\mathbf{B} =$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) & -\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) & \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \\ B_x & B_y & B_z \end{vmatrix} \tag{1.6.4}$$

Now we consider only the x–component, writing $B_i = \mu H_i$:

$$\begin{aligned}(\nabla\times\mathbf{H})\times\mathbf{B}\|_{x-\text{component}} &= \mu\left[-H_z\left(\frac{\partial H_z}{\partial x}-\frac{\partial H_x}{\partial z}\right)-H_y\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\right]= \\ &= \mu\left[-H_z\frac{\partial H_z}{\partial x}+H_z\frac{\partial H_x}{\partial z}-H_y\frac{\partial H_y}{\partial x}+H_y\frac{\partial H_x}{\partial y}\right]\end{aligned} \quad (1.6.5)$$

From the forth Maxwell's equation (i.e. $\nabla\cdot\mathbf{B}=0$) and the constitutive relation $\mathbf{B}=\mu\mathbf{H}$, we can add to eq. (1.6.5) the term $H_x(\nabla\cdot\mathbf{H})$ and the couple of terms $\pm H_x\frac{\partial H_x}{\partial x}$, because they're both zero:

$$\begin{aligned}&(\nabla\times\mathbf{H})\times\mathbf{B}\|_{x-\text{component}} = \\ &= \mu\left[\begin{array}{l}\underline{\underline{-H_z\frac{\partial H_z}{\partial x}}}+H_z\frac{\partial H_x}{\partial z}\underline{\underline{-H_y\frac{\partial H_y}{\partial x}}}+H_y\frac{\partial H_x}{\partial y}+H_x\frac{\partial H_x}{\partial x}+ \\ +H_x\frac{\partial H_y}{\partial y}+H_x\frac{\partial H_z}{\partial z}+H_x\frac{\partial H_x}{\partial x}\underline{\underline{-H_x\frac{\partial H_x}{\partial x}}}\end{array}\right]=\end{aligned} \quad (1.6.6)$$

Let us consider the only emphasized terms of eq. (1.6.6):

$$-\mu\left[H_z\frac{\partial H_z}{\partial x}+H_y\frac{\partial H_y}{\partial x}+H_x\frac{\partial H_x}{\partial x}\right]=-\frac{1}{2}\mu\left[2H_z\frac{\partial H_z}{\partial x}+2H_y\frac{\partial H_y}{\partial x}+2H_x\frac{\partial H_x}{\partial x}\right]=$$

$$-\frac{1}{2}\mu\left[\frac{\partial H_z^2}{\partial x}+\frac{\partial H_y^2}{\partial x}+\frac{\partial H_x^2}{\partial x}\right]=-\hat{\mathbf{x}}\frac{1}{2}\mu\nabla H^2$$

and now consider the remaining terms:

$$\mu\left[H_z\frac{\partial H_x}{\partial z}+H_y\frac{\partial H_x}{\partial y}+H_x\frac{\partial H_x}{\partial x}+H_x\frac{\partial H_y}{\partial y}+H_x\frac{\partial H_z}{\partial z}+H_x\frac{\partial H_x}{\partial x}\right]\underset{\text{order them}}{\overset{\uparrow}{=}}$$

$$=\mu\left[2H_x\frac{\partial H_x}{\partial x}+\left(H_x\frac{\partial H_y}{\partial y}+H_y\frac{\partial H_x}{\partial y}\right)+\left(H_z\frac{\partial H_x}{\partial z}+H_x\frac{\partial H_z}{\partial z}\right)\right]=$$

$$=\mu\left[\frac{\partial H_x^2}{\partial x}+\frac{\partial(H_yH_x)}{\partial y}+\frac{\partial(H_xH_z)}{\partial z}\right]=\mu\nabla\cdot\left(H_x\left(H_x\hat{\mathbf{x}}+H_y\hat{\mathbf{y}}+H_z\hat{\mathbf{z}}\right)\right)=\mu\nabla\cdot(H_x\mathbf{H})$$

So we have that eq. (1.6.6) can be written as:

$$\begin{aligned}(\nabla\times\mathbf{H})\times\mathbf{B}\|_{x-\text{component}} &= \mu\nabla\cdot(H_x\mathbf{H})-\hat{\mathbf{x}}\frac{1}{2}\mu\nabla H^2 = \\ &= \nabla\cdot\left(\mu H_x\mathbf{H}-\hat{\mathbf{x}}\frac{1}{2}\mu H^2\right)\end{aligned} \quad (1.6.7)$$

that is the right–hand side of eq. (1.6.2).

Eq. (1.6.1) is obtained in similar way. In the left–hand side, there is the term $\left(\mathbf{D}\times\frac{\partial \mathbf{B}}{\partial t}\right)_x$ that suggests us the cross product of the first Maxwell's equation (i.e. $\nabla\times\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$) with the vector $\mathbf{D}$ and then we extract the x–component. So we have to demonstrate the right–hand side of eq. (1.6.1). We can apply the cross product to the first Maxwell's equation:

$$\mathbf{D}\times(\nabla\times\mathbf{E}) = -\mathbf{D}\times\frac{\partial \mathbf{B}}{\partial t}$$

From the properties of the cross product, it's possible to invert the order of the terms in the left–hand side and change the sign in the right–hand–side:

$$(\nabla\times\mathbf{E})\times\mathbf{D} = \mathbf{D}\times\frac{\partial \mathbf{B}}{\partial t}$$

Now consider the term $(\nabla\times\mathbf{E})\times\mathbf{D}$:

$$\begin{aligned}(\nabla\times\mathbf{E})\times\mathbf{D} &= \left[\hat{\mathbf{x}}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)-\hat{\mathbf{y}}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)+\hat{\mathbf{z}}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\right]\times\mathbf{D} = \\ &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right) & -\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right) & \left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right) \\ D_x & D_y & D_z\end{vmatrix}\end{aligned} \tag{1.6.8}$$

Now we consider only the x–component, writing $D_i = \varepsilon E_i$:

$$\begin{aligned}(\nabla\times\mathbf{E})\times\mathbf{D}\Big\|_{x-\text{component}} &= \varepsilon\left[-E_z\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)-E_y\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)\right] = \\ &= \varepsilon\left[-E_z\frac{\partial E_z}{\partial x}+E_z\frac{\partial E_x}{\partial z}-E_y\frac{\partial E_y}{\partial x}+E_y\frac{\partial E_x}{\partial y}\right]\end{aligned} \tag{1.6.9}$$

As for eq. (1.6.5), we can add to eq. (1.6.9) the third Maxwell's equation (i.e. $\nabla\cdot\mathbf{D}-\rho=0$), but in this case there is the term ρ and it's correct for the results that we want to obtain. In fact, multiplying it with E_x, the term $-\rho E_x$ completes the left–hand side of eq. (1.6.1), changing its sign.

With these considerations, we can add to eq. (1.6.9) the term $E_x\nabla\cdot\mathbf{D}$ and the couple of terms $\pm E_x\frac{\partial E_x}{\partial x}$:

$$(\nabla\times\mathbf{E})\times\mathbf{D}\big\|_{x-\text{component}} =$$

$$=\varepsilon\begin{bmatrix} \underline{\underline{-E_z\frac{\partial E_z}{\partial x}}}+E_z\frac{\partial E_x}{\partial z}\underline{\underline{-E_y\frac{\partial E_y}{\partial x}}}+E_y\frac{\partial E_x}{\partial y}+E_x\frac{\partial E_x}{\partial x}+\\ +E_x\frac{\partial E_y}{\partial y}+E_x\frac{\partial E_z}{\partial z}+E_x\frac{\partial E_x}{\partial x}\underline{\underline{-E_x\frac{\partial E_x}{\partial x}}} \end{bmatrix} \tag{1.6.10}$$

Let us consider the only emphasized terms of eq.(1.6.10):

$$\varepsilon\left[-E_z\frac{\partial E_z}{\partial x}-E_y\frac{\partial E_y}{\partial x}-E_x\frac{\partial E_x}{\partial x}\right]=-\frac{1}{2}\varepsilon\left[2E_z\frac{\partial E_z}{\partial x}+2E_y\frac{\partial E_y}{\partial x}+2E_x\frac{\partial E_x}{\partial x}\right]=$$

$$=-\frac{1}{2}\varepsilon\left[\frac{\partial E_z^2}{\partial x}+\frac{\partial E_y^2}{\partial x}+\frac{\partial E_x^2}{\partial x}\right]=-\hat{\mathbf{x}}\frac{1}{2}\varepsilon\nabla E^2$$

and now consider the remaining terms:

$$\varepsilon\left[E_z\frac{\partial E_x}{\partial z}+E_y\frac{\partial E_x}{\partial y}+E_x\frac{\partial E_x}{\partial x}+E_x\frac{\partial E_y}{\partial y}+E_x\frac{\partial E_z}{\partial z}+E_x\frac{\partial E_x}{\partial x}\right]\underset{\substack{\uparrow\\ \text{order them}}}{=}$$

$$=\varepsilon\left[2E_x\frac{\partial E_x}{\partial x}+\left(E_y\frac{\partial E_x}{\partial y}+E_x\frac{\partial E_y}{\partial y}\right)+\left(E_z\frac{\partial E_x}{\partial z}+E_x\frac{\partial E_z}{\partial z}\right)\right]=$$

$$=\varepsilon\left[\frac{\partial E_x^2}{\partial x}+\frac{\partial(E_xE_y)}{\partial y}+\frac{\partial(E_xE_z)}{\partial z}\right]=\varepsilon\nabla\cdot\left(E_x\left(E_x\hat{\mathbf{x}}+E_y\hat{\mathbf{y}}+E_z\hat{\mathbf{z}}\right)\right)=\nabla\cdot(\varepsilon E_x\mathbf{E})$$

So we have that eq. (1.6.10) can be written as:

$$\begin{aligned}(\nabla\times\mathbf{E})\times\mathbf{D}\big\|_{x-\text{component}} &=\nabla\cdot(\varepsilon E_x\mathbf{E})-\hat{\mathbf{x}}\frac{1}{2}\varepsilon\nabla E^2=\\ &=\nabla\cdot\left(\varepsilon E_x\mathbf{E}-\hat{\mathbf{x}}\frac{1}{2}\varepsilon E^2\right)\end{aligned} \tag{1.6.11}$$

that is the right–hand side of eq.(1.6.1).

- Question n° 2

The identity (1.6.3) is obtained adding eq.(1.6.1) and (1.6.2) as follow:

$$\rho E_x+(\mathbf{J}\times\mathbf{B})_x+\left(\mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}\right)_x+\left(\frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}\right)_x=\nabla\cdot\left(\varepsilon E_x\mathbf{E}+\mu H_x\mathbf{H}-\hat{\mathbf{x}}\frac{1}{2}\varepsilon E^2-\hat{\mathbf{x}}\frac{1}{2}\mu H^2\right)$$

$$\rho E_x+(\mathbf{J}\times\mathbf{B})_x+\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})\Big|_x=\nabla\cdot\left(\varepsilon E_x\mathbf{E}+\mu H_x\mathbf{H}-\hat{\mathbf{x}}\frac{1}{2}\varepsilon E^2-\hat{\mathbf{x}}\frac{1}{2}\mu H^2\right)$$

It is easy to note the presence of f_x, $\frac{\partial G_x}{\partial t}$ and T_x as defined in the text of the exercise.

- Question n° 3

Operating in the similar way to question n°1, it is possible to demonstrate that the relationships (1.6.1) and (1.6.2) can be written for the y and z–component as follow:

$$\begin{cases} \left(\mathbf{D}\times\dfrac{\partial\mathbf{B}}{\partial t}\right)_y - \rho E_y = \nabla\cdot\left(\hat{\mathbf{y}}\dfrac{1}{2}\varepsilon E^2 - \varepsilon E_y\mathbf{E}\right) \\ \left(\mathbf{J}\times\mathbf{B}\right)_y + \left(\dfrac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}\right)_y = \nabla\cdot\left(\hat{\mathbf{y}}\dfrac{1}{2}\mu H^2 - \mu H_y\mathbf{H}\right) \end{cases} \tag{1.6.12}$$

$$\begin{cases} \left(\mathbf{D}\times\dfrac{\partial\mathbf{B}}{\partial t}\right)_z + \rho E_z = \nabla\cdot\left(\varepsilon E_z\mathbf{E} - \hat{\mathbf{z}}\dfrac{1}{2}\varepsilon E^2\right) \\ \left(\mathbf{J}\times\mathbf{B}\right)_z + \left(\dfrac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}\right)_z = \nabla\cdot\left(\mu H_z\mathbf{H} - \hat{\mathbf{z}}\dfrac{1}{2}\mu H^2\right) \end{cases} \tag{1.6.13}$$

From eq. (1.6.12) and (1.6.13) as in question n°2, we can derive the relationship that represents momentum conservation for y and z–component:

$$f_y + \frac{\partial G_y}{\partial t} = \nabla\cdot T_y \tag{1.6.14}$$

$$f_z + \frac{\partial G_z}{\partial t} = \nabla\cdot T_z \tag{1.6.15}$$

where

$$\begin{cases} f_y = \left(\mathbf{J}\times\mathbf{B}\right)_y - \rho E_y \\ G_y = \left(\dfrac{\partial}{\partial t}\left(\mathbf{D}\times\mathbf{B}\right)\right)_y \\ T_y = \hat{\mathbf{y}}\dfrac{1}{2}\left(\varepsilon E^2 + \mu H^2\right) - \varepsilon E_y\mathbf{E} - \mu H_y\mathbf{H} \end{cases} \qquad \begin{cases} f_z = \left(\mathbf{J}\times\mathbf{B}\right)_z + \rho E_z \\ G_z = \left(\dfrac{\partial}{\partial t}\left(\mathbf{D}\times\mathbf{B}\right)\right)_z \\ T_z = \varepsilon E_z\mathbf{E} + \mu H_z\mathbf{H} - \hat{\mathbf{z}}\dfrac{1}{2}\left(\varepsilon E^2 - \mu H^2\right) \end{cases}$$

1.7 Exercise

Consider the permittivity of a dispersive material

$$\varepsilon(\omega) = \varepsilon_0 + \frac{\varepsilon_0 \omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma} \tag{1.7.1}$$

where ω_p is the so–called plasma frequency of the material defined by:

$$\omega_p = \frac{Ne^2}{\varepsilon_0 m} \tag{1.7.2}$$

and γ measures the rate of collisions per unit of time.

Show that the casual and stable time–domain dielectric response of eq. (1.7.1) is given as follows:

$$\begin{cases} \varepsilon(t) = \varepsilon_0 \delta(t) + \varepsilon_0 \chi(t) \\ \chi(t) = \dfrac{\omega_p^2}{\bar{\omega}_0} e^{-\gamma t/2} \sin(\bar{\omega}_0 t) u(t) \end{cases}$$

where $u(t)$ is the unit–step function and $\bar{\omega}_0 = \sqrt{\omega_0^2 - \gamma^2/4}$, and we must assume that $\gamma < 2\omega_0$, as typically the case in practice. Discuss the solution for the case $\gamma/2 > \omega_0$.

Solution

For the linearity of Fourier transform, we have

$$\begin{aligned} \Im^{-1}\{\varepsilon(\omega)\} &= \Im^{-1}\left\{\varepsilon_0 + \frac{\varepsilon_0 \omega_p^2}{\omega_0^2 - \omega^2 + j\omega\gamma}\right\} = \\ &= \Im^{-1}\{\varepsilon_0\} + \varepsilon_0 \omega_p^2 \Im^{-1}\left\{\frac{1}{\omega_0^2 - \omega^2 + j\omega\gamma}\right\} \end{aligned} \tag{1.7.3}$$

where $\Im^{-1}$ denotes the inverse Fourier transform operator.

The first term of eq. (1.7.3) is constant, so it's easy to calculate:

$$\Im^{-1}\{\varepsilon_0\} = \varepsilon_0 \delta(t) \tag{1.7.4}$$

because the Fourier transform of Dirac delta function is:

$$\Im\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1$$

The second term of eq. (1.7.3) is more complicate and it is necessary to simplify the argument. First of all, we can reduce the denominator in the product of two polynomials of first degree. So we have to find the solutions of the equation $\omega_0^2 - \omega^2 + j\omega\gamma = 0$ in ω and we obtain:

$$\omega_{1,2} = \frac{-j\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{-2} \tag{1.7.5}$$

Assuming that $\sqrt{-\gamma^2 + 4\omega_0^2} = 2\bar{\omega}_0$ and that $2\omega_0 > \gamma$, we can rewrite eq. (1.7.5) as follow.

$$\omega_{1,2} = j\frac{\gamma}{2} \mp \bar{\omega}_0 \tag{1.7.6}$$

where it's important to note that ± has been substituted by ∓ because of the minus sign of the denominator. Now we can write:

$$\Im^{-1}\left\{\frac{1}{\omega_0^2 - \omega^2 + j\omega\gamma}\right\} = \Im^{-1}\left\{\frac{1}{(\omega-\omega_1)(\omega-\omega_2)}\right\} = \Im^{-1}\left\{\frac{A}{(\omega-\omega_1)} + \frac{B}{(\omega-\omega_2)}\right\}$$

where A and B are two constant that we calculate applying the method of weighted residuals:

$$A = \lim_{\omega\to\omega_1}\frac{1}{(\omega-\omega_2)} = \frac{1}{(\omega_1-\omega_2)} = \frac{1}{\left(j\frac{\gamma}{2} - \bar{\omega}_0 - j\frac{\gamma}{2} - \bar{\omega}_0\right)} = -\frac{1}{2\bar{\omega}_0}$$

$$B = \lim_{\omega\to\omega_2}\frac{1}{(\omega-\omega_1)} = \frac{1}{(\omega_2-\omega_1)} = \frac{1}{\left(j\frac{\gamma}{2} + \bar{\omega}_0 - j\frac{\gamma}{2} + \bar{\omega}_0\right)} = \frac{1}{2\bar{\omega}_0}$$

so:

$$\Im^{-1}\left\{\frac{1}{\omega_0^2 - \omega^2 + j\omega\gamma}\right\} = \Im^{-1}\left\{\frac{-1/2\bar{\omega}_0}{(\omega-\omega_1)} + \frac{1/2\bar{\omega}_0}{(\omega-\omega_2)}\right\} \tag{1.7.7}$$

Now the problem is only to transform the trivial expression $1/(\omega - \omega_i)$ and then to apply the result to eq. (1.7.7). To solve this problem, consider:

$$\Im\left\{je^{-j\omega_i t}u(t)\right\},$$

where $u(t)$ is the unit step function, and we obtain:

$$\begin{aligned}\Im\left\{je^{-j\omega_i t}u(t)\right\} &= j\int_{-\infty}^{+\infty} e^{-j\omega_i t}u(t)e^{j\omega t}dt = j\int_{0}^{+\infty} e^{j(\omega-\omega_i)t}dt = \\ &= \cancel{j}\frac{1}{\cancel{j}(\omega-\omega_i)}\left[e^{j(\omega-\omega_i)t}\right]_0^{+\infty} = \\ &= \frac{1}{(\omega-\omega_i)}\left[e^{j(\omega-\omega_i)t}\Big|_{t\to+\infty} - e^{-j(\omega-\omega_i)t}\Big|_{t\to 0}\right] = \\ &= \frac{1}{(\omega-\omega_i)}[0-1] = -\frac{1}{(\omega-\omega_i)}\end{aligned} \tag{1.7.8}$$

So it's possible to assume that:

$$\Im^{-1}\left\{\frac{1}{\omega-\omega_i}\right\} = -je^{j\omega_i t}u(t) \tag{1.7.9}$$

Using eq. (1.7.9) in eq. (1.7.7), we have:

$$\begin{aligned}\Im^{-1}\left\{\frac{-\frac{1}{2\bar{\omega}_0}}{(\omega-\omega_1)}+\frac{\frac{1}{2\bar{\omega}_0}}{(\omega-\omega_2)}\right\} &= \frac{1}{2\bar{\omega}_0}je^{j\omega_1 t}u(t)-\frac{1}{2\bar{\omega}_0}je^{j\omega_2 t}u(t) = \\ &= \frac{1}{2\bar{\omega}_0}j\left[e^{j\omega_1 t}-e^{j\omega_2 t}\right]u(t)\end{aligned} \tag{1.7.10}$$

Substituting the solutions (1.7.6) in eq. (1.7.10), we have:

$$\begin{aligned}\frac{1}{2\bar{\omega}_0}j\left[e^{j\omega_1 t}-e^{j\omega_2 t}\right]u(t) &= \frac{1}{2\bar{\omega}_0}j\left[e^{j\left(j\frac{\gamma}{2}-\bar{\omega}_0\right)t}-e^{j\left(j\frac{\gamma}{2}+\bar{\omega}_0\right)t}\right]u(t) = \\ &= \frac{1}{2\bar{\omega}_0}j\left[e^{-\gamma\frac{t}{2}}e^{-j\bar{\omega}_0 t}-e^{-\gamma\frac{t}{2}}e^{+j\bar{\omega}_0 t}\right]u(t) = \\ &= \frac{1}{2\bar{\omega}_0}je^{-\gamma\frac{t}{2}}\left[e^{-j\bar{\omega}_0 t}-e^{+j\bar{\omega}_0 t}\right]u(t) = \\ &= \frac{1}{\cancel{2}\bar{\omega}_0}\cancel{j}e^{-\gamma\frac{t}{2}}\left[-\cancel{2j}\mathrm{Sin}(\bar{\omega}_0 t)\right]u(t) = \\ &= \frac{1}{\bar{\omega}_0}e^{-\gamma\frac{t}{2}}\mathrm{Sin}(\bar{\omega}_0 t)u(t)\end{aligned} \tag{1.7.11}$$

Using the result in (1.7.4) and (1.7.11), we have:

$$\begin{aligned}\Im^{-1}\{\varepsilon(\omega)\} &= \Im^{-1}\{\varepsilon_0\}+\varepsilon_0\omega_p^2\Im^{-1}\left\{\frac{1}{\omega_0^2-\omega^2+j\omega\gamma}\right\} = \\ &= \varepsilon_0\delta(t)+\varepsilon_0\frac{\omega_p^2}{\bar{\omega}_0}e^{-\gamma\frac{t}{2}}\mathrm{Sin}(\bar{\omega}_0 t)u(t) = \varepsilon_0\delta(t)+\varepsilon_0\chi(t)\end{aligned} \tag{1.7.12}$$

1.8 Exercise

Show that the plasma frequency for electrons can be expressed in the simple numerical form:

$$f_p = 9\sqrt{N} \tag{1.8.1}$$

where f_p is in Hz and N is the electron density in electrons/m^3. What is f_p for the ionosphere if $N = 10^{12}$?

Solution

Plasma frequency id defined as

$$f_p = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{\varepsilon_0 m}} \tag{1.8.2}$$

where e is the electron charge, ε_0 is the permittivity of vacuum and m is the mass of electron. So we have to demonstrate the follow identity:

$$\frac{1}{2\pi}\sqrt{\frac{e^2}{\varepsilon_0 m}} = 9 \tag{1.8.3}$$

The charge of an electron is $1,602 \cdot 10^{-19}\,\mathrm{C}$ and its mass is about $9,10 \cdot 10^{-31}\,\mathrm{Kg}$. The electric permittivity is about $8,85 \cdot 10^{-12}\,\mathrm{F/m}$. So:

$$\frac{1}{2 \cdot 3,14}\sqrt{\frac{\left(1,602 \cdot 10^{-19}\right)^2}{9,10 \cdot 10^{-31} \cdot 8,85 \cdot 10^{-12}}} \approx 8,988925....$$

With $N = 10^{12}$ the plasma frequency of the ionosphere is:

$$f_p^{iono} = 9\sqrt{10^{12}} = 9 \cdot 10^6 = 9\mathrm{MHz}$$

1.9 Exercise

Show that the relaxation equation

$$\ddot{\rho}(\mathbf{r},t)+\gamma\dot{\rho}(\mathbf{r},t)+\omega_p^2\rho(\mathbf{r},t)=0 \qquad (1.9.1)$$

where ρ is the charge density in the conductor, γ is the measurement of collisions per unit of time and ω_p is the plasma frequency, can be written in the following form in term of dc–conductivity $\sigma=\varepsilon_0\omega_p^2/\gamma=\mathrm{Ne}^2/\mathrm{m}\gamma$:

$$\frac{1}{\gamma}\ddot{\rho}(\mathbf{r},t)+\dot{\rho}(\mathbf{r},t)+\frac{\sigma}{\varepsilon_0}\rho(\mathbf{r},t)=0 \qquad (1.9.2)$$

Then show that it reduces to the naive relaxation equation

$$\frac{\partial\rho}{\partial t}+\frac{\sigma}{\varepsilon}\rho=0 \qquad (1.9.3)$$

in the limit $\tau=1/\gamma\to 0$. Show also that in this limit, Ohm's law

$$\mathbf{J}(\mathbf{r},t)=\omega_p^2\int_{-\infty}^{t}e^{-\gamma(t-t')}\varepsilon_0\mathbf{E}(\mathbf{r},t)dt' \qquad (1.9.4)$$

takes the instantaneous form $\mathbf{J}=\sigma\mathbf{E}$, from which the naive relaxation constant $\tau_{relax}=\varepsilon_0/\sigma$ was derived.

Solution

Eq. (1.9.2) is obtained dividing eq. (1.9.1) by γ:

$$\frac{1}{\gamma}\ddot{\rho}(\mathbf{r},t)+\dot{\rho}(\mathbf{r},t)+\frac{\omega_p^2}{\gamma}\rho(\mathbf{r},t)=0$$

where $\omega_p^2/\gamma=\sigma/\varepsilon_0$. It's easy to note that if $\tau=1/\gamma\to 0$, then the term $\ddot{\rho}(\mathbf{r},t)/\gamma\to 0$ and the eq. (1.9.2) is reduced to eq. (1.9.3).

The Ohm's law (1.9.4) can be written, highlighting we have to solve only the integral of an exponential:

$$\mathbf{J}(\mathbf{r},t)=\omega_p^2\varepsilon_0\mathbf{E}(\mathbf{r},t)\int_{-\infty}^{t}e^{-\gamma(t-t')}dt'$$

So

$$\int_{-\infty}^{t} e^{-\gamma(t-t')}dt' = \frac{1}{\gamma}\int_{-\infty}^{t} e^{-\gamma(t-t')}d[-\gamma(t-t')] = \frac{1}{\gamma}\left[e^{-\gamma(t-t')}\right]_{-\infty}^{t} = \frac{1}{\gamma}[1-0] = \frac{1}{\gamma}$$

and we can write:

$$\mathbf{J}(\mathbf{r},t) = \frac{\omega_p^2\varepsilon_0}{\gamma}\mathbf{E}(\mathbf{r},t) = \sigma\mathbf{E}(\mathbf{r},t) \qquad (1.9.5)$$

1.10 Exercise

Conductors and plasmas exhibit anisotropic and birefringent behavior when they are in the presence of an external magnetic field. The equation of motion of conduction electrons in a constant external magnetic field is

$$m\dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - m\gamma\mathbf{v} \tag{1.10.1}$$

with the collisional term included. Assume the magnetic field is in the z–direction, $\mathbf{B} = \hat{\mathbf{z}}B$, and that $\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y$ and $\mathbf{v} = \hat{\mathbf{x}}v_x + \hat{\mathbf{y}}v_y$.

1. Show that in component form, the above equations of motion read:

$$\begin{aligned}\dot{v}_x &= \frac{e}{m}E_x + \omega_B v_y - \gamma v_x \\ \dot{v}_y &= \frac{e}{m}E_y - \omega_B v_x - \gamma v_y\end{aligned} \tag{1.10.2}$$

   where $\omega_B = eB/m$ is the cyclotron frequency.

   What is the cyclotron frequency in Hz for electrons in the Earth' magnetic field $B = 0.4\text{ gauss} = 0.4\times 10^{-4}\text{ Tesla}$?

2. To solve this system, work with the combinations $v_x \pm jv_y$. Assuming harmonic time–dependence, show that the solution is:

$$v_x \pm jv_y = \frac{\frac{e}{m}\left(E_x \pm jE_y\right)}{\gamma + j\left(\omega \pm \omega_B\right)} \tag{1.10.3}$$

3. Define the induced currents as $\mathbf{J} = Ne\mathbf{v}$. Show that:

$$J_x \pm jJ_y = \sigma_\pm(\omega)\left(E_x \pm jE_y\right) \tag{1.10.4}$$

   where $\sigma_\pm(\omega) = \dfrac{\gamma\sigma_0}{\gamma + j\left(\omega \pm \omega_B\right)}$ with $\sigma_0 = Ne^2/m\gamma$, that is the dc value of the conductivity.

4. Show that the time–domain version of eq. (1.10.3) is:

$$J_x(t) \pm jJ_y(t) = \int_0^t \sigma_\pm(t-t')\left(E_x(t') \pm jE_y(t')\right)dt' \tag{1.10.5}$$

   where $\sigma_\pm(t) = \gamma\sigma_0 e^{-\gamma t}e^{\mp j\omega_B t}u(t)$ is the inverse Fourier transform of $\sigma_\pm(\omega)$ and $u(t)$ is the unit–step function.

5. Rewrite eq. (1.10.5) in component form:

$$J_x(t) = \int_0^t \left[\sigma_{xx}(t-t')E_x(t') + \sigma_{xy}(t-t')E_y(t')\right]dt'$$
$$J_y(t) = \int_0^t \left[\sigma_{yx}(t-t')E_x(t') + \sigma_{yy}(t-t')E_y(t')\right]dt' \tag{1.10.6}$$

and identify the quantities $\sigma_{xx}(t), \sigma_{xy}(t), \sigma_{yx}(t), \sigma_{yy}(t)$.

6. Evaluate eq. (1.10.6) in the special case $E_x(t) = E_x u(t)$ and $E_y(t) = E_y u(t)$, where E_x and E_y are constants, and show that after a long time the steady–state version of eq. (1.10.6) will be:

$$J_x = \sigma_0 \frac{E_x + bE_y}{1+b^2}$$
$$J_y = \sigma_0 \frac{E_y - bE_x}{1+b^2} \tag{1.10.7}$$

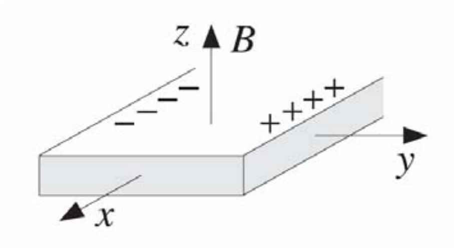

Fig. 1.10.1: Conductor with finite extent in y–direction.

where $b = \omega_B/\gamma$. If the conductor has finite extent in the y–direction, as show in Fig. 1.10.1, then no steady current can flow in this direction, $J_y = 0$. This implies that if an electric field is applied in the x–direction, an electric field will develop across the y–ends of the conductor, $E_y = bE_x$. The conduction charges will tend to accumulate either on the right or the left side of the conductor, depending on the sign of b, which depends on the sign of the electric charge e. This is the Hall effect and is used to determinate the sign of the conduction charges in semiconductors, e.g. positive holes for p–type, or negative electrons for n–type.

What is the numerical value of b for electrons in copper if B is 1 gauss?

7. For a collisionless plasma ($\gamma = 0$), show that its dielectric behavior is determined from $D_x \pm jD_y = \varepsilon_\pm(\omega)\left(E_x \pm jE_y\right)$, where

$$\varepsilon_\pm(\omega) = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega\left(\omega \pm \omega_B\right)}\right) \tag{1.10.8}$$

where ω_p is the plasma frequency. Thus, the plasma exhibits birefringence.

Solution

- Question n°1

First of all, divide eq. (1.10.1) by m:

$$\dot{\mathbf{v}} = \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \gamma \mathbf{v}$$

and expand the terms

$$\dot{v}_x \hat{\mathbf{x}} + \dot{v}_y \hat{\mathbf{y}} = \frac{e}{m}\left(E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}} + v_y B \hat{\mathbf{x}} - v_x B \hat{\mathbf{y}}\right) - \gamma\left(v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}\right)$$

Now it is possible to separate x and y–component as follow:

$$\begin{cases} \dot{v}_x = \frac{e}{m}\left(E_x + v_y B\right) - \gamma v_x \\ \dot{v}_y = \frac{e}{m}\left(E_y - v_x B\right) - \gamma v_y \end{cases} \Rightarrow \begin{cases} \dot{v}_x = \frac{e}{m} E_x + \frac{eB}{m} v_y - \gamma v_x \\ \dot{v}_y = \frac{e}{m} E_y - \frac{eB}{m} v_x - \gamma v_y \end{cases}$$

where is easy to note the cyclotron frequency ω_B.

The cyclotron frequency for electrons ($e = 1,602 \times 10^{-19}\,\text{C}$, $m = 9,10 \times 10^{-31}\,\text{Kg}$) in the Earth's magnetic field is:

$$f_B = \frac{1,602 \times 10^{-19} \times 0,4 \times 10^{-4}}{2 \times 3,14 \times 9,10 \times 10^{-31}} = \frac{1,602 \times 0,4}{2 \times 3,14 \times 9,10} \times 10^{8} = 1,12\text{MHz}$$

- Question n°2

Assuming harmonic time dependence means that

$$v_i(t) = v_i e^{j\omega t} \quad \Rightarrow \quad \dot{v}_i(t) = j\omega v_i e^{j\omega t}$$

so we have:

$$\begin{aligned} j\omega v_x &= \frac{e}{m} E_x + \omega_B v_y - \gamma v_x \\ j\omega v_y &= \frac{e}{m} E_y - \omega_B v_x - \gamma v_y \end{aligned} \tag{1.10.9}$$

Now combine the equations:

$$j\omega\left(v_x \pm j v_y\right) = \left(\frac{e}{m} E_x + \omega_B v_y - \gamma v_x\right) \pm j\left(\frac{e}{m} E_y - \omega_B v_x - \gamma v_y\right)$$

that is

$$j\omega\left(v_x \pm j v_y\right) = \frac{e}{m}\left(E_x \pm j E_y\right) + \omega_B\left(v_y \mp j v_x\right) - \gamma\left(v_x \pm j v_y\right) \tag{1.10.10}$$

In Eq. (1.10.10) there is the term $\omega_B\left(v_y \mp jv_x\right)$ that we have to express in the form $C\left(v_x \pm jv_y\right)$, where the constant C is to be found. If we take out of the parentheses $\pm j$, we obtain $\mp j\omega_B\left(v_x \pm jv_y\right)$ and the constant $C = \mp j\omega_B$. So eq. (1.10.10) becomes:

$$j\omega\left(v_x \pm jv_y\right) = \frac{e}{m}\left(E_x \pm jE_y\right) \mp j\omega_B\left(v_x \pm jv_y\right) - \gamma\left(v_x \pm jv_y\right)$$

and we obtain:

$$j\omega\left(v_x \pm jv_y\right) \pm j\omega_B\left(v_x \pm jv_y\right) + \gamma\left(v_x \pm jv_y\right) = \frac{e}{m}\left(E_x \pm jE_y\right)$$

$$\left(v_x \pm jv_y\right)\left(\gamma + j\left(\omega \pm \omega_B\right)\right) = \frac{e}{m}\left(E_x \pm jE_y\right)$$

$$\left(v_x \pm jv_y\right) = \frac{\frac{e}{m}\left(E_x \pm jE_y\right)}{\gamma + j\left(\omega \pm \omega_B\right)} \quad (1.10.11)$$

- Question n°3

Substituting eq. (1.10.11) in the expression for the induced currents ($\mathbf{J} = Ne\mathbf{v}$), we have:

$$J_x \pm jJ_y = Ne\left(v_x \pm jv_y\right) = \frac{\frac{Ne^2}{m}}{\gamma + j\left(\omega \pm \omega_B\right)}\left(E_x \pm jE_y\right) \quad (1.10.12)$$

where we can identify $\sigma_{\pm}(\omega)$ as

$$\sigma_{\pm}(\omega) = \frac{\frac{Ne^2}{m}}{\gamma + j\left(\omega \pm \omega_B\right)} = \frac{\gamma\sigma_0}{\gamma + j\left(\omega \pm \omega_B\right)} \quad (1.10.13)$$

where $\sigma_0 = \dfrac{Ne^2}{\gamma m}$ is the dc value of the conductivity.

- Question n°4

We have to calculate the Fourier transform:

$$\Im^{-1}\left\{\sigma_{\pm}(\omega)\right\} = \Im^{-1}\left\{\frac{\gamma\sigma_0}{\gamma + j\left(\omega \pm \omega_B\right)}\right\} \quad (1.10.14)$$

Eq. (1.10.14) can be written as:

$$\gamma\sigma_0\Im^{-1}\left\{\frac{1}{j\omega + \left(\gamma \pm j\omega_B\right)}\right\} = j\gamma\sigma_0\Im^{-1}\left\{\frac{1}{\omega + j\left(\gamma \pm j\omega_B\right)}\right\} = j\gamma\sigma_0\Im^{-1}\left\{\frac{1}{\omega + \omega_{\pm}}\right\} \quad (1.10.15)$$

where $\omega_{\pm} = j(\gamma \pm j\omega_B)$. It's easy to note that the inverse Fourier transform in (1.10.15) is already calculated in exercise 1.7 (eq. (1.7.9)). So we have:

$$\begin{aligned}\sigma_{\pm}(t) &= j\gamma\sigma_0\mathfrak{I}^{-1}\left\{\frac{1}{\omega+\omega_{\pm}}\right\} = j\gamma\sigma_0\left(-je^{-j\omega_{\pm}t}u(t)\right) = \\ &= \gamma\sigma_0 e^{-j\omega_{\pm}t}u(t) = \gamma\sigma_0 e^{-\gamma t}e^{\mp j\omega_B t}u(t)\end{aligned} \tag{1.10.16}$$

where $u(t)$ is the unit step function.

Now we can write eq. (1.10.12) in the time–domain version:

$$J_x(t) \pm jJ_y(t) = \int_0^t \sigma_{\pm}(t-t')\left(E_x(t') \pm jE_y(t')\right)dt' \tag{1.10.17}$$

- Question n°5

It's possible to decompose eq. (1.10.17) in its two component as follow:

$$\begin{aligned}J_x(t) + jJ_y(t) &= \int_0^t \sigma_{+}(t-t')\left(E_x(t') + jE_y(t')\right)dt' \\ J_x(t) - jJ_y(t) &= \int_0^t \sigma_{-}(t-t')\left(E_x(t') - jE_y(t')\right)dt'\end{aligned} \tag{1.10.18}$$

Combining them, we obtain:

$$2J_x(t) = \int_0^t \left[\sigma_{+}(t-t')\left(E_x(t') + jE_y(t')\right) + \sigma_{-}(t-t')\left(E_x(t') - jE_y(t')\right)\right]dt' \tag{1.10.19}$$

$$2jJ_y(t) = \int_0^t \left[\sigma_{+}(t-t')\left(E_x(t') + jE_y(t')\right) - \sigma_{-}(t-t')\left(E_x(t') - jE_y(t')\right)\right]dt' \tag{1.10.20}$$

Manipulating the expression in the brackets, we have:

$$J_x(t) = \int_0^t \left[\left(\frac{\sigma_{+}(t-t') + \sigma_{-}(t-t')}{2}\right)E_x(t') + j\left(\frac{\sigma_{+}(t-t') - \sigma_{-}(t-t')}{2}\right)E_y(t')\right]dt' \tag{1.10.21}$$

$$J_y(t) = \int_0^t \left[\left(\frac{\sigma_{+}(t-t') - \sigma_{-}(t-t')}{2j}\right)E_x(t') + j\left(\frac{\sigma_{+}(t-t') + \sigma_{-}(t-t')}{2j}\right)E_y(t')\right]dt' \tag{1.10.22}$$

and it's easy to identify the follow quantities:

$$\begin{cases} \sigma_{xx}(t) = \dfrac{\sigma_+(t)+\sigma_-(t)}{2} \\ \sigma_{xy}(t) = j\left(\dfrac{\sigma_+(t)-\sigma_-(t)}{2}\right) \\ \sigma_{yx}(t) = \left(\dfrac{\sigma_+(t)-\sigma_-(t)}{2j}\right) \\ \sigma_{yy}(t) = \left(\dfrac{\sigma_+(t)+\sigma_-(t)}{2}\right) \end{cases} \tag{1.10.23}$$

- Question n°6

Consider the expression of $J_x(t)$ in eq. (1.10.21) and divide it in two integrals:

$$J_x(t) = \int_0^t \left(\frac{\sigma_+(t-t')+\sigma_-(t-t')}{2}\right) E_x(t')dt' + j\int_0^t \left(\frac{\sigma_+(t-t')-\sigma_-(t-t')}{2}\right) E_y(t')dt' = \\ = I_1 + I_2$$

I_1 and I_2 can be solved separately. Let's start with I_1 substituting $E_x(t) = E_x u(t)$ and the definition of $\sigma_{xx}(t)$. So we obtain:

$$I_1 = \int_0^t \left(\frac{\sigma_+(t-t')+\sigma_-(t-t')}{2}\right) E_x u(t')dt' = \\ = \frac{1}{2}\left[\int_0^t \sigma_+(t-t')E_x u(t')dt' + \int_0^t \sigma_-(t-t')E_x(t')dt'\right] = \frac{1}{2}\left(I_{11}+I_{12}\right)$$

Now we solve separately I_{11} and I_{12}, substituting the definitions of $\sigma_+(t)$ and $\sigma_-(t)$ respectively:

$$I_{11} = \int_0^t \sigma_+(t-t')E_x u(t')dt' = \int_0^t \gamma\sigma_0 e^{-j\omega_+(t-t')} u(t-t') E_x u(t')dt' = \\ = \gamma\sigma_0 E_x \int_0^t e^{-j\omega_+(t-t')}dt' = -\gamma\sigma_0 E_x \int_0^t e^{-j\omega_+(t-t')}d(t-t') = \\ = -\gamma\sigma_0 E_x \left[\frac{e^{-j\omega_+(t-t')}}{-j\omega_+}\right]_0^t = -\gamma\sigma_0 E_x \left(\frac{1-e^{-j\omega_+ t}}{-j\omega_+}\right)$$

After long time, i.e. $t \to \infty$, the integral I_{11} results:

$$I_{11} = -\gamma\sigma_0 E_x \left(\frac{1}{-j\omega_+}\right) = \frac{\gamma\sigma_0}{(\gamma + j\omega_B)} E_x \tag{1.10.24}$$

In the same way we can solve I_{12} and obtain:

$$I_{12} = -\gamma\sigma_0 E_x \left(\frac{1}{-j\omega_-}\right) = \frac{\gamma\sigma_0}{(\gamma - j\omega_B)} E_x \tag{1.10.25}$$

Now combining eq. (1.10.24) and (1.10.25), we have the solution of integral I_1:

$$I_1 = \frac{1}{2}(I_{11}+I_{12}) = \frac{1}{2}\left(\frac{\gamma\sigma_0 E_x}{(\gamma+j\omega_B)} + \frac{\gamma\sigma_0 E_x}{(\gamma-j\omega_B)}\right) =$$
$$= \frac{\gamma\sigma_0 E_x}{2}\left(\frac{1}{(\gamma+j\omega_B)} + \frac{1}{(\gamma-j\omega_B)}\right) = \frac{\gamma\sigma_0 E_x}{2}\left(\frac{\gamma - \cancel{j\omega_B} + \gamma + \cancel{j\omega_B}}{\gamma^2+\omega_B^2}\right) = \quad (1.10.26)$$
$$= \frac{\gamma\sigma_0 E_x}{\cancel{2}}\left(\frac{\cancel{2}\gamma}{\gamma^2+\omega_B^2}\right) = \frac{\gamma^2\sigma_0}{\gamma^2+\omega_B^2}E_x$$

As for integral I_1, we can solve integral I_2:

$$I_2 = j\int_0^t \left(\frac{\sigma_+(t-t') - \sigma_-(t-t')}{2}\right)E_y u(t')dt' =$$
$$= \frac{j}{2}\left[\int_0^t \sigma_+(t-t')E_y u(t')dt' - \int_0^t \sigma_-(t-t')E_y u(t')dt'\right] = \frac{j}{2}\left(I_{21}-I_{22}\right)$$

The integrals I_{21} and I_{22} have the same structure of integrals I_{11} and I_{12}. So the results are known:

$$I_{21} = \frac{\gamma\sigma_0}{(\gamma+j\omega_B)}E_y$$
$$I_{22} = \frac{\gamma\sigma_0}{(\gamma-j\omega_B)}E_y \quad (1.10.27)$$

It is easy to calculate I_2 as:

$$I_2 = \frac{j}{2}\left(I_{21}-I_{22}\right) = \frac{j}{2}\left(\frac{\gamma\sigma_0}{(\gamma+j\omega_B)}E_y - \frac{\gamma\sigma_0}{(\gamma-j\omega_B)}E_y\right) =$$
$$= \frac{j\gamma\sigma_0 E_y}{2}\left(\frac{1}{(\gamma+j\omega_B)} - \frac{1}{(\gamma-j\omega_B)}\right) = \frac{j\gamma\sigma_0 E_y}{2}\left(\frac{\cancel{\gamma} - j\omega_B - \cancel{\gamma} - j\omega_B}{\gamma^2+\omega_B^2}\right) = (1.10.28)$$
$$= \frac{j\gamma\sigma_0 E_y}{\cancel{2}}\left(\frac{-\cancel{2}j\omega_B}{\gamma^2+\omega_B^2}\right) = \frac{\omega_B\sigma_0}{\gamma^2+\omega_B^2}E_y$$

Now it is possible to write $J_x(t)$ in steady state when a constant electric field is applied:

$$J_x(t) = \frac{\gamma^2\sigma_0}{\gamma^2+\omega_B^2}E_x + \frac{\omega_B\sigma_0}{\gamma^2+\omega_B^2}E_y = \sigma_0\frac{\gamma^2 E_x + \omega_B E_y}{\gamma^2+\omega_B^2} =$$
$$= \sigma_0 \frac{\dfrac{\gamma^2 E_x + \omega_B E_y}{\gamma^2}}{\dfrac{\gamma^2+\omega_B^2}{\gamma^2}} = \sigma_0\frac{E_x + bE_y}{1+b^2}$$

where $b = \omega_B/\gamma$.

Consider now the expression of $J_y(t)$ in eq. (1.10.22) and divide it in two integrals:

$$J_y(t) = \int_0^t \left(\frac{\sigma_+(t-t') - \sigma_-(t-t')}{2j} \right) E_x(t')dt' + \int_0^t \left(\frac{\sigma_+(t-t') + \sigma_-(t-t')}{2} \right) E_y(t')dt' = \\ = I_3 + I_4$$

I_3 and I_4 can be solved separately. Let's start with I_3 substituting $E_x(t) = E_x u(t)$ and the definition of $\sigma_{yx}(t)$. So we obtain:

$$I_3 = \int_0^t \left(\frac{\sigma_+(t-t') - \sigma_-(t-t')}{2j} \right) E_x u(t')dt' = \\ = \frac{1}{2j} \left[\int_0^t \sigma_+(t-t')E_x u(t')dt' - \int_0^t \sigma_-(t-t')E_x(t')dt' \right] = \frac{1}{2j}\left(I_{31} - I_{32}\right)$$

Now we solve separately I_{31} and I_{32}, substituting the definitions of $\sigma_+(t)$ and $\sigma_-(t)$ respectively:

$$I_{31} = \int_0^t \sigma_+(t-t')E_x u(t')dt' = \int_0^t \gamma\sigma_0 e^{-j\omega_+(t-t')} u\left(t-t'\right) E_x u(t')dt' = \\ = \gamma\sigma_0 E_x \int_0^t e^{-j\omega_+(t-t')}dt' = -\gamma\sigma_0 E_x \int_0^t e^{-j\omega_+(t-t')}d(t-t') = \\ = -\gamma\sigma_0 E_x \left[\frac{e^{-j\omega_+(t-t')}}{-j\omega_+} \right]_0^t = -\gamma\sigma_0 E_x \left(\frac{1 - e^{-j\omega_+ t}}{-j\omega_+} \right)$$

After long time, i.e. $t \to \infty$, the integral I_{31} results:

$$I_{31} = -\gamma\sigma_0 E_x \left(\frac{1}{-j\omega_+} \right) = \frac{\gamma\sigma_0}{(\gamma + j\omega_B)} E_x \tag{1.10.29}$$

In the same way we can solve I_{32} and obtain:

$$I_{32} = -\gamma\sigma_0 E_x \left(\frac{1}{-j\omega_-} \right) = \frac{\gamma\sigma_0}{(\gamma - j\omega_B)} E_x \tag{1.10.30}$$

Now combining eq. (1.10.29)and (1.10.30), we have the solution of integral I_3:

$$I_3 = \frac{1}{2j}(I_{31} - I_{32}) = \frac{1}{2j}\left(\frac{\gamma\sigma_0 E_x}{(\gamma + j\omega_B)} - \frac{\gamma\sigma_0 E_x}{(\gamma - j\omega_B)} \right) = \\ = \frac{\gamma\sigma_0 E_x}{2j}\left(\frac{1}{(\gamma + j\omega_B)} - \frac{1}{(\gamma - j\omega_B)} \right) = \frac{\gamma\sigma_0 E_x}{2j}\left(\frac{\cancel{\gamma} - j\omega_B - \cancel{\gamma} - j\omega_B}{\gamma^2 + \omega_B^2} \right) = \\ = \frac{\gamma\sigma_0 E_x}{\cancel{2}j}\left(\frac{-\cancel{2}j\omega_B}{\gamma^2 + \omega_B^2} \right) = -\frac{\omega_B\sigma_0}{\gamma^2 + \omega_B^2} E_x \tag{1.10.31}$$

As for integral I_3, we can solve integral I_4:

$$I_4 = \int_0^t \left(\frac{\sigma_+(t-t') + \sigma_-(t-t')}{2} \right) E_y(t')dt' =$$
$$= \frac{1}{2}\left[\int_0^t \sigma_+(t-t')E_y u(t')dt' + \int_0^t \sigma_-(t-t')E_y u(t')dt' \right] = \frac{1}{2}\left(I_{41} + I_{42} \right)$$

The integrals I_{41} and I_{42} have the same structure of integrals I_{31} and I_{32}. So the results are known:

$$I_{41} = \frac{\gamma\sigma_0}{(\gamma + j\omega_B)} E_y$$
$$I_{42} = \frac{\gamma\sigma_0}{(\gamma - j\omega_B)} E_y \tag{1.10.32}$$

It is easy to calculate I_4 as:

$$I_4 = \frac{1}{2}\left(I_{41} - I_{42} \right) = \frac{1}{2}\left(\frac{\gamma\sigma_0}{(\gamma + j\omega_B)} E_y + \frac{\gamma\sigma_0}{(\gamma - j\omega_B)} E_y \right) =$$
$$= \frac{\gamma\sigma_0 E_y}{2}\left(\frac{1}{(\gamma + j\omega_B)} + \frac{1}{(\gamma - j\omega_B)} \right) = \frac{\gamma\sigma_0 E_y}{2}\left(\frac{\gamma - \cancel{j\omega_B} + \gamma + \cancel{j\omega_B}}{\gamma^2 + \omega_B^2} \right) = \tag{1.10.33}$$
$$= \frac{\gamma\sigma_0 E_y}{\cancel{2}}\left(\frac{-\cancel{2}\gamma}{\gamma^2 + \omega_B^2} \right) = \frac{\gamma^2\sigma_0}{\gamma^2 + \omega_B^2} E_y$$

Now it is possible to write $J_y(t)$ in steady state when a constant electric field is applied:

$$J_y(t) = -\frac{\omega_B\sigma_0}{\gamma^2 + \omega_B^2} E_x + \frac{\gamma^2\sigma_0}{\gamma^2 + \omega_B^2} E_y = \sigma_0 \frac{\gamma^2 E_y - \omega_B E_x}{\gamma^2 + \omega_B^2} =$$
$$= \sigma_0 \frac{\dfrac{\gamma^2 E_y - \omega_B E_x}{\gamma^2}}{\dfrac{\gamma^2 + \omega_B^2}{\gamma^2}} = \sigma_0 \frac{E_y + bE_x}{1 + b^2}$$

where $b = \omega_B / \gamma$.

- Question n°7

To solve this point it is necessary to obtain the expression of $\sigma_\pm(\omega)$ for a collisionless plasma because it is note the relationship:

$$\varepsilon_\pm(\omega) = \varepsilon_0 + \frac{\sigma_\pm(\omega)}{j\omega} \tag{1.10.34}$$

The definition of the conductibility $\sigma_\pm(\omega)$ has just obtained in this exercise, i.e. eq. (1.10.13). We have only to set $\gamma=0$ for indicating that there is no collision between the electrons and the medium structure:

$$\sigma_{\pm}(\omega)=\frac{\frac{Ne^2}{m}}{j(\omega\pm\omega_B)} \tag{1.10.35}$$

Now we substitute eq. (1.10.35) in (1.10.34) and we have:

$$\varepsilon_{\pm}(\omega)=\varepsilon_0-\frac{\frac{Ne^2}{m}}{\omega(\omega\pm\omega_B)}=\varepsilon_0\left(1-\frac{\omega_p^2}{\omega(\omega\pm\omega_B)}\right) \tag{1.10.36}$$

where $\varepsilon_0\omega_p^2=\frac{Ne^2}{m}$.

The numerical value of b for electrons in copper can be find out using:

$$b=\omega_B\gamma^{-1}=\frac{eB}{m\gamma}$$

where $e=1,6\times10^{-19}C$, $m=9,1\times10^{-31}Kg$, $\gamma=4,1\times10^{13}s^{-1}$. If $B=1\text{ gauss}=10^{-4}\text{ Tesla}$, then

$$b=\frac{1,6\times10^{-19}\times10^{-4}}{9,1\times10^{-31}\times4,1\times10^{13}}=\frac{1,6}{9,1\times4,1}\times10^{5}=4288$$

The result is different from the one in the text which is 43.

1.11 Exercise

This problem deals with various properties of the Kramers–Kronig dispersion relations for the electric susceptibility, given by:

$$\chi_r(\omega) = \frac{1}{\pi} \wp \int_{-\infty}^{+\infty} \frac{\chi_i(\omega')}{\omega' - \omega} d\omega'$$
$$\chi_i(\omega) = -\frac{1}{\pi} \wp \int_{-\infty}^{+\infty} \frac{\chi_r(\omega')}{\omega' - \omega} d\omega' \qquad (1.11.1)$$

where $\wp$ denotes the "principal value" and $\chi(\omega) = \chi_r(\omega) + j\chi_i(\omega)$ is the Fourier transform of $\chi(t)$. Because the time–response $\chi(t)$ is real–valued, its Fourier transform $\chi(\omega)$ will satisfy the Hermitian symmetry property $\chi(-\omega) = \chi^*(\omega)$, which is equivalent to the even symmetry of its real part, $\chi_r(-\omega) = \chi_r(\omega)$, and the odd symmetry of its imaginary part, $\chi_i(-\omega) = -\chi_i(\omega)$.

1. Using the symmetry properties, show that eq. (1.11.1) can be written in the folded form:

$$\chi_r(\omega) = \frac{2}{\pi} \wp \int_0^{\infty} \frac{\omega' \chi_i(\omega')}{\omega'^2 - \omega^2} d\omega'$$
$$\chi_i(\omega) = -\frac{2}{\pi} \wp \int_0^{\infty} \frac{\omega \chi_r(\omega')}{\omega'^2 - \omega^2} d\omega' \qquad (1.11.2)$$

2. Using the definition of the principal–value integrals, show the following integral:

$$\wp \int_0^{\infty} \frac{d\omega'}{\omega'^2 - \omega^2} = 0 \qquad (1.11.3)$$

   Hint: You may use the following indefinite integral: $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right|$.

3. Using eq. (1.11.3), show that the relations (1.11.2) may be rewritten as ordinary integrals (without the $\wp$ instruction) as follows:

$$\chi_r(\omega) = \frac{2}{\pi} \int_0^{\infty} \frac{\omega' \chi_i(\omega') - \omega \chi_i(\omega)}{\omega'^2 - \omega^2} d\omega'$$
$$\chi_i(\omega) = -\frac{2}{\pi} \int_0^{\infty} \frac{\omega \chi_r(\omega') - \omega \chi_r(\omega)}{\omega'^2 - \omega^2} d\omega' \qquad (1.11.4)$$

   Hint: You will need to argue that the integrands have no singularity at $\omega' = \omega$.

4. For a simple oscillator model of dielectric polarization, the susceptibility is given by:

$$\begin{aligned}\chi(\omega) &= \chi_r(\omega) - j\chi_i(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + j\gamma\omega} = \\ &= \frac{\omega_p^2\left(\omega_0^2 - \omega^2\right)}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2} - j\frac{\gamma\omega\omega_p^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\end{aligned} \tag{1.11.5}$$

Show that for this model the quantities $\chi_r(\omega)$ and $\chi_i(\omega)$ satisfy the modified Kramers–Kronig relationships (1.11.4).

Hint: You may use the following definite integrals, for which you may assume that $0 < \gamma < 2\omega_0$:

$$\frac{2}{\pi}\int_0^\infty \frac{dx}{\left(\omega_0^2 - x^2\right)^2 + \gamma^2 x^2} = \frac{1}{\gamma\omega_0^2}, \qquad \frac{2}{\pi}\int_0^\infty \frac{x^2 dx}{\left(\omega_0^2 - x^2\right)^2 + \gamma^2 x^2} = \frac{1}{\gamma}$$

Indeed, show that these integrals may be reduced to the following ones, which can be found in standard tables of integrals:

$$\frac{2}{\pi}\int_0^\infty \frac{dy}{1 - 2y^2 C\cos\theta + y^4} = \frac{2}{\pi}\int_0^\infty \frac{y^2 dy}{1 - 2y^2 C\cos\theta + y^4} = \frac{1}{\sqrt{2\left(1 - C\cos\theta\right)}}$$

where $\sin(\theta/2) = \gamma/(2\omega_0)$.

5. Consider the limit of Eq. (1.11.5) as $\gamma \to 0$. Show that in this case the functions χ_r, χ_i are given as follows, and that they still satisfy the Kramers–Kronig relations:

$$\chi_r(\omega) = \wp\frac{\omega_p^2}{\omega_0 - \omega} + \wp\frac{\omega_p^2}{\omega_0 + \omega}, \qquad \chi_i(\omega) = \frac{\pi\omega_p^2}{2\omega_0}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right].$$

Solution

The Cauchy's principal value is defined for the integration of a function $f(x)$ with limited value in the interval $[a,b]$ except for the $x = x_0$. This is a special case of Reimann's principal value:

$$\wp\int_a^b f(x)dx = \lim_{\varsigma_1 \to 0^+}\int_a^{x_0 - \varsigma_1} f(x)dx + \lim_{\varsigma_2 \to 0^+}\int_{x_0 + \varsigma_2}^b f(x)dx$$

where ς_1 and ς_2 are independent of each other, whereas in the Cauchy's principal value $\varsigma_1 = \varsigma_2$, so:

$$\wp\int_a^b f(x)dx = \lim_{\varsigma \to 0^+}\int_a^{x_0 - \varsigma} f(x)dx + \int_{x_0 + \varsigma}^b f(x)dx$$

In the case of the exercise the singularity is at $\omega' = \omega$.

- Question n°1

Let us decompose $\chi_r(\omega)$ as follow

$$\chi_r(\omega) = \chi_{r+}(\omega) + \chi_{r-}(\omega) = \frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_i(\omega')}{\omega'-\omega}d\omega' + \frac{1}{\pi}\wp\int_{-\infty}^{0}\frac{\chi_i(\omega')}{\omega'-\omega}d\omega'$$

and we change the variable in the second term ($\omega' \to -\omega'$):

$$\chi_r(\omega) = \frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_i(\omega')}{\omega'-\omega}d\omega' + \frac{1}{\pi}\wp\int_{+\infty}^{0}\frac{\chi_i(-\omega')}{-\omega'-\omega}d(-\omega') \underset{\substack{\uparrow \\ \text{Simmetry property of } \chi_i}}{=}$$

$$= \frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_i(\omega')}{\omega'-\omega}d\omega' + \frac{1}{\pi}\wp\int_{+\infty}^{0}\frac{-\chi_i(\omega')}{-\omega'-\omega}d(-\omega') =$$

$$= \frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_i(\omega')}{\omega'-\omega}d\omega' + \frac{1}{\pi}\wp\int_{+\infty}^{0}\frac{\chi_i(\omega')}{-\omega'-\omega}d\omega' \underset{\substack{\uparrow \\ \text{Invert integral's limits}}}{=}$$

$$= \frac{1}{\pi}\wp\int_0^{+\infty}\left(\frac{\chi_i(\omega')}{\omega'-\omega} + \frac{\chi_i(\omega')}{\omega'+\omega}\right)d\omega' =$$

$$= \frac{1}{\pi}\wp\int_0^{+\infty}\left(\frac{(\omega'+\omega)\chi_i(\omega') + (\omega'-\omega)\chi_i(\omega')}{\omega'^2-\omega^2}\right)d\omega' =$$

$$= \frac{1}{\pi}\wp\int_0^{+\infty}\left(\frac{\omega'\chi_i(\omega') + \cancel{\omega\chi_i(\omega')} + \omega'\chi_i(\omega') - \cancel{\omega\chi_i(\omega')}}{\omega'^2-\omega^2}\right)d\omega' =$$

$$= \frac{2}{\pi}\wp\int_0^{+\infty}\frac{\omega'\chi_i(\omega')}{\omega'^2-\omega^2}d\omega'$$

In the same way, it's possible to decompose $\chi_i(\omega)$:

$$\chi_i(\omega) = \chi_{i+}(\omega) + \chi_{i-}(\omega) = -\frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_r(\omega')}{\omega'-\omega}d\omega' - \frac{1}{\pi}\wp\int_{-\infty}^{0}\frac{\chi_r(\omega')}{\omega'-\omega}d\omega'$$

and we change the variable in the second term ($\omega' \to -\omega'$):

$$\chi_{\mathrm{i}}(\omega) = -\frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_{\mathrm{r}}(\omega')}{\omega'-\omega}\mathrm{d}\omega' - \frac{1}{\pi}\wp\int_{+\infty}^{0}\frac{\chi_{\mathrm{r}}(-\omega')}{-\omega'-\omega}\mathrm{d}(-\omega') \underset{\text{Simmetry property of } \chi_{\mathrm{r}}}{\overset{\uparrow}{=}}$$

$$= -\frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_{\mathrm{r}}(\omega')}{\omega'-\omega}\mathrm{d}\omega' - \frac{1}{\pi}\wp\int_{+\infty}^{0}\frac{\chi_{\mathrm{r}}(\omega')}{-\omega'-\omega}\mathrm{d}(-\omega') =$$

$$= -\frac{1}{\pi}\wp\int_0^{+\infty}\frac{\chi_{\mathrm{r}}(\omega')}{\omega'-\omega}\mathrm{d}\omega' + \frac{1}{\pi}\wp\int_{+\infty}^{0}\frac{\chi_{\mathrm{r}}(\omega')}{-\omega'-\omega}\mathrm{d}\omega' \underset{\text{Invert integral's limits}}{\overset{\uparrow}{=}}$$

$$= -\frac{1}{\pi}\wp\int_0^{+\infty}\left(\frac{\chi_{\mathrm{r}}(\omega')}{\omega'-\omega} - \frac{\chi_{\mathrm{r}}(\omega')}{\omega'+\omega}\right)\mathrm{d}\omega' =$$

$$= \frac{1}{\pi}\wp\int_0^{+\infty}\left(\frac{(\omega'+\omega)\chi_{\mathrm{i}}(\omega') - (\omega'-\omega)\chi_{\mathrm{i}}(\omega')}{\omega'^2-\omega^2}\right)\mathrm{d}\omega' =$$

$$= \frac{1}{\pi}\wp\int_0^{+\infty}\left(\frac{\cancel{\omega'\chi_{\mathrm{i}}(\omega')} + \omega\chi_{\mathrm{i}}(\omega') - \cancel{\omega'\chi_{\mathrm{i}}(\omega')} + \omega\chi_{\mathrm{i}}(\omega')}{\omega'^2-\omega^2}\right)\mathrm{d}\omega' =$$

$$= \frac{2}{\pi}\wp\int_0^{+\infty}\frac{\omega\chi_{\mathrm{i}}(\omega')}{\omega'^2-\omega^2}\mathrm{d}\omega'$$

- Question n°2

Using the definition of the principal–value integrals and the hint, we write:

$$\wp\int_0^{\infty}\frac{\mathrm{d}\omega'}{\omega'^2-\omega^2} = \lim_{\varsigma\to 0}\left[\int_0^{\omega-\varsigma}\frac{\mathrm{d}\omega'}{\omega'^2-\omega^2} + \int_{\omega+\varsigma}^{\infty}\frac{\mathrm{d}\omega'}{\omega'^2-\omega^2}\right] =$$

$$= \frac{1}{2\omega}\lim_{\varsigma\to\infty}\left[\ln\left|\frac{\omega+\omega'}{\omega-\omega'}\right|_0^{\omega-\varsigma} + \ln\left|\frac{\omega+\omega'}{\omega-\omega'}\right|_{\omega+\varsigma}^{\infty}\right] = \tag{1.11.6}$$

$$= \frac{1}{2\omega}\lim_{\varsigma\to\infty}\left[\ln\left|\frac{2\omega-\varsigma}{\varsigma}\right| \cancel{-\ln|1|} \cancel{+\ln|1|} - \ln\left|-\frac{2\omega+\varsigma}{\varsigma}\right|\right] =$$

$$= \frac{1}{2\omega}\lim_{\varsigma\to\infty}\left[\ln\left|\frac{2\omega-\varsigma}{\varsigma}\right| - \ln\left|-\frac{2\omega+\varsigma}{\varsigma}\right|\right] = 0$$

- Question n°3

The integrands of equations (1.11.2) have a singularity in $\omega' = \omega$ and we have to use the principal–value for solving the integral. But if we also introduce a singularity at the numerator in $\omega' = \omega$, we will have the integrand that will not diverge. Besides subtracting eq. (1.11.3), that is zero, to eq. (1.11.2), we obtain:

$$\chi_r(\omega)=\frac{2}{\pi}\wp\int_0^\infty \frac{\omega'\chi_i(\omega')}{\omega'^2-\omega^2}d\omega'-\wp\int_0^\infty \frac{d\omega'}{\omega'^2-\omega^2}$$
$$\chi_i(\omega)=-\frac{2}{\pi}\wp\int_0^\infty \frac{\omega\chi_r(\omega')}{\omega'^2-\omega^2}d\omega'-\wp\int_0^\infty \frac{d\omega'}{\omega'^2-\omega^2}$$

Being equal to zero, the second term can be multiplied by a constant and we choose $2\omega\chi(\omega)/\pi$. So we have:

$$\chi_r(\omega)=\frac{2}{\pi}\left[\wp\int_0^\infty\left(\frac{\omega'\chi_i(\omega')}{\omega'^2-\omega^2}-\frac{\omega\chi_i(\omega)}{\omega'^2-\omega^2}\right)d\omega'\right]$$
$$\chi_i(\omega)=-\frac{2}{\pi}\left[\wp\int_0^\infty\left(\frac{\omega\chi_r(\omega')}{\omega'^2-\omega^2}-\frac{\omega\chi_r(\omega)}{\omega'^2-\omega^2}\right)d\omega'\right] \quad (1.11.7)$$

Because now the integrands don't present a singularity in $\omega'=\omega$, we can cancel $\wp$ instruction and obtain:

$$\chi_r(\omega)=\frac{2}{\pi}\int_0^\infty\left(\frac{\omega'\chi_i(\omega')-\omega\chi_i(\omega)}{\omega'^2-\omega^2}\right)d\omega'$$
$$\chi_i(\omega)=-\frac{2}{\pi}\int_0^\infty\left(\frac{\omega\chi_r(\omega')-\omega\chi_r(\omega)}{\omega'^2-\omega^2}\right)d\omega' \quad (1.11.8)$$

- Question n°4

Substitute the quantity $\chi_i(\omega)$ expressed in eq. (1.11.5) inside the modified Kramers–Kronig relationships (1.11.4) for $\chi_r(\omega)$:

$$\chi_r(\omega)=\frac{2}{\pi}\int_0^\infty \frac{x\chi_i(x)-\omega\chi_i(\omega)}{x^2-\omega^2}dx, \quad \text{where} \quad \chi_i(\omega)=\frac{\gamma\omega\omega_p^2}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$$

where we have substituted $\omega'\rightarrow x$ to distinguish better the arguments of the integral. Let us denote the denominator of $\chi_i(\omega)$ as $\mathrm{Den}[\omega]$. So we have that:

$$\chi_r(\omega)=\frac{2}{\pi}\int_0^\infty\frac{1}{x^2-\omega^2}\left[x\frac{\gamma x\omega_p^2}{\mathrm{Den}[x]}-\omega\frac{\gamma\omega\omega_p^2}{\mathrm{Den}[\omega]}\right]dx=$$
$$=\frac{2\gamma\omega_p^2}{\pi}\int_0^\infty\frac{1}{x^2-\omega^2}\left[\frac{x^2}{\mathrm{Den}[x]}-\frac{\omega^2}{\mathrm{Den}[\omega]}\right]dx=$$
$$=\frac{2\gamma\omega_p^2}{\pi}\int_0^\infty\frac{1}{x^2-\omega^2}\left[\frac{x^2\mathrm{Den}[\omega]-\omega^2\mathrm{Den}[x]}{\mathrm{Den}[x]\mathrm{Den}[\omega]}\right]dx$$

Now we can expand the numerator $x^2\mathrm{Den}[\omega]-\omega^2\mathrm{Den}[x]$ inside the integral using the extended form for $\mathrm{Den}[x]$ and $\mathrm{Den}[\omega]$:

$$\begin{aligned}x^2\mathrm{Den}[\omega]-\omega^2\mathrm{Den}[x] &= x^2\left[\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2\right]-\omega^2\left[\left(\omega_0^2-x^2\right)^2+\gamma^2x^2\right]=\\ &= x^2\left[\omega_0^4+\omega^4-2\omega_0^2\omega^2+\gamma^2\omega^2\right]-\omega^2\left[\omega_0^4+x^4-2\omega_0^2x^2+\gamma^2x^2\right]=\\ &= x^2\omega_0^4+x^2\omega^4\cancel{-2x^2\omega_0^2\omega^2}\cancel{+x^2\gamma^2\omega^2}-\omega^2\omega_0^4-\omega^2x^4\cancel{+2\omega^2\omega_0^2x^2}\cancel{-\omega^2\gamma^2x^2}=\\ &= x^2\omega_0^4+x^2\omega^4-\omega^2\omega_0^4-\omega^2x^4=\omega_0^4\left(x^2-\omega^2\right)-\omega^2x^2\left(x^2-\omega^2\right)=\\ &=\left(x^2-\omega^2\right)\left(\omega_0^4-\omega^2x^2\right)\end{aligned}$$

and substitute it inside the integral:

$$\begin{aligned}\chi_r(\omega) &= \frac{2\gamma\omega_p^2}{\pi}\int_0^\infty \frac{1}{\cancel{x^2-\omega^2}}\left[\frac{\cancel{\left(x^2-\omega^2\right)}\left(\omega_0^4-\omega^2x^2\right)}{\mathrm{Den}[x]\mathrm{Den}[\omega]}\right]dx=\\ &=\frac{\gamma\omega_p^2}{\mathrm{Den}[\omega]}\frac{2}{\pi}\int_0^\infty\frac{\omega_0^4-\omega^2x^2}{\mathrm{Den}[x]}dx=\frac{\gamma\omega_p^2}{\mathrm{Den}[\omega]}\left[\frac{2}{\pi}\int_0^\infty\frac{\omega_0^4}{\mathrm{Den}[x]}dx-\frac{2}{\pi}\int_0^\infty\frac{\omega^2x^2}{\mathrm{Den}[x]}dx\right]=\\ &=\frac{\gamma\omega_p^2}{\mathrm{Den}[\omega]}\left[\omega_0^4\frac{2}{\pi}\int_0^\infty\frac{1}{\mathrm{Den}[x]}dx-\omega^2\frac{2}{\pi}\int_0^\infty\frac{x^2}{\mathrm{Den}[x]}dx\right]\end{aligned}$$

Now it is possible to use the note results suggested by the text of the exercise:

$$\begin{aligned}\chi_r(\omega) &= \frac{\cancel{\gamma}\omega_p^2}{\mathrm{Den}[\omega]}\left[\omega_0^4\frac{1}{\cancel{\gamma}\omega_0^2}-\omega^2\frac{1}{\cancel{\gamma}}\right]=\frac{\omega_p^2\left(\omega_0^2-\omega^2\right)}{\mathrm{Den}[\omega]}=\\ &=\frac{\omega_p^2\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\end{aligned}$$

that is exactly the expression of $\chi_r(\omega)$ for a simple oscillator model of dielectric polarization in eq. (1.11.5).

Now we have to demonstrate the dual expression for $\chi_i(\omega)$:

$$\chi_i(\omega)=-\frac{2}{\pi}\int_0^\infty\frac{\omega\chi_r(x)-\omega\chi_r(\omega)}{x^2-\omega^2}dx \quad \text{where} \quad \chi_r(\omega)=\frac{\omega_p^2\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$$

Let us use the same notation as above. So we have:

$$\chi_i(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{1}{x^2-\omega^2}\left[\omega\frac{\omega_p^2\left(\omega_0^2-x^2\right)}{\text{Den}[x]} - \omega\frac{\omega_p^2\left(\omega_0^2-\omega^2\right)}{\text{Den}[\omega]}\right]dx =$$

$$= -\frac{2\omega\omega_p^2}{\pi}\int_0^\infty \frac{1}{x^2-\omega^2}\left[\frac{\left(\omega_0^2-x^2\right)\text{Den}[\omega] - \left(\omega_0^2-\omega^2\right)\text{Den}[x]}{\text{Den}[x]\text{Den}[\omega]}\right]dx$$

where $\text{Den}[x] = \left(\omega_0^2 - x^2\right)^2 + \gamma^2 x^2$ and $\text{Den}[\omega] = \left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2$.

Now we can expand the numerator $\left(\omega_0^2 - x^2\right)\text{Den}[\omega] - \left(\omega_0^2 - \omega^2\right)\text{Den}[x]$ inside the integral using the extended form for $\text{Den}[x]$ and $\text{Den}[\omega]$:

$$\left(\omega_0^2-x^2\right)\text{Den}[\omega] - \left(\omega_0^2-\omega^2\right)\text{Den}[x] =$$

$$\left(\omega_0^2-x^2\right)\left[\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2\right] - \left(\omega_0^2-\omega^2\right)\left[\left(\omega_0^2-x^2\right)^2+\gamma^2x^2\right] =$$

$$\left(\omega_0^2-x^2\right)\left[\omega_0^4+\omega^4-2\omega_0^2\omega^2+\gamma^2\omega^2\right] - \left(\omega_0^2-\omega^2\right)\left[\omega_0^4+x^4-2\omega_0^2x^2+\gamma^2x^2\right] =$$

$$\cancel{+\omega_0^6} + \omega^4\omega_0^2 - 2\omega_0^4\omega^2 + \gamma^2\omega^2\omega_0^2 - x^2\omega_0^4 - x^2\omega^4 \cancel{+2x^2\omega_0^2\omega^2} \cancel{-\gamma^2x^2\omega^2} +$$

$$\cancel{-\omega_0^6} - x^4\omega_0^2 + 2\omega_0^4x^2 - \gamma^2x^2\omega_0^2 + \omega^2\omega_0^4 + x^4\omega^2 \cancel{-2\omega_0^2x^2\omega^2} \cancel{+\gamma^2x^2\omega^2} =$$

$$-\omega_0^2\left(x^4-\omega^4\right) + 2\omega_0^4\left(x^2-\omega^2\right) - \gamma^2\omega_0^2\left(x^2-\omega^2\right) - \omega_0^4\left(x^2-\omega^2\right) + x^2\omega^2\left(x^2-\omega^2\right) =$$

$$-\omega_0^2\left(x^2-\omega^2\right)\left(x^2+\omega^2\right) + 2\omega_0^4\left(x^2-\omega^2\right) - \gamma^2\omega_0^2\left(x^2-\omega^2\right) - \omega_0^4\left(x^2-\omega^2\right) + x^2\omega^2\left(x^2-\omega^2\right) =$$

$$\left(x^2-\omega^2\right)\left[-\omega_0^2\left(x^2+\omega^2\right) - \gamma^2\omega_0^2 + \omega_0^4 + x^2\omega^2\right]$$

and substitute it inside the integral:

$$\chi_i(\omega) = -\frac{2\omega\omega_p^2}{\pi}\int_0^\infty \frac{1}{x^2-\omega^2}\left[\frac{\left(x^2-\omega^2\right)\left[-\omega_0^2\left(x^2+\omega^2\right)-\gamma^2\omega_0^2+\omega_0^4+x^2\omega^2\right]}{\text{Den}[x]\text{Den}[\omega]}\right]dx =$$

$$= -\frac{\omega\omega_p^2}{\text{Den}[\omega]}\frac{2}{\pi}\int_0^\infty \frac{-\omega_0^2\left(x^2+\omega^2\right)-\gamma^2\omega_0^2+\omega_0^4+x^2\omega^2}{\text{Den}[x]}dx =$$

$$= -\frac{\omega\omega_p^2}{\text{Den}[\omega]}\left[\frac{2}{\pi}\int_0^\infty\left(-\omega_0^2\frac{x^2}{\text{Den}[x]}-\omega_0^2\omega^2\frac{1}{\text{Den}[x]}-\gamma^2\omega_0^2\frac{1}{\text{Den}[x]}+\omega_0^4\frac{1}{\text{Den}[x]}+\omega^2\frac{x^2}{\text{Den}[x]}\right)dx\right] =$$

$$= -\frac{\omega\omega_p^2}{\text{Den}[\omega]}\left[-\frac{\omega_0^2}{\gamma}-\frac{\omega_0^2\omega^2}{\gamma\omega_0^2}-\frac{\gamma^2\omega_0^2}{\gamma\omega_0^2}+\frac{\omega_0^4}{\gamma\omega_0^2}+\frac{\omega^2}{\gamma}\right] =$$

$$= \frac{\omega\omega_p^2}{\text{Den}[\omega]}\left[\frac{\omega_0^2}{\gamma}+\frac{\omega^2}{\gamma}+\frac{\gamma^2}{\gamma}-\frac{\omega_0^2}{\gamma}-\frac{\omega^2}{\gamma}\right] = \frac{\gamma\omega\omega_p^2}{\text{Den}[\omega]} = \frac{\gamma\omega\omega_p^2}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$$

that is exactly the expression of $\chi_i(\omega)$ for a simple oscillator model of dielectric polarization in eq. (1.11.5).

- Question n° 5

Let us compare the first integrals in x and y respectively:

$$\frac{2}{\pi}\int_0^\infty \frac{dx}{\left(\omega_0^2-x^2\right)^2+\gamma^2x^2} \equiv \frac{2}{\pi}\int_0^\infty \frac{dy}{1-2y^2\cos\theta+y^4}$$

They differ only in the denominator, so we can compare them to find the condition of equality, substituting $y \to x$:

$$\left(\omega_0^2-x^2\right)^2+\gamma^2x^2 = 1-2x^2\text{C}\text{os}\,\theta+x^4$$
$$\omega_0^4+x^4-2\omega_0^2x^2+\gamma^2x^2 = 1-2x^2\text{C}\text{os}\,\theta+x^4$$

that is:

$$\omega_0^4-2x^2\left(\omega_0^2-\frac{\gamma^2}{2}\right)+x^4 = 1-2x^2\text{C}\text{os}\,\theta+x^4 \tag{1.11.9}$$

Using the suggested relation $\text{Sin}(\theta/2) = \gamma/(2\omega_0)$ and the relation $\text{C}\text{os}\,\theta = 1-2\text{Sin}(\theta/2)$, we have:

$$\text{C}\text{os}\,\theta = 1-2\text{Sin}^2\left(\frac{\theta}{2}\right) = 1-2\frac{\gamma^2}{4\omega_0^2} = 1-\frac{\gamma^2}{2\omega_0^2} \tag{1.11.10}$$

Now it is possible substitute eq. (1.11.10) in eq. (1.11.9):

$$\omega_0^4-2\omega_0^2x^2\text{C}\text{os}\,\theta+x^4 = 1-2x^2\text{C}\text{os}\,\theta+x^4 \tag{1.11.11}$$

In order to match the right– and left–side of eq. (1.11.11), it is necessary that $\omega_0 \to 1$ and it is correct because the integrals in y are a reduced form of the integrals in x. Indeed the result of integrals in y can be written as follow:

$$\frac{1}{\sqrt{2(1-\mathrm{C}\,\mathrm{os}\,\theta)}} = \frac{1}{\sqrt{2\left(1-\left(1-\frac{\gamma^2}{4\omega_0^2}\right)\right)}} = \frac{1}{\sqrt{\frac{\gamma^2}{2\omega_0^2}}} = \left.\frac{1}{\gamma}\right|_{\omega_0 \to 1}$$

that is exactly the result of integrals in x when $\omega_0 \to 1$.

- Question n° 6

Starting from the expression of $\chi(\omega)$ as in eq. (1.11.5) and applying the limit as $\gamma \to 0$ we get:

$$\begin{aligned}\chi^0(\omega) &= \lim_{\gamma\to 0}\chi(\omega) = \lim_{\gamma\to 0}\frac{\omega_p^2}{\omega_0^2-\omega^2+\mathrm{j}\gamma\omega} = \\ &= \lim_{\gamma\to 0}\frac{\left(\omega_0^2-\omega^2\right)^2}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\frac{\omega_p^2}{\omega_0^2-\omega^2} - \mathrm{j}\lim_{\gamma\to 0}\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\omega_p^2\end{aligned} \tag{1.11.12}$$

$\chi_r^0(\omega)$ and $\chi_i^0(\omega)$ represent the real and imaginary part of $\chi^0(\omega)$ for $\gamma \to 0$, respectively.

It is easy to note that the real part converges as:

$$\begin{aligned}\chi_r^0(\omega) &= \lim_{\gamma\to 0}\frac{\left(\omega_0^2-\omega^2\right)^2}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\frac{\omega_p^2}{\omega_0^2-\omega^2} = \\ &= \wp\frac{\omega_p^2}{\omega_0^2-\omega^2} = \frac{1}{2\omega_0}\left[\wp\frac{\omega_p^2}{\omega_0-\omega} - \wp\frac{\omega_p^2}{\omega_0+\omega}\right]\end{aligned} \tag{1.11.13}$$

For what concerns $\chi_i^0(\omega) = \omega_p^2 \lim_{\gamma\to 0}\frac{\gamma\omega}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$ we have to note that it is very similar to the following definition of the Dirac delta function:

$$\delta(\mathrm{x}) = \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\frac{\varepsilon}{\mathrm{x}^2+\varepsilon^2} \tag{1.11.14}$$

So, we first manipulate $\chi_i^0(\omega)$ in order to apply eq. (1.11.14). Dividing and multiplying numerator and denominator by ω^2 and π respectively, we have:

$$\chi_i^0(\omega) = \omega_p^2 \lim_{\gamma \to 0} \frac{\dfrac{\gamma \not{\omega}}{\omega^{\not{2}}}}{\dfrac{\left(\omega_0^2 - \omega^2\right)^2}{\omega^2} + \dfrac{\gamma^2 \not{\omega}^{\not{2}}}{\not{\omega}^{\not{2}}}} =$$
$$= \frac{\pi \omega_p^2}{\omega} \lim_{\gamma \to 0} \frac{1}{\pi} \frac{\gamma}{\left(\dfrac{\omega_0^2 - \omega^2}{\omega}\right)^2 + \gamma^2} \tag{1.11.15}$$

The limit $\gamma \to 0$ of eq. (1.11.15) has the same form of eq. (1.11.14), assuming $\varepsilon \to \gamma$ and $x \to \dfrac{\omega_0^2 - \omega^2}{\omega}$. So it is possible to write:

$$\chi_i^0(\omega) = \frac{\pi \omega_p^2}{\omega} \delta\left(\frac{\omega_0^2 - \omega^2}{\omega}\right) \tag{1.11.16}$$

In order to write (1.11.16) in a simpler form, we apply the following two properties of the Dirac delta function:

1. consider a function $f(x)$ with n zeros:

$$\begin{cases} f(x) = 0 \\ f'(x) \neq 0 \end{cases} \quad \text{in } x_i = x_1,\ x_2,\ \ldots,\ x_n$$

   then

$$\delta(f(x)) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{\left|f'(x_i)\right|} \tag{1.11.17}$$

   As an example, if we consider $f(x) = x^2 - a^2$, we have:

$$\delta\left(x^2 - a^2\right) = \frac{\delta(x-a)}{\left|\dfrac{d}{dx}\left(x^2 - a^2\right)\right|_{x=a}} + \frac{\delta(x+a)}{\left|\dfrac{d}{dx}\left(x^2 - a^2\right)\right|_{x=-a}} =$$
$$= \frac{1}{2|a|}\delta(x-a) + \frac{1}{2|-a|}\delta(x+a) = \frac{1}{2|a|}\left[\delta(x-a) + \delta(x+a)\right]$$

2. Consider a function $g(x)$:

$$g(x)\delta(x - x_0) = g(x_0)\delta(x - x_0) \tag{1.11.18}$$

In order to apply (1.11.17) to (1.11.16), first of all we have to evaluate the zeros of the function $f(\omega) = \dfrac{\omega_0^2 - \omega^2}{\omega}$:

$$f(\omega) = 0 \quad \Leftrightarrow \quad \omega = \{+\omega_0, -\omega_0\} \tag{1.11.19}$$

and its first derivate respect to ω :

$$\begin{aligned} f'(\omega) &= \frac{d}{d\omega} f(\omega) = \frac{d}{d\omega}\left(\frac{\omega_0^2-\omega^2}{\omega}\right) = \left(\omega_0^2-\omega^2\right)\frac{d(1/\omega)}{d\omega} + \left(\frac{1}{\omega}\right)\frac{d\left(\omega_0^2-\omega^2\right)}{d\omega} = \\ &= -\left(\frac{\omega_0^2-\omega^2}{\omega^2}\right) - \left(\frac{1}{\cancel{\omega}}\right)2\cancel{\omega} = -2-\frac{\omega_0^2-\omega^2}{\omega^2} \end{aligned} \tag{1.11.20}$$

Now it is possible to write:

$$\begin{aligned} \chi_i^0(\omega) &= \frac{\pi\omega_p^2}{\omega}\delta\left(\frac{\omega_0^2-\omega^2}{\omega}\right) = \\ &= \frac{\pi\omega_p^2}{\omega}\sum_{i=1}^{2}\frac{\delta(\omega-\omega_i)}{\left|f'(\omega_i)\right|} = \\ &= \frac{\pi\omega_p^2}{\omega}\left[\frac{\delta(\omega-\omega_0)}{\left|-2-\cancel{\frac{\omega_0^2-\omega_0^2}{\omega_0^2}}\right|} + \frac{\delta(\omega+\omega_0)}{\left|-2-\cancel{\frac{\omega_0^2-\omega_0^2}{\omega_0^2}}\right|}\right] = \\ &= \frac{\pi\omega_p^2}{2\omega}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \end{aligned} \tag{1.11.21}$$

Let us, then, apply the second property of the Dirac delta function (1.11.18):

$$\begin{aligned} \chi_i^0(\omega) &= \frac{\pi\omega_p^2}{2}\left[\frac{1}{\omega}\delta(\omega-\omega_0)+\frac{1}{\omega}\delta(\omega+\omega_0)\right] = \\ &= \frac{\pi\omega_p^2}{2}\left[\frac{1}{\omega_0}\delta(\omega-\omega_0)-\frac{1}{\omega_0}\delta(\omega+\omega_0)\right] = \\ &= \frac{\pi\omega_p^2}{2\omega_0}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right] \end{aligned} \tag{1.11.22}$$

This expression still satisfies the Kramers–Kronig relations. Substitute the quantity $\chi_i^0(\omega)$ expressed in eq. (1.11.22) inside the modified Kramers–Kronig relationship (1.11.4) for $\chi_r(\omega)$:

$$\chi_r(\omega) = \frac{2}{\pi}\int_0^{\infty}\frac{x\chi_i^0(x)-\omega\chi_i^0(\omega)}{x^2-\omega^2}dx$$

where we have substituted $\omega' \rightarrow x$ to distinguish better the arguments of the integral. So:

$$\chi_r(\omega) = \frac{2}{\pi}\frac{\pi\omega_p^2}{2\omega_0}\int_0^\infty \frac{1}{x^2-\omega^2}\left[x\left(\delta(x-\omega_0)-\delta(x+\omega_0)\right)-\omega\left(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right)\right]dx =$$

$$= \frac{\omega_p^2}{\omega_0}\left[\begin{array}{l}\int_0^\infty \frac{x}{x^2-\omega^2}\delta(x-\omega_0)dx - \int_0^\infty \frac{x}{x^2-\omega^2}\delta(x+\omega_0)dx - \\ +\int_0^\infty \frac{\omega}{x^2-\omega^2}\delta(\omega-\omega_0)dx + \int_0^\infty \frac{\omega}{x^2-\omega^2}\delta(\omega+\omega_0)dx\end{array}\right] = \frac{\omega_p^2}{\omega_0}\left[I_1+I_2+I_3+I_4\right]$$

The integrals I_3 and I_4 vanishe as demonstrated in (1.11.6). Also the integral I_2 vanishes because it is over the real–positive values of x while the Dirac delta function is always zero on this interval (it is not zero only for $x=-\omega_0$, which is real negative). So we can write:

$$\chi_r(\omega) = \wp\frac{\omega_p^2}{\omega_0}\frac{\omega_0}{\omega_0{}^2-\omega^2} = \wp\frac{\omega_p^2}{\omega_0{}^2-\omega^2} = \chi_r^0(\omega) \qquad (1.11.23)$$

The Kramers–Kronig relationship (1.11.4) for $\chi_i(\omega)$ with $\chi_r(\omega)=\chi_r^0(\omega)$ can be written using the definition of $\chi_r^0(\omega)$ as in eq. (1.11.12):

$$\chi_r^0(\omega) = \lim_{\gamma\to 0}\frac{\omega_p^2\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}$$

So:

$$\chi_i^0(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{\omega}{x^2-\omega^2}\lim_{\gamma\to 0}\left[\frac{\omega_p^2\left(\omega_0^2-x^2\right)}{\left(\omega_0^2-x^2\right)^2+\gamma^2x^2} - \frac{\omega_p^2\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\right]dx$$

and, exchanging the limit with the integral, we can write:

$$\chi_i^0(\omega) = \lim_{\gamma\to 0}\left\{-\frac{2}{\pi}\int_0^\infty \frac{\omega}{x^2-\omega^2}\left[\frac{\omega_p^2\left(\omega_0^2-x^2\right)}{\left(\omega_0^2-x^2\right)^2+\gamma^2x^2} - \frac{\omega_p^2\left(\omega_0^2-\omega^2\right)}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\right]dx\right\}$$

Now it is easy to note that the argument of the limit has already been solved in the solution of question n°4 of this exercise resulting in $\chi_i(\omega)$. Then, the limit $\gamma\to 0$ to $\chi_i(\omega)$ should be applied, which is given by eq. (1.11.14).

1.12 Exercise

Derive the Kramers–Kronig relationship:

$$\chi(\omega) = \frac{1}{\pi j} \wp \int_{-\infty}^{+\infty} \frac{\chi(\omega')}{\omega - \omega'} d\omega' \tag{1.12.1}$$

by starting with the causality condition $\chi(t)u(-t) = 0$ and translating it to the frequency domain, that is, expressing it as the convolution of the Fourier transforms of $\chi(t)$ and $u(-t)$.

Solution

The convolution is a mathematical operation on two functions f and g, producing a third function that is typically viewed as a modified version of one of the original functions. In signal theory, it represents the transformation obtained when a signal passes through a black–box system with a known impulse response. In similar way, in frequency domain the output of the system is the product of the Fourier transformations of the input signal and the impulse response. So the convolution in time domain is also the corresponding operation of the product in frequency domain and vice versa. It is defined as:

$$(f * g)(t) \triangleq \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) d\tau = \int_{-\infty}^{+\infty} g(\tau) f(t-\tau) d\tau$$

Using the causality condition $\chi(t) = \chi(t)u(t)$, we have that

$$\chi(\omega) = \Im\{\chi(t)\} = \Im\{\chi(t)u(t)\} = \Im\{\chi(t)\} * \Im\{u(t)\} = \frac{1}{2\pi}\chi(\omega) * U(\omega)$$

where $\chi(\omega)$ and $U(\omega)$ are the Fourier transforms of $\chi(t)$ and $u(t)$, respectively.

The Fourier transformation of Heaviside step function is:

$$U(\omega) = \wp \frac{1}{j\omega} + \pi\delta(\omega)$$

So we have:

$$\begin{aligned}
\chi(\omega) &= \frac{1}{2\pi} \wp \int_{-\infty}^{+\infty} \chi(\omega') U(\omega - \omega') d\omega' \\
&= \frac{1}{2\pi} \wp \int_{-\infty}^{+\infty} \chi(\omega') \frac{1}{j(\omega - \omega')} d\omega' + \frac{1}{2\cancel{\pi}} \cancel{\pi} \int_{-\infty}^{+\infty} \chi(\omega') \delta(\omega - \omega') d\omega' = \\
&= \frac{1}{2\pi j} \wp \int_{-\infty}^{+\infty} \chi(\omega') \frac{1}{(\omega - \omega')} d\omega' + \frac{1}{2}\chi(\omega)
\end{aligned}$$

Rearranging terms and canceling a factor of $1/2$, we obtain the Kramers–Kronig relation in its complex–value form:

$$\chi(\omega) = \frac{1}{\pi \mathrm{j}} \wp \int_{-\infty}^{+\infty} \frac{\chi(\omega')}{(\omega - \omega')} \mathrm{d}\omega' .$$

1.13 Exercise

An isotropic homogeneous lossless dielectric medium is moving with uniform velocity $\mathbf{v}$ with respect to a fixed coordinate frame S. In the frame S' moving with dielectric, the constitutive relations are assumed to be the usual ones, that is, $\mathbf{D}' = \varepsilon\mathbf{E}'$ and $\mathbf{B}' = \mu\mathbf{H}'$. Using the Lorentz transformations:

$$\boxed{\begin{aligned} \mathbf{E}'_{\perp} &= \gamma\left(\mathbf{E}_{\perp} + c\boldsymbol{\beta}\times\mathbf{B}_{\perp}\right) \\ \mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B}_{\perp} - \frac{1}{c}\boldsymbol{\beta}\times\mathbf{E}_{\perp}\right) \\ \mathbf{E}'_{//} &= \mathbf{E}_{//} \\ \mathbf{B}'_{//} &= \mathbf{B}_{//} \end{aligned}} \qquad \boxed{\begin{aligned} \mathbf{H}'_{\perp} &= \gamma\left(\mathbf{H}_{\perp} - c\boldsymbol{\beta}\times\mathbf{D}_{\perp}\right) \\ \mathbf{D}'_{\perp} &= \gamma\left(\mathbf{D}_{\perp} + \frac{1}{c}\boldsymbol{\beta}\times\mathbf{H}_{\perp}\right) \\ \mathbf{H}'_{//} &= \mathbf{H}_{//} \\ \mathbf{D}'_{//} &= \mathbf{D}_{//} \end{aligned}} \tag{1.13.1}$$

where $c\boldsymbol{\beta} = \mathbf{v}$, $\boldsymbol{\beta}/c = \mathbf{v}/c^2$ and $\gamma = 1\big/\sqrt{1-|\boldsymbol{\beta}|^2}$, show that the constitutive relations take the following form in the fixed frame S:

$$\mathbf{D} = \varepsilon\mathbf{E} + a\mathbf{v}\times\left(\mathbf{H} - \varepsilon\mathbf{v}\times\mathbf{E}\right) \tag{1.13.2}$$

$$\mathbf{B} = \mu\mathbf{H} - a\mathbf{v}\times\left(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}\right) \tag{1.13.3}$$

where $a = \dfrac{\varepsilon\mu - \varepsilon_0\mu_0}{1-\varepsilon\mu v^2}$.

Solution

It is possible to express the constitutive relations $\mathbf{D}' = \varepsilon\mathbf{E}'$ and $\mathbf{B}' = \mu\mathbf{H}'$ as follow:

$$\begin{aligned} \mathbf{D}' &= \mathbf{D}'_{\perp} + \mathbf{D}'_{//} = \varepsilon\left(\mathbf{E}'_{\perp} + \mathbf{E}'_{//}\right) \\ \mathbf{B}' &= \mathbf{B}'_{\perp} + \mathbf{B}'_{//} = \mu\left(\mathbf{H}'_{\perp} + \mathbf{H}'_{//}\right) \end{aligned} \tag{1.13.4}$$

where the subscripts $\perp$ and // indicate the component perpendicular and parallel at the velocity vector $\mathbf{v}$.

Considering the first equation of set (1.13.4), we can substitute to all of component with the superscript with the correspondent definition given by the Lorentz transformation:

$$\begin{aligned}
&\mathbf{D}'_{\perp}+\mathbf{D}'_{//}=\varepsilon\left(\mathbf{E}'_{\perp}+\mathbf{E}'_{//}\right)\\
&\gamma\left(\mathbf{D}_{\perp}+\frac{1}{c}\boldsymbol{\beta}\times\mathbf{H}_{\perp}\right)+\mathbf{D}_{//}=\varepsilon\left[\gamma\left(\mathbf{E}_{\perp}+c\boldsymbol{\beta}\times\mathbf{B}_{\perp}\right)+\mathbf{E}_{//}\right]\\
&\gamma\mathbf{D}_{\perp}+\frac{\gamma}{c}\boldsymbol{\beta}\times\mathbf{H}_{\perp}+\mathbf{D}_{//}=\varepsilon\gamma\left(\mathbf{E}_{\perp}+c\boldsymbol{\beta}\times\mathbf{B}_{\perp}\right)+\varepsilon\mathbf{E}_{//}\\
&\gamma\mathbf{D}_{\perp}+\frac{\gamma}{c}\boldsymbol{\beta}\times\mathbf{H}_{\perp}+\mathbf{D}_{//}=\varepsilon\gamma\left(\mathbf{E}_{\perp}+c\boldsymbol{\beta}\times\mathbf{B}_{\perp}\right)+\varepsilon\mathbf{E}_{//}
\end{aligned}\tag{1.13.5}$$

Now it is possible to substitute $c\boldsymbol{\beta}=\mathbf{v}$ and $\boldsymbol{\beta}/c=\mathbf{v}/c^2$ and separate the component parallel and perpendicular at the velocity vector $\mathbf{v}$:

$$\begin{cases}\mathbf{D}_{\perp}+\dfrac{1}{c^2}\mathbf{v}\times\mathbf{H}_{\perp}=\varepsilon\mathbf{E}_{\perp}+\varepsilon\mathbf{v}\times\mathbf{B}_{\perp} & (\perp-\text{component})\\ \mathbf{D}_{//}=\varepsilon\mathbf{E}_{//} & (//-\text{component})\end{cases}\tag{1.13.6}$$

The relation between the parallel components of vector **D** and vector **E** is the same in the two frame S and S'. On the contrary, the perpendicular component depends on both electric and magnetic field. In the fixed frame S the constitutive relation $\mathbf{B}=\mu\mathbf{H}$ is not valid, so we have to evaluate it using the set (1.13.1) and the constitutive relation $\mathbf{B}'=\mu\mathbf{H}'$ in the frame S' where it is valid. Considering only the perpendicular component, we have:

$$\begin{aligned}
&\mathbf{B}'_{\perp}=\mu\mathbf{H}'_{\perp}\\
&\not{\gamma}\left(\mathbf{B}_{\perp}-\frac{1}{c}\boldsymbol{\beta}\times\mathbf{E}_{\perp}\right)=\mu\not{\gamma}\left(\mathbf{H}_{\perp}-c\boldsymbol{\beta}\times\mathbf{D}_{\perp}\right)\\
&\mathbf{B}_{\perp}-\frac{1}{c}\boldsymbol{\beta}\times\mathbf{E}_{\perp}=\mu\mathbf{H}_{\perp}-\mu c\boldsymbol{\beta}\times\mathbf{D}_{\perp}\\
&\mathbf{B}_{\perp}=\mu\mathbf{H}_{\perp}+\frac{1}{c}\boldsymbol{\beta}\times\mathbf{E}_{\perp}-\mu c\boldsymbol{\beta}\times\mathbf{D}_{\perp}
\end{aligned}$$

and express it as function of velocity **v**:

$$\mathbf{B}_{\perp}=\mu\mathbf{H}_{\perp}+\frac{1}{c^2}\mathbf{v}\times\mathbf{E}_{\perp}-\mu\mathbf{v}\times\mathbf{D}_{\perp}\tag{1.13.7}$$

Now we can substitute eq. (1.13.7) in eq. (1.13.6) and obtain:

$$\mathbf{D}_{\perp}+\frac{1}{c^2}\mathbf{v}\times\mathbf{H}_{\perp}=\varepsilon\mathbf{E}_{\perp}+\varepsilon\mathbf{v}\times\left(\mu\mathbf{H}_{\perp}+\frac{1}{c^2}\mathbf{v}\times\mathbf{E}_{\perp}-\mu\mathbf{v}\times\mathbf{D}_{\perp}\right)$$

$$\mathbf{D}_{\perp}=\varepsilon\mathbf{E}_{\perp}-\frac{1}{c^2}\mathbf{v}\times\mathbf{H}_{\perp}+\varepsilon\mu\mathbf{v}\times\mathbf{H}_{\perp}+\frac{\varepsilon}{c^2}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_{\perp}-\varepsilon\mu\mathbf{v}\times\mathbf{v}\times\mathbf{D}_{\perp}$$

$$\mathbf{D}_{\perp}+\varepsilon\mu\mathbf{v}\times\mathbf{v}\times\mathbf{D}_{\perp}=\varepsilon\mathbf{E}_{\perp}+\left(\varepsilon\mu-\frac{1}{c^2}\right)\mathbf{v}\times\mathbf{H}_{\perp}+\frac{\varepsilon}{c^2}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_{\perp}$$

The double cross product $\mathbf{v}\times\mathbf{v}\times\mathbf{D}_\perp$ can be evaluated using the BAC-CAB rule:

$$\mathbf{v}\times\mathbf{v}\times\mathbf{D}_\perp = \mathbf{v}\left(\mathbf{v}\cdot\mathbf{D}_\perp\right) - \mathbf{D}_\perp\left(\mathbf{v}\cdot\mathbf{v}\right) = 0 - \mathrm{v}^2\mathbf{D}_\perp = -\mathrm{v}^2\mathbf{D}_\perp$$

so

$$\mathbf{D}_\perp - \varepsilon\mu\mathrm{v}^2\mathbf{D}_\perp = \varepsilon\mathbf{E}_\perp + \left(\varepsilon\mu - \frac{1}{\mathrm{c}^2}\right)\mathbf{v}\times\mathbf{H}_\perp + \frac{\varepsilon}{\mathrm{c}^2}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp$$

$$\left(1-\varepsilon\mu\mathrm{v}^2\right)\mathbf{D}_\perp = \varepsilon\mathbf{E}_\perp + \left(\varepsilon\mu - \varepsilon_0\mu_0\right)\mathbf{v}\times\mathbf{H}_\perp + \frac{\varepsilon}{\mathrm{c}^2}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp \qquad (1.13.8)$$

$$\mathbf{D}_\perp = \frac{\varepsilon}{\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{E}_\perp + \frac{\left(\varepsilon\mu - \varepsilon_0\mu_0\right)}{\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{v}\times\mathbf{H}_\perp + \frac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp$$

It is easy to note that the term $\mathbf{v}\times\mathbf{H}_\perp$ is multiplied by the coefficient a, so:

$$\mathbf{D}_\perp = \frac{\varepsilon}{\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{E}_\perp + \mathrm{a}\mathbf{v}\times\mathbf{H}_\perp + \frac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp \qquad (1.13.9)$$

Comparing eq. (1.13.9) and eq. (1.13.2), we can note that the coefficient of $\mathbf{E}_\perp$ and $\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp$ are different. It is possible to think that probably we have to add and subtract a unknown quantity. To find it, we will compare the our expression and the expression suggested in eq. (1.13.2), that is:

$$\begin{cases} \dfrac{\varepsilon}{\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{E}_\perp + \dfrac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp \\ \varepsilon\mathbf{E}_\perp - \varepsilon\dfrac{\varepsilon\mu - \varepsilon_0\mu_0}{\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp \end{cases}$$

where we already substituted $\dfrac{\varepsilon\mu - \varepsilon_0\mu_0}{\left(1-\varepsilon\mu\mathrm{v}^2\right)} = \mathrm{a}$.

Now we can identify with X the unknown quantity and we can impose that:

$$\begin{cases} \dfrac{\varepsilon}{\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{E}_\perp + \mathrm{X}\mathbf{E}_\perp = \varepsilon\mathbf{E}_\perp \\ \dfrac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp + \mathrm{X}\mathbf{E}_\perp = -\varepsilon\dfrac{\varepsilon\mu - \varepsilon_0\mu_0}{\left(1-\varepsilon\mu\mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp \end{cases}$$

which can be simplified as:

$$\begin{cases} \dfrac{\varepsilon}{\left(1-\varepsilon\mu \mathrm{v}^2\right)} + \mathrm{X} = \varepsilon \\ -\dfrac{\varepsilon \mathrm{v}^2}{\mathrm{c}^2\left(1-\varepsilon\mu \mathrm{v}^2\right)} + \mathrm{X} = +\varepsilon \mathrm{v}^2 \dfrac{\varepsilon\mu - \varepsilon_0\mu_0}{\left(1-\varepsilon\mu \mathrm{v}^2\right)} \end{cases} \tag{1.13.10}$$

where in the second equation is present the term $-\mathrm{v}^2$ because of $\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp = -\mathrm{v}^2\mathbf{E}_\perp$.

From the first of (1.13.10), we obtain that:

$$\mathrm{X} = \varepsilon - \frac{\varepsilon}{\left(1-\varepsilon\mu \mathrm{v}^2\right)} = \varepsilon\frac{\cancel{+1} - \varepsilon\mu \mathrm{v}^2 \cancel{-1}}{\left(1-\varepsilon\mu \mathrm{v}^2\right)} = -\varepsilon\frac{\varepsilon\mu \mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)} \tag{1.13.11}$$

and from the second:

$$\begin{aligned} \mathrm{X} &= \frac{\varepsilon \mathrm{v}^2}{\mathrm{c}^2\left(1-\varepsilon\mu \mathrm{v}^2\right)} \varepsilon \mathrm{v}^2 \frac{\varepsilon\mu - \varepsilon_0\mu_0}{\left(1-\varepsilon\mu \mathrm{v}^2\right)} = \frac{\varepsilon \mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\left[\frac{1}{\mathrm{c}^2} + \varepsilon\mu - \varepsilon_0\mu_0\right] = \\ &= \varepsilon\frac{\mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\left[\cancel{\varepsilon_0\mu_0} + \varepsilon\mu \cancel{-\varepsilon_0\mu_0}\right] = \varepsilon\frac{\varepsilon\mu \mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)} \end{aligned} \tag{1.13.12}$$

It is easy to note that X has the same module, but opposite sign. This confirms our argument of finding a quantity to add and subtract at eq. (1.13.9). So finally we can write:

$$\begin{aligned} \mathbf{D}_\perp &= \frac{\varepsilon}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp + \mathrm{a}\mathbf{v}\times\mathbf{H}_\perp + \frac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp + \mathrm{X}\mathbf{E}_\perp - \mathrm{X}\mathbf{E}_\perp = \\ &= \frac{\varepsilon}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp + \mathrm{a}\mathbf{v}\times\mathbf{H}_\perp + \frac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp + \varepsilon\frac{\varepsilon\mu \mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp - \varepsilon\frac{\varepsilon\mu \mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp \end{aligned} \tag{1.13.13}$$

Now we can simplify eq. (1.13.13) as follow:

$$\frac{\varepsilon}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp - \varepsilon\frac{\varepsilon\mu \mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp = \frac{\varepsilon}{\cancel{\left(1-\varepsilon\mu \mathrm{v}^2\right)}}\cancel{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp = \varepsilon\mathbf{E}_\perp \tag{1.13.14}$$

$$\begin{aligned} &\frac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp + \varepsilon\frac{\varepsilon\mu \mathrm{v}^2}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{E}_\perp = \\ &\frac{\varepsilon}{\mathrm{c}^2\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp - \varepsilon\frac{\varepsilon\mu}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp = \\ &\frac{\varepsilon}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\left(\frac{1}{\mathrm{c}^2} - \varepsilon\mu\right)\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp \\ &= \varepsilon\frac{\varepsilon_0\mu_0 - \varepsilon\mu}{\left(1-\varepsilon\mu \mathrm{v}^2\right)}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp = -\varepsilon \mathrm{a}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_\perp \end{aligned} \tag{1.13.15}$$

Substituting (1.13.14) and (1.13.15) in eq. (1.13.13), we obtain:

$$\mathbf{D}_{\perp} = \varepsilon\mathbf{E}_{\perp} + \mathrm{a}\mathbf{v}\times\mathbf{H}_{\perp} - \varepsilon\mathrm{a}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_{\perp} \tag{1.13.16}$$

Now using the second equation of (1.13.6) and eq. (1.13.16), it is possible write the expression of vector **D** in the fixed frame S:

$$\begin{aligned}\mathbf{D} &= \mathbf{D}_{\perp} + \mathbf{D}_{//}\\ &= \varepsilon\mathbf{E}_{\perp} + \mathrm{a}\mathbf{v}\times\mathbf{H}_{\perp} - \varepsilon\mathrm{a}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_{\perp} + \varepsilon\mathbf{E}_{//} =\\ &= \varepsilon\left(\mathbf{E}_{\perp} + \mathbf{E}_{//}\right) + \mathrm{a}\mathbf{v}\times\mathbf{H}_{\perp} - \varepsilon\mathrm{a}\mathbf{v}\times\mathbf{v}\times\mathbf{E}_{\perp} + \underbrace{\mathrm{a}\mathbf{v}\times\mathbf{H}_{//}}_{\text{null}} - \underbrace{\varepsilon\mathrm{a}\,\mathbf{v}\times\mathbf{v}\times\mathbf{E}_{//}}_{\text{null}} =\\ &= \varepsilon\mathbf{E} + \mathrm{a}\mathbf{v}\times\mathbf{H} - \varepsilon\mathrm{a}\mathbf{v}\times\mathbf{v}\times\mathbf{E} = \varepsilon\mathbf{E} + \mathrm{a}\mathbf{v}\times\left(\mathbf{H} - \varepsilon\mathbf{v}\times\mathbf{E}\right)\end{aligned} \tag{1.13.17}$$

It is easy to note that eq. (1.13.17) and eq. (1.13.2) are identical. In the same way, we can demonstrate eq. (1.13.3), but we kwon that the relations are dual and we can obtain eq. (1.13.3) just operating a changing of variables as follow:

$$\begin{cases}\mathbf{D}\to\mathbf{B}\\ \mathbf{E}\to\mathbf{H}\\ \mathbf{v}\times\mathbf{H}\to -\mathbf{v}\times\mathbf{E}\\ \varepsilon\to\mu\end{cases}$$

So:

$$\mathbf{D} = \varepsilon\mathbf{E} + \mathrm{a}\mathbf{v}\times\left(\mathbf{H} - \varepsilon\mathbf{v}\times\mathbf{E}\right) \quad\to\quad \mathbf{B} = \mu\mathbf{H} - \mathrm{a}\mathbf{v}\times\left(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}\right) \tag{1.13.18}$$

Electromagnetic Waves and Antennas

Exercise book
Chapter 2: Uniform Plane Waves

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2.1 Exercise

A function $E(z,t)$ may be thought of as a function $E(\zeta,\xi)$ of the independent variables $\zeta = z - ct$ and $\xi = z + ct$. Show that the wave equation:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(z,t) = 0 \tag{2.1.1}$$

and the forward-backward equations:

$$\begin{aligned}\frac{\partial \mathbf{E}_+}{\partial z} &= -\frac{1}{c}\frac{\partial \mathbf{E}_+}{\partial t}\\ \frac{\partial \mathbf{E}_-}{\partial z} &= +\frac{1}{c}\frac{\partial \mathbf{E}_-}{\partial t}\end{aligned} \tag{2.1.2}$$

become in these variables:

$$\frac{\partial^2 \mathbf{E}}{\partial \zeta \partial \xi} = 0, \quad \frac{\partial \mathbf{E}_+}{\partial \xi} = 0, \quad \frac{\partial \mathbf{E}_-}{\partial \zeta} = 0 \tag{2.1.3}$$

Thus, $\mathbf{E}_+$ may depend only on ζ and $\mathbf{E}_-$ only on ξ.

Solution

First of all, we have to evaluate the derivates:

$$\begin{cases} \frac{\partial\zeta}{\partial z} = \frac{\partial}{\partial z}(z-ct) = 1 \quad \Rightarrow \quad \partial\zeta = \partial z & \text{(a)} \\ \frac{\partial\zeta}{\partial t} = \frac{\partial}{\partial t}(z-ct) = -c \quad \Rightarrow \quad \partial\zeta = -c\partial t & \text{(b)} \\ \frac{\partial\xi}{\partial z} = \frac{\partial}{\partial z}(z+ct) = 1 \quad \Rightarrow \quad \partial\xi = \partial z & \text{(c)} \\ \frac{\partial\xi}{\partial t} = \frac{\partial}{\partial t}(z+ct) = +c \quad \Rightarrow \quad \partial\xi = +c\partial t & \text{(d)} \end{cases} \tag{2.1.4}$$

so, multiplying eq. (a) and (c) of (2.1.4), we have $\partial^2 z = \partial\zeta\partial\xi$ and, multiplying eq. (b) and (d) of (2.1.4), we have $-c^2\partial t = \partial\zeta\partial\xi$. Now we can substitute them inside eq. (2.1.1) to obtain:

$$\left(\frac{\partial^2}{\partial\zeta\partial\xi} + \frac{\partial^2}{\partial\zeta\partial\xi}\right)\mathbf{E}(\zeta,\xi) = 0$$

that is:

$$\cancel{2}\,\frac{\partial^2}{\partial\zeta\partial\xi}\mathbf{E}(\zeta,\xi) = 0$$

Using the relationships (a) and (b) of (2.1.4), we can rewrite the forward equation in (2.1.2) as follow:

$$\frac{\partial\mathbf{E}_+}{\partial\xi} = -\frac{\partial\mathbf{E}_+}{\partial\xi}$$

that is verified only when

$$\frac{\partial\mathbf{E}_+}{\partial\xi} = 0 \tag{2.1.5}$$

In the same way, using the relationships (c) and (d) of (2.1.4), the backward equation in (2.1.2) becomes:

$$\frac{\partial\mathbf{E}_-}{\partial\zeta} = -\frac{\partial\mathbf{E}_-}{\partial\zeta}$$

and, consequently

$$\frac{\partial\mathbf{E}_-}{\partial\zeta} = 0 \tag{2.1.6}$$

2.2 ExerciseEquation Section (Next)

A source located at $z=0$ generates an electromagnetic pulse of duration of T seconds, given by $\mathbf{E}(0,t)=\hat{\mathbf{x}}E_0\left[u(t)-u(t-T)\right]$, where $u(t)$ is the unit step function and E_0 is a constant. The pulse is launched towards the positive z–direction. Determine expressions for $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ and sketch them versus z at any given t.

Solution

For a forward–moving wave, we have $\mathbf{E}(z,t)=\mathbf{F}(z-ct)=\mathbf{F}\big(0-c(t-z/c)\big)$, which implies that $\mathbf{E}(z,t)$ is completely determined by $\mathbf{E}(z,0)$ or, alternatively, by $\mathbf{E}(0,t)$:

$$\mathbf{E}(z,t)=\mathbf{E}(z-ct,0)=\mathbf{E}(0,t-z/c)$$

Using this property , we find for the electric and magnetic fields:

$$\begin{aligned}\mathbf{E}(z,t)&=\mathbf{E}(0,t-z/c)=\hat{\mathbf{x}}E_0\left[u(t-z/c)-u(t-z/c-T)\right]\\ \mathbf{H}(z,t)&=\frac{1}{Z_0}\hat{\mathbf{z}}\times\mathbf{E}(z,t)=\hat{\mathbf{y}}\frac{E_0}{Z_0}\left[u(t-z/c)-u(t-z/c-T)\right]\end{aligned} \tag{2.2.1}$$

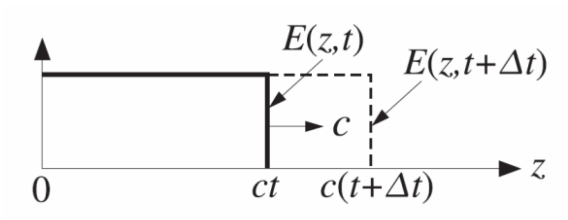

Fig. 2.2.1: Expanding wave–font at time t and $t+\Delta t$.

Because of the unit step, the non–zero values of the fields are restricted to $t-z/c\geq 0$, or, $z\leq ct$, that is, at the time t the wave–front has propagated only up to the position $z=ct$. Fig. 2.2.1 shows the expanding wave–fronts at time t and $t+\Delta t$.

2.3 ExerciseEquation Section (Next)

Show that for a single–frequency wave propagating along z–direction the corresponding transverse fields $\mathbf{E}(z)$, $\mathbf{H}(z)$ satisfy the system of equations:

$$\frac{\partial}{\partial z}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\times\hat{\mathbf{z}}\end{bmatrix}=\begin{bmatrix}0 & -j\omega\mu\\ -j\omega\varepsilon & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\times\hat{\mathbf{z}}\end{bmatrix} \tag{2.3.1}$$

where the matrix is meant to apply individually to the x, y components of the vector entries. Show that the following similarity transformation diagonalizes the transition matrix, and discuss its role in decoupling and solving the above system in terms of forward and backward waves:

$$\begin{bmatrix}1 & Z_0\\ 1 & -Z_0\end{bmatrix}\begin{bmatrix}0 & -j\omega\mu\\ -j\omega\varepsilon & 0\end{bmatrix}\begin{bmatrix}1 & Z_0\\ 1 & -Z_0\end{bmatrix}^{-1}=\begin{bmatrix}-jk & 0\\ 0 & jk\end{bmatrix} \tag{2.3.2}$$

where $k=\omega/c$, $c=1/\sqrt{\mu\varepsilon}$, and $Z_0=\sqrt{\mu/\varepsilon}$.

Solution

For a single–frequency wave, we can assume a time–dependence as $e^{j\omega t}$. So the electric and magnetic field can be expressed as $\mathbf{E}(x,y,z,t)=\mathbf{E}(x,y,z)e^{j\omega t}$, $\mathbf{H}(x,y,z,t)=\mathbf{H}(x,y,z)e^{j\omega t}$, respectively. The Maxwell's equations can be written in the form:

$$\begin{cases}\nabla\times\mathbf{E}(x,y,z)e^{j\omega t}=-\mu\dfrac{\partial\left(\mathbf{H}(x,y,z)e^{j\omega t}\right)}{\partial t}\\[2ex] \nabla\times\mathbf{H}(x,y,z)e^{j\omega t}=-\mu\dfrac{\partial\left(\mathbf{E}(x,y,z)e^{j\omega t}\right)}{\partial t}\end{cases} \tag{2.3.3}$$

Evaluating the derivate in the right–hand side of both equations, the term $e^{j\omega t}$ can be simplified:

$$\begin{cases}\nabla\times\mathbf{E}(x,y,z)\,\cancel{e^{j\omega t}}=-j\omega\mu\mathbf{H}(x,y,z)\,\cancel{e^{j\omega t}}\\ \nabla\times\mathbf{H}(x,y,z)\,\cancel{e^{j\omega t}}=j\omega\varepsilon\mathbf{E}(x,y,z)\,\cancel{e^{j\omega t}}\end{cases}$$

The curl of the electric (or magnetic) field can be written as determinant of the following matrix:

$$\nabla\times\mathbf{E}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial_x & \partial_y & \partial_z\\ E_x(x,y,z) & E_y(x,y,z) & E_z(x,y,z)\end{vmatrix}$$

where ∂_i with $i=x,y,z$ are the partial derivates. But we are in presence of an uniform plane waves propagating along z–direction, so the electric filed vector lies on the x–y plane, i.e. the z–

component of the electric field is null($E_z = 0$), and also in each plane the vector has constant amplitude, i.e. the derivates along x and y are null. The curl becomes:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & \partial_z \\ E_x(x,y,z) & E_y(x,y,z) & 0 \end{vmatrix} \tag{2.3.4}$$

The only applicable derivate is ∂_z, so (2.3.4) is similar to:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ \dfrac{\partial E_x(x,y,z)}{\partial z} & \dfrac{\partial E_y(x,y,z)}{\partial z} & 0 \end{vmatrix} \tag{2.3.5}$$

that is simply the cross–product of $\hat{\mathbf{z}}$ and $\dfrac{\partial \mathbf{E}(x,y,z)}{\partial z}$:

$$\hat{\mathbf{z}} \times \frac{\partial \mathbf{E}(x,y,z)}{\partial z} = -j\omega\mu\mathbf{H}(x,y,z) \tag{2.3.6}$$

$$\hat{\mathbf{z}} \times \frac{\partial \mathbf{H}(x,y,z)}{\partial z} = +j\omega\varepsilon\mathbf{E}(x,y,z) \tag{2.3.7}$$

Consider eq. (2.3.6) and apply the cross–product with $\hat{\mathbf{z}}$ to both of side:

$$\left(\hat{\mathbf{z}} \times \frac{\partial \mathbf{E}}{\partial z}\right) \times \hat{\mathbf{z}} = -j\omega\mu\mathbf{H} \times \hat{\mathbf{z}} \tag{2.3.8}$$

and, using BAC-CAB rule, the left–hand side simplifies into:

$$\left(\hat{\mathbf{z}} \times \frac{\partial \mathbf{E}}{\partial z}\right) \times \hat{\mathbf{z}} = \frac{\partial \mathbf{E}}{\partial z}(\hat{\mathbf{z}} \cdot \hat{\mathbf{z}}) - \hat{\mathbf{z}}\left(\hat{\mathbf{z}} \cdot \frac{\partial \mathbf{E}}{\partial z}\right) = \frac{\partial \mathbf{E}}{\partial z} - \hat{\mathbf{z}}\frac{\partial E_z}{\partial z} = \frac{\partial \mathbf{E}}{\partial z}$$

where we used the condition $\partial_z E_z = 0$ for a plane wave.

So eq.(2.3.8) can be written as follow:

$$\frac{\partial \mathbf{E}}{\partial z} = -j\omega\mu\mathbf{H} \times \hat{\mathbf{z}} \tag{2.3.9}$$

On the contrary, eq. (2.3.7) needs only to invert the cross product at the left–hand side and to change the sign at the right–hand side:

$$\frac{\partial \mathbf{H}}{\partial z} \times \hat{\mathbf{z}} = -j\omega\varepsilon\mathbf{E} \tag{2.3.10}$$

Now it is easy to write in matrix form eq. (2.3.9) and (2.3.10) to obtain (2.3.1).

The transition matrix has to be diagonalized and we need of eigenvectors to create the matrix for the base change. The eigenvectors are found from the eigenvalues that are the roots of the characteristic polynomial:

$$\text{Det}\left(\mathbf{A}-\alpha\mathbf{I}\right)=0$$

where $\mathbf{A}=\begin{bmatrix}0 & -\text{j}\omega\mu\\ -\text{j}\omega\varepsilon & 0\end{bmatrix}$, $\mathbf{I}$ is the identity matrix and α are the eigenvalues.

So we have:

$$\text{Det}\left(\begin{bmatrix}0 & -\text{j}\omega\mu\\ -\text{j}\omega\varepsilon & 0\end{bmatrix}-\alpha\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=\text{Det}\begin{pmatrix}-\alpha & -\text{j}\omega\mu\\ -\text{j}\omega\varepsilon & -\alpha\end{pmatrix}=\alpha^2+\omega^2\mu\varepsilon=0$$

which gives:

$$\begin{aligned}\alpha_1&=-\text{j}\omega\sqrt{\mu\varepsilon}=-\text{jk}\\ \alpha_2&=+\text{j}\omega\sqrt{\mu\varepsilon}=+\text{jk}\end{aligned} \tag{2.3.11}$$

It has two separate eigenvalues, so it is diagonalizable. The diagonal matrix is simply:

$$\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}=\begin{bmatrix}-\text{jk} & 0\\ 0 & +\text{jk}\end{bmatrix}$$

that is the right–hand side of (2.3.2). The left–hand side is composed by the product of the matrix $\mathbf{A}$ and two matrixes for the base change. These matrixes are made putting in row the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ calculated as follow:

$$\mathbf{A}\cdot\mathbf{v}_\text{i}=\alpha_\text{i}\mathbf{v}_\text{i} \qquad \text{with } \text{i}=1,2 \text{ and } \mathbf{v}_\text{i}=\begin{pmatrix}v_{\text{i}1}\\ v_{\text{i}2}\end{pmatrix}$$

Expanding it two different systems of equations, one for each eigenvalue, we have:

$$\begin{bmatrix}0 & -\text{j}\omega\mu\\ -\text{j}\omega\varepsilon & 0\end{bmatrix}\begin{pmatrix}v_{11}\\ v_{12}\end{pmatrix}=\alpha_1\begin{pmatrix}v_{11}\\ v_{12}\end{pmatrix}$$

$$\begin{bmatrix}0 & -\text{j}\omega\mu\\ -\text{j}\omega\varepsilon & 0\end{bmatrix}\begin{pmatrix}v_{21}\\ v_{22}\end{pmatrix}=\alpha_2\begin{pmatrix}v_{21}\\ v_{22}\end{pmatrix}$$

or equivalently,

$$\begin{cases}-\text{j}\omega\mu v_{12}=\alpha_1 v_{11}\\ -\text{j}\omega\varepsilon v_{11}=\alpha_1 v_{12}\end{cases} \tag{2.3.12}$$

$$\begin{cases}-\text{j}\omega\mu v_{22}=\alpha_2 v_{21}\\ -\text{j}\omega\varepsilon v_{21}=\alpha_2 v_{22}\end{cases} \tag{2.3.13}$$

Solving (2.3.12) and (2.3.13), we find that the eigenvectors are given by:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ Z_0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -Z_0 \end{pmatrix} \tag{2.3.14}$$

Now we can write the matrix for the base change as:

$$\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \end{bmatrix} = \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix} = \begin{bmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{bmatrix} \tag{2.3.15}$$

where the superscript T indicate the vector transpose.

In order to verify that eq. (2.3.2) is correct, we have to calculate the inverse of the matrix (2.3.15):

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{bmatrix}^{-1} = \frac{1}{\mathrm{Det}[\mathbf{P}]}\left(\mathbf{C}_{ij}(\mathbf{P})\right)^T$$

where $\mathbf{C}_{ij}(\mathbf{P})$ is the matrix of cofactors of $\mathbf{P}$. The cofactor in position (i, j) is defined as follow:

$$C_{ij}(P) = (-1)^{i+j}\,\mathrm{Det}\left(\mathrm{Minor}(\mathbf{P}, i, j)\right)$$

where $\mathrm{Minor}(\mathbf{P}, i, j)$ represents the matrix obtained by $\mathbf{P}$ cancelling the i–th row and j–th column.

It is easy to evaluate it:

$$\mathbf{C}_{ij}(\mathbf{P}) = \begin{bmatrix} -Z_0 & -1 \\ -Z_0 & 1 \end{bmatrix} \tag{2.3.16}$$

and now we can write:

$$\mathbf{P}^{-1} = \begin{bmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{bmatrix}^{-1} = \frac{1}{-2Z_0}\begin{bmatrix} -Z_0 & -Z_0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/(2Z_0) & -1/(2Z_0) \end{bmatrix}$$

So:

$$\begin{aligned}
\mathbf{PAP}^{-1} &= \begin{bmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{bmatrix}\begin{bmatrix} 0 & -j\omega\mu \\ -j\omega\varepsilon & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/(2Z_0) & -1/(2Z_0) \end{bmatrix} = \\
&= \begin{bmatrix} -j\omega\varepsilon Z_0 & -j\omega\mu \\ j\omega\varepsilon Z_0 & -j\omega\mu \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/(2Z_0) & -1/(2Z_0) \end{bmatrix} = \\
&= \begin{bmatrix} -j\left(\dfrac{\omega\varepsilon Z_0}{2} + \dfrac{\omega\mu}{2Z_0}\right) & -j\left(\dfrac{\omega\varepsilon Z_0}{2} - \dfrac{\omega\mu}{2Z_0}\right) \\ +j\left(\dfrac{\omega\varepsilon Z_0}{2} - \dfrac{\omega\mu}{2Z_0}\right) & +j\left(\dfrac{\omega\varepsilon Z_0}{2} + \dfrac{\omega\mu}{2Z_0}\right) \end{bmatrix}
\end{aligned}$$

where $Z_0 = \sqrt{\mu/\varepsilon}$. It can be simplified to obtain:

$$\begin{bmatrix} 1 & Z_0 \\ 1 & -Z_0 \end{bmatrix}\begin{bmatrix} 0 & -j\omega\mu \\ -j\omega\varepsilon & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/(2Z_0) & -1/(2Z_0) \end{bmatrix} = \begin{bmatrix} -jk & 0 \\ 0 & +jk \end{bmatrix} \tag{2.3.17}$$

where $k = \omega\sqrt{\mu\varepsilon}$ and (2.3.17) is equivalent to (2.3.2).

The diagonal matrix given in (2.3.17) can be substituted in (2.3.1) as follow:

$$\frac{\partial}{\partial z}\begin{bmatrix} \mathbf{E} \\ \mathbf{H}\times\hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} -jk & 0 \\ 0 & +jk \end{bmatrix}\begin{bmatrix} \mathbf{E} \\ \mathbf{H}\times\hat{\mathbf{z}} \end{bmatrix} \tag{2.3.18}$$

that is

$$\begin{cases} \dfrac{\partial}{\partial z}\mathbf{E} = -jk\mathbf{E} \\ \dfrac{\partial}{\partial z}\mathbf{H}\times\hat{\mathbf{z}} = +jk\mathbf{H}\times\hat{\mathbf{z}} \end{cases} \tag{2.3.19}$$

The electric and magnetic field are related by the characteristic impedance of the medium, in this case vacuum, as follow:

$$\mathbf{E} = Z_0\mathbf{H}\times\hat{\mathbf{k}} = Z_0\mathbf{H}\times\hat{\mathbf{z}} \tag{2.3.20}$$

where $\hat{\mathbf{k}} = \hat{\mathbf{z}}$, being the electromagnetic wave propagating along z–direction.

So using (2.3.20) in the set (2.3.19), we obtain:

$$\begin{cases} \dfrac{\partial}{\partial z}\mathbf{E} = -jk\mathbf{E} \\ \cancel{\dfrac{1}{Z_0}}\dfrac{\partial}{\partial z}\mathbf{E} = +\cancel{\dfrac{1}{Z_0}}jk\mathbf{E} \end{cases} \tag{2.3.21}$$

It is possible to note that the diagonalization (2.3.2) allow us to decouple the electric and the magnetic field as in (2.3.19) and, using the relationship (2.3.20), we are able to express the electric field in term of forward and backward wave.

2.4 ExerciseEquation Section (Next)

The visible spectrum has the wavelength range 380–780 nm. What is this in THz? In particular, determine the frequencies of red, orange, yellow, green, blue, and violet having the nominal wavelengths of 700, 610, 590, 530, 470, and 420 nm.

Solution

The wavelength λ is the distance by which the phase of the sinusoidal wave changes by 2π radians. Since the propagation factor e^{-jkz} accumulates a phase of k radians per meter, we have by definition that $k\lambda = 2\pi$. The wavelength λ can be expressed via the frequency of the wave in Hertz, $f = \omega/2\pi$, as follows:

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{\cancel{2\pi} c}{\cancel{2\pi} f} = \frac{c}{f} \tag{2.4.1}$$

Using the relation (2.4.1), we can calculate the frequency range for the electromagnetic visible spectrum:

$$380\times10^{-9} < \lambda < 780\times10^{-9} \quad \Rightarrow \quad \frac{c}{380\times10^{-9}} > f > \frac{c}{780\times10^{-9}}$$

that is:

$$789.5 \text{ THz} > f > 384.6 \text{ THz} \tag{2.4.2}$$

The frequencies of the colours are:

Colour	**Wavelength (nm)**	**Frequency (THz)**

Red	700	428.5
Orange	610	491.8
Yellow	590	508.5
Green	530	566.0
Blue	470	638.3
Violet	420	714.3

Table 2.4.1: Wavelength and frequency of colours in the visibility region.

2.5 ExerciseEquation Section (Next)

What is the frequency in THz of a typical CO_2 laser (used in laser surgery) having the far infrared wavelength of 20 μm?

Solution

Using eq. (2.4.1), it is easy to obtain the result:

$$f = \frac{c}{\lambda} = \frac{3\times 10^8}{20\times 10^{-6}} = 15\times 10^{12} = 15\ \text{THz}$$

2.6 ExerciseEquation Section (Next)

What is the wavelength in meters or cm of a wave with the frequencies of 10 KHz, 10 MHz, and 10 GHz?

What is the frequency in GHz of the 21–cm hydrogen line observed in the cosmos?

What is the wavelength in cm of the typical microwave oven frequency of 2.45 GHz?

Solution

Using eq. (2.4.1), we have:

$$\lambda_{(10\text{KHz})} = \frac{c}{f} = \frac{3\times 10^8}{10\times 10^3} = 30000\ \text{m} = 30\ \text{Km}$$

$$\lambda_{(10\text{MHz})} = \frac{c}{f} = \frac{3\times 10^8}{10\times 10^6} = 30\ \text{m}$$

$$\lambda_{(10\text{GHz})} = \frac{c}{f} = \frac{3\times 10^8}{10\times 10^{12}} = 0.030\ \text{m} = 30\ \text{mm}$$

The frequency in GHz of the 21–cm hydrogen line observed in the cosmos is:

$$f = \frac{c}{\lambda} = \frac{3\times 10^8}{21\times 10^{-2}} = 1.43\ \text{GHz}$$

The wavelength in cm of the typical microwave oven frequency of 2.45 GHz is:

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{2.45\times 10^9} = 0.122\ \text{m} =\ 12.2\ \text{cm}$$

2.7 ExerciseEquation Section (Next)

Suppose you start with $\mathbf{E}(z,t) = \hat{\mathbf{x}}E_0 e^{j\omega t - jkz}$, but you don't yet know the relationship between k and ω (you may assume they are both positive.) By inserting $\mathbf{E}(z,t)$ into Maxwell's equations, determine the k–ω relationship as a consequence of these equations. Determine also the magnetic field $\mathbf{H}(z,t)$ and verify that all Maxwell's equations are satisfied.

Repeat the problem if $\mathbf{E}(z,t) = \hat{\mathbf{x}}E_0 e^{j\omega t + jkz}$ and $\mathbf{E}(z,t) = \hat{\mathbf{y}}E_0 e^{j\omega t - jkz}$.

Solution

Consider the source–free Maxwell's equations:

$$\begin{cases} \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

and now substitute the expression of $\mathbf{E}(z,t)$ in the first Maxwell's equation. Assuming valid the constitutive relation $\mathbf{B} = \mu \mathbf{H}$:

$$\nabla \times \hat{\mathbf{x}}E_0 e^{j\omega t - jkz} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{2.7.1}$$

The cross–product in the left–hand side of eq. (2.7.1) can be expanded as follow:

$$\hat{\mathbf{y}} jk E_0 e^{j\omega t - jkz} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

It is easy to note that $\mathbf{H}(z,t)$ has to depend by z and t in the same way as $\mathbf{E}(z,t)$ because the variables t and z are presented only in the exponential. So $\mathbf{H}(z,t) = \hat{\mathbf{y}}H_0 e^{j\omega t - jkz}$:

$$\hat{\mathbf{y}} jkE_0 e^{j\omega t - jkz} = -\mu \hat{\mathbf{y}} H_0 \frac{\partial}{\partial t} e^{j\omega t - jkz}$$

$$\cancel{\hat{\mathbf{y}}} jkE_0 \cancel{e^{j\omega t - jkz}} = -\mu \cancel{\hat{\mathbf{y}}} H_0 \cancel{j} \omega \cancel{e^{j\omega t - jkz}}$$

which gives:

$$kE_0 = -\omega\mu H_0 \tag{2.7.2}$$

Assuming valid the constitutive relation $\mathbf{D} = \varepsilon \mathbf{E}$, substitute the expression of $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ in the second Maxwell's equation:

$$\nabla \times \hat{\mathbf{y}} H_0 e^{j\omega t - jkz} = \varepsilon \hat{\mathbf{x}} E_0 \frac{\partial e^{j\omega t - jkz}}{\partial t} \tag{2.7.3}$$

The cross–product in the left–hand side of eq. (2.7.3) and the derivate in the right–hand side can be expanded as follow:

$$\hat{\mathbf{x}} \frac{\partial}{\partial z}\left(H_0 e^{j\omega t - jkz}\right) + \cancel{\hat{\mathbf{z}} \frac{\partial}{\partial x}\left(H_0 e^{j\omega t - jkz}\right)} = j\omega\varepsilon \hat{\mathbf{x}} E_0 e^{j\omega t - jkz}$$

which gives:

$$kH_0 = -\omega\varepsilon E_0 \tag{2.7.4}$$

Thanks to the relationships (2.7.2) and (2.7.4) we can find the k–ω relation:

$$\begin{cases} kE_0 = -\omega\mu H_0 \\ kH_0 = -\omega\varepsilon E_0 \end{cases}$$

$$k^2 \cancel{E_0 H_0} = \omega^2 \mu\varepsilon \cancel{E_0 H_0}$$

and, consequently:

$$k = \pm\omega\sqrt{\mu\varepsilon} \tag{2.7.5}$$

2.8 ExerciseEquation Section (Next)

Determine the polarization types of the following waves, and indicate the direction, if linear, and sense of rotation, if circular or elliptic:

$$\begin{bmatrix} \text{a.} & \mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz} \\ \text{b.} & \mathbf{E} = E_0(\hat{\mathbf{x}} - \sqrt{3}\hat{\mathbf{y}})e^{-jkz} \\ \text{c.} & \mathbf{E} = E_0(j\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz} \\ \text{d.} & \mathbf{E} = E_0(\hat{\mathbf{x}} - 2j\hat{\mathbf{y}})e^{-jkz} \end{bmatrix} \begin{bmatrix} \text{e.} & \mathbf{E} = E_0(\hat{\mathbf{x}} - \hat{\mathbf{y}})e^{-jkz} \\ \text{f.} & \mathbf{E} = E_0(\sqrt{3}\hat{\mathbf{x}} - \hat{\mathbf{y}})e^{-jkz} \\ \text{g.} & \mathbf{E} = E_0(j\hat{\mathbf{x}} - \hat{\mathbf{y}})e^{jkz} \\ \text{h.} & \mathbf{E} = E_0(\hat{\mathbf{x}} + 2j\hat{\mathbf{y}})e^{jkz} \end{bmatrix} \tag{2.8.1}$$

Solution

The polarization of a plane wave is defined to be the direction of the time–varying real–valued field $\mathbf{E}(z,t) = \Re e\left[\mathbf{E}(z)e^{j\omega t}\right]$ where $\mathbf{E}(z) = \underline{\mathbf{E}}_0 e^{\pm jkz}$. At any fixed point z, the vector $\mathbf{E}(z,t)$ may be along a fixed linear direction or it may be rotating as a function of t, tracing a circle or an ellipse.

Consider the following expression for the electric field:

$$\mathbf{E}(z,t) = \left(\hat{\mathbf{x}}A_x e^{j\phi_x} + \hat{\mathbf{y}}A_y e^{j\phi_y}\right)e^{j\omega t \pm jkz} = \hat{\mathbf{x}}A_x e^{j(\omega t \pm jkz + \phi_x)} + \hat{\mathbf{y}}A_y e^{j(\omega t \pm jkz + \phi_y)} \tag{2.8.2}$$

where A_x and A_y are real–positive quantities.

Extracting the real part for each component, we find the corresponding real–valued x,y components:

$$\begin{aligned} E_x(z,t) &= A_x \cos(\omega t \pm kz + \phi_x) \\ E_y(z,t) &= A_y \cos(\omega t \pm kz + \phi_y) \end{aligned} \tag{2.8.3}$$

The sign of kz is defined by the direction of propagation of the wave: forward–moving fields have the minus sign, e.g. $-kz$, backward–moving fields the plus sign, e.g. $+kz$.

In order to determine the polarization type of the waves, we consider the time–dependence of these fields at some fixed point along z–axis. For convenience we choose $z = 0$:

$$\begin{aligned} E_x(z,t) &= A_x \cos(\omega t + \phi_x) \\ E_y(z,t) &= A_y \cos(\omega t + \phi_y) \end{aligned} \tag{2.8.4}$$

The parameters A_x, A_y, ϕ_x, ϕ_y allow us to determine the type of polarization:

- <u>Linear polarization</u> ($\phi_a = \phi_b = 0$ or $\phi_a = 0$, $\phi_b = -\pi$): the two components E_x, E_y are in phase and the electric field vector oscillates along a straight line. It is of interest the direction, respect the x–axis, along which the electric field oscillates with angular frequency ω. It is directly related to the amplitudes of the components E_x, E_y:

$$\vartheta = \arctan\left(\frac{A_y}{A_x}\right) \tag{2.8.5}$$

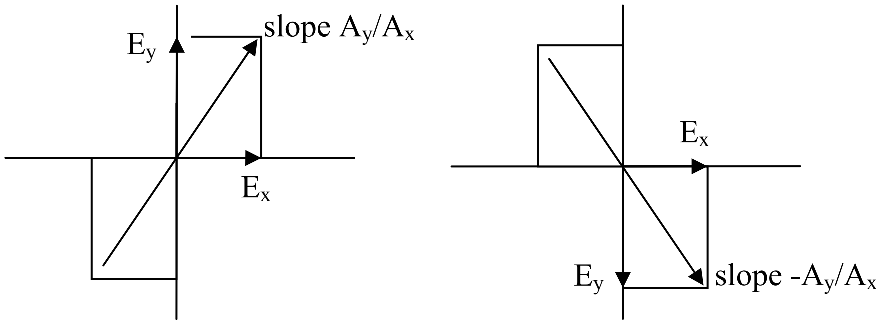

Fig. 2.8.1:Directions along the electric field oscillates in linear polarization.

- Elliptical polarization ($\phi_x - \phi_y = \pm\pi/2$): E_x and E_y have different amplitudes and are in quadrature phase because one is always 90° out of phase respect to other.
- Circular polarization ($A_x = A_y$ and $\phi_x - \phi_y = \pm\pi/2$): this is a particular case of elliptical polarization when the amplitudes of the components are equal.

The sign of the relative phase $\phi = \phi_x - \phi_y$ suggests the sense of rotation: counter–clockwise ($\phi = -\pi/2$) and clockwise ($\phi = +\pi/2$) and consequently, according to the direction of propagation, left or right elliptical polarization (or circular in particular cases).

$$E_x(t) = A\cos\omega t$$
$$E_y(t) = A\cos(\omega t + \pi/2) = A\sin\omega t$$

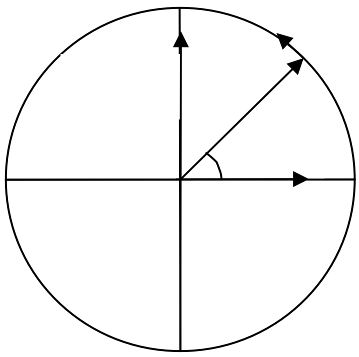

Fig. 2.8.2: Counter–clockwise rotation of the electric field vector.

$$E_x(t) = A\cos\omega t$$
$$E_y(t) = A\cos(\omega t - \pi/2) = -A\sin\omega t$$

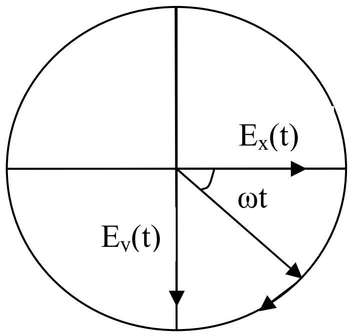

Fig. 2.8.3: Clockwise rotation of the electric field vector.

To decide whether this represents a right or left polarization, we use the IEEE convention. Curl the fingers of your left and right hands into a fist and point both thumbs towards the direction of propagation. If the fingers of your right (left) hand are curling in the direction of rotation of the electric field, then the polarization is right (left) polarized.

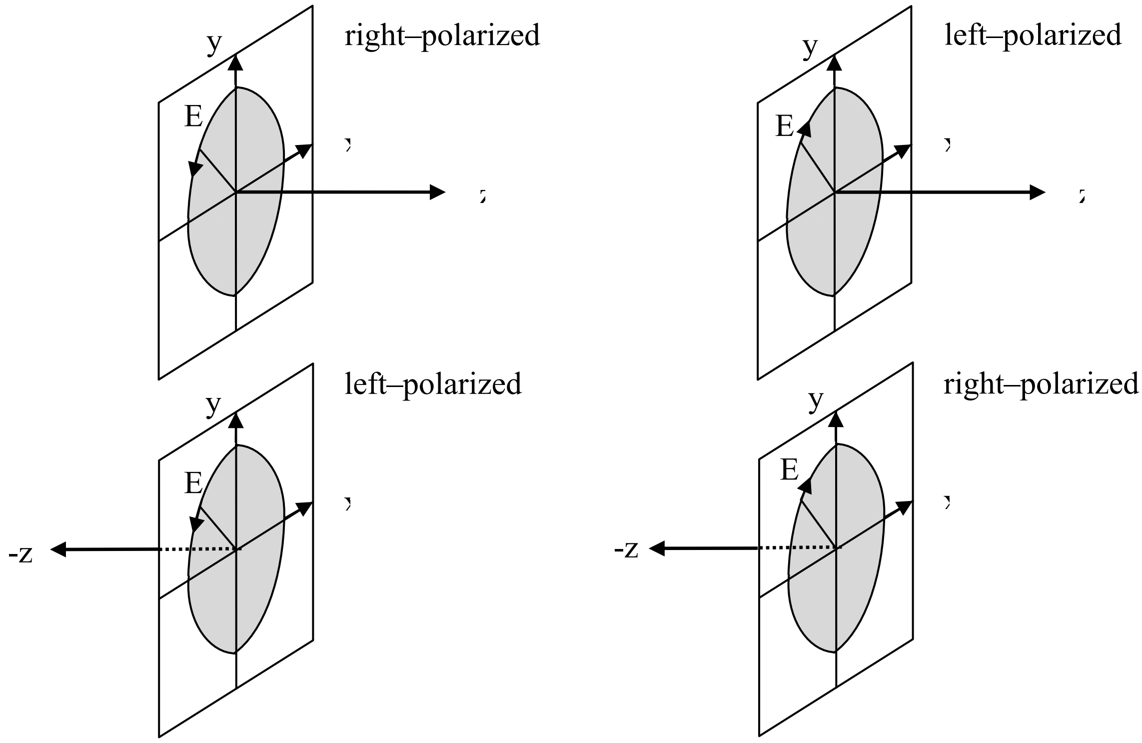

Fig. 2.8.4: Left and right circular polarizations.

Let us solve the exercise for the case (a): $\mathbf{E}(z)=E_0(\hat{\mathbf{x}}+\hat{\mathbf{y}})e^{-jkz}$. First of all we have to express the field in its real–valued form in $z=0$, in order to obtain an expression similar to eq. (2.8.4):

$$\mathbf{E}(t,z)=E_0\Re e\left[(\hat{\mathbf{x}}+\hat{\mathbf{y}})e^{j(\omega t-kz)}\right]=E_0\hat{\mathbf{x}}\cos(\omega t-kz)+E_0\hat{\mathbf{y}}\cos(\omega t-kz)$$

so

$$E_x(t,z)=E_0\cos(\omega t-kz)$$
$$E_y(t,z)=E_0\cos(\omega t-kz)$$

It is easy to note that it is a forward–moving wave, because of the term $-kz$, and linear polarized, being $\phi_x = \phi_y = 0$. Using eq. (2.8.5), the direction ϑ of electric field vector is 45°.

On the contrary, let us solve the exercise for the case (c): $\mathbf{E} = E_0(j\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz}$. Its real–valued form in $z = 0$ is:

$$\mathbf{E}(t,z) = E_0 \Re e\left[(j\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{j(\omega t - kz)}\right] = E_0 \Re e\left[\left(\hat{\mathbf{x}}e^{j\pi/2} + \hat{\mathbf{y}}\right)e^{j(\omega t - kz)}\right] =$$
$$= E_0 \hat{\mathbf{x}} \cos(\omega t - kz + \pi/2) + E_0 \hat{\mathbf{y}} \cos(\omega t - kz)$$

so

$$E_x(t,z) = E_0 \cos(\omega t - kz + \pi/2)$$
$$E_y(t,z) = E_0 \cos(\omega t - kz)$$

The wave is still forward–moving, but the relative phase $\phi = \phi_x - \phi_y = \pi/2$, so it is in general elliptical polarized. In this case $|E_x| = |E_y|$, i.e. $A_x = A_y$, then it is circular polarized. According to Fig. 2.8.2 and Fig. 2.8.3, the sense of rotation is counterclockwise. Now we apply the IEEE convention and the find that the field is right–circular polarized.

Table 2.8.1 contains the results of the exercise for each given electric field:

#	Expression	Polarization Type	Direction/ Sense of Rotation
a	$\mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz}$	Linear	45°
b	$\mathbf{E} = E_0\left(\hat{\mathbf{x}} - \sqrt{3}\hat{\mathbf{y}}\right)e^{-jkz}$	Linear	-60°
c	$\mathbf{E} = E_0(j\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{-jkz}$	Circular	Counter–clockwise
d	$\mathbf{E} = E_0(\hat{\mathbf{x}} - 2j\hat{\mathbf{y}})e^{-jkz}$	Elliptical	Counter–clockwise
e	$\mathbf{E} = E_0(\hat{\mathbf{x}} - \hat{\mathbf{y}})e^{-jkz}$	Linear	-45°
f	$\mathbf{E} = E_0\left(\sqrt{3}\hat{\mathbf{x}} - \hat{\mathbf{y}}\right)e^{-jkz}$	Linear	-30°
g	$\mathbf{E} = E_0(j\hat{\mathbf{x}} - \hat{\mathbf{y}})e^{jkz}$	Circular	Clockwise
h	$\mathbf{E} = E_0(\hat{\mathbf{x}} + 2j\hat{\mathbf{y}})e^{jkz}$	Elliptical	Clockwise

Table 2.8.1: Results of exercise n° 2.8.

2.9 ExerciseEquation Section (Next)

A uniform plane wave, propagating in the z–direction in vacuum, has the following electric field:

$$\mathbf{E}(z,t) = 2\hat{\mathbf{x}}\mathrm{Cos}(\omega t - kz) + 4\hat{\mathbf{y}}\mathrm{Sin}(\omega t - kz) \tag{2.9.1}$$

1. Determine the vector phasor representing $\mathbf{E}(z,t)$ in the complex form $\mathbf{E} = \mathbf{E}_0 e^{j\omega t - jkz}$.
2. Determine the polarization of this electric field (linear, circular, elliptic, left–handed, right–handed).
3. Determine the magnetic field $\mathbf{H}(z,t)$ in its real–valued form.

Solution

- Question n°1

First of all, we need to manipulate (2.9.1) in order to obtain an expression with components similar to:

$$\begin{aligned} E_x(z,t) &= A_x \cos(\omega t \pm kz + \phi_x) \\ E_y(z,t) &= A_y \cos(\omega t \pm kz + \phi_y) \end{aligned} \tag{2.9.2}$$

So we can write:

$$\begin{aligned} E_x(z,t) &= 2\cos(\omega t - kz) \\ E_y(z,t) &= 4\cos(\omega t - kz - \pi/2) \end{aligned} \tag{2.9.3}$$

from which, we can obtain the complex–valued electric field:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}2e^{j(\omega t - jkz)} + \hat{\mathbf{y}}4e^{j(\omega t - jkz - \pi/2)} = \left(2\hat{\mathbf{x}} + 4\hat{\mathbf{y}}e^{-j\pi/2}\right)e^{j\omega t - jkz} =$$
$$= (2\hat{\mathbf{x}} - j4\hat{\mathbf{y}})e^{j(\omega t - kz)}$$

- Question n°2

In (2.9.3) it is easy to note that the relative phase $\phi = \phi_x - \phi_y = \pi/2$, so according to Fig. 2.8.2 and Fig. 2.8.3, the sense of rotation is counter–clockwise. The field is forward–moving and, using the IEEE convention, the field is right–elliptical polarized, being $|E_x| \neq |E_y|$.

- Question n°3

Using the relation:

$$\mathbf{E} = Z_0(\mathbf{H} \times \hat{\mathbf{z}})$$

where Z_0 is the characteristic impedance of vacuum, we find:

$$\mathbf{H}(z,t) = \frac{1}{Z_0}\hat{\mathbf{z}}\times\mathbf{E}(z,t) = \frac{1}{Z_0}\hat{\mathbf{z}}\times\left(E_x(z,t)\,\hat{\mathbf{x}} + E_y(z,t)\,\hat{\mathbf{y}}\right) =$$

$$= \frac{1}{Z_0}\left(E_x(z,t)\,\hat{\mathbf{y}} - E_y(z,t)\,\hat{\mathbf{x}}\right) =$$

$$= \frac{1}{Z_0}\left(2\cos(\omega t - kz)\,\hat{\mathbf{y}} - 4\cos(\omega t - kz - \pi/2)\,\hat{\mathbf{x}}\right)$$

2.10 Exercise Equation Section (Next)

A uniform plane wave propagating in vacuum along the z–direction has real–valued electric field components:

$$E_x = \cos(\omega t - kz), \quad E_y = 2\sin(\omega t - kz) \tag{2.10.1}$$

1. Its phasor form has the form $\mathbf{E} = (A\hat{\mathbf{x}} + B\hat{\mathbf{y}})e^{\pm jkz}$. Determine the numerical values of the complex–valued coefficients A, B and the correct sign of the exponent.
2. Determine the polarization of this wave (left, right, linear, etc.). Explain your reasoning.

Solution

- Question n°1

First of all, we need to manipulate (2.10.1) in order to obtain an expression with components similar to (2.9.2):

$$E_x(z,t) = \cos(\omega t - kz)$$
$$E_y(z,t) = 2\cos(\omega t - kz - \pi/2)$$

from which, we can obtain the complex–valued electric field:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}e^{j(\omega t - jkz)} + \hat{\mathbf{y}}2e^{j(\omega t - jkz - \pi/2)} = \left(\hat{\mathbf{x}} + 2\hat{\mathbf{y}}e^{-j\pi/2}\right)e^{j\omega t - jkz} =$$
$$= (\hat{\mathbf{x}} - j2\hat{\mathbf{y}})e^{j(\omega t - kz)}$$

- Question n°2

The polarization of the wave is elliptical because the module of x, y components are different. It is also right polarized because the relative phase $\phi = \phi_x - \phi_y = \pi/2$.

2.11 Exercise Equation Section (Next)

Consider the two electric fields, one given in its real–valued form, and the other, in its phasor form:

$$\begin{aligned} &\text{a.}\quad \mathbf{E}(z,t) = \hat{\mathbf{x}}\sin(\omega t + kz) + 2\hat{\mathbf{y}}\cos(\omega t + kz) \\ &\text{b.}\quad \mathbf{E}(z) = \left[(1+j)\hat{\mathbf{x}} - (1-j)\hat{\mathbf{y}}\right]e^{-jkz} \end{aligned} \tag{2.11.1}$$

For both cases, determine the polarization of the wave (linear, circular, left, right, etc.) and the direction of propagation.

For the case (a), determine the field in its phasors form. For the case (b), determine the field in its real–valued form as a function of t, z.

Solution

- Case (a)

First of all, we rewrite the first field of (2.11.1) as follow:

$$\begin{aligned} E_x(z,t) &= \cos(\omega t + kz - \pi/2) \\ E_y(z,t) &= 2\cos(\omega t + kz) \end{aligned} \tag{2.11.2}$$

from which, we can obtain the complex–valued electric field:

$$\begin{aligned} \mathbf{E}(z,t) &= \hat{\mathbf{x}}e^{j(\omega t + jkz - \pi/2)} + \hat{\mathbf{y}}2e^{j(\omega t + jkz)} = \left(\hat{\mathbf{x}}e^{-j\pi/2} + 2\hat{\mathbf{y}}\right)e^{j\omega t + jkz} = \\ &= (2\hat{\mathbf{y}} - j\hat{\mathbf{x}})e^{j(\omega t + kz)} \end{aligned}$$

In (2.11.2) it is easy to note that the relative phase $\phi = \phi_x - \phi_y = -\pi/2$, so according to Fig. 2.8.2 and Fig. 2.8.3, the sense of rotation is clockwise. The field is backward–moving and, using the IEEE convention, the field is right–elliptical polarized ($|E_x| \neq |E_y|$).

- Case (b)

In this case, we have to write $\mathbf{E}(z)$ in its real–valued form as a function of t, z. So:

$$\begin{aligned} \mathbf{E}(z,t) &= \Re e\left[\left((1+j)\hat{\mathbf{x}} - (1-j)\hat{\mathbf{y}}\right)e^{j(\omega t - kz)}\right] = \\ &= \Re e\left[\left((1+j)\hat{\mathbf{x}} - (1-j)\hat{\mathbf{y}}\right)e^{j(\omega t - kz)}\right] = \\ &= \Re e\left[\left((1+j)\hat{\mathbf{x}} - (1-j)\hat{\mathbf{y}}\right)\left(\cos(\omega t - kz) + j\sin(\omega t - kz)\right)\right] = \\ &= (\hat{\mathbf{x}} - \hat{\mathbf{y}})\cos(\omega t - kz) - (\hat{\mathbf{x}} + \hat{\mathbf{y}})\sin(\omega t - kz) = \\ &= (\hat{\mathbf{x}} - \hat{\mathbf{y}})\cos(\omega t - kz) - (\hat{\mathbf{x}} + \hat{\mathbf{y}})\cos(\omega t - kz - \pi/2) = \\ &= \hat{\mathbf{x}}\left(\cos(\omega t - kz) - \cos(\omega t - kz - \pi/2)\right) - \hat{\mathbf{y}}\left(\cos(\omega t - kz) + \cos(\omega t - kz - \pi/2)\right) \end{aligned} \tag{2.11.3}$$

It is necessary to apply to (2.11.3) the sum–to–product identity or Prosthaphaeresis formula:

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad (2.11.4)$$

and, because $\cos(x) = -\cos(x \pm \pi)$, we obtain:

$$\begin{aligned}\mathbf{E}(z,t) &= \hat{\mathbf{x}}\left(\cos(\omega t - kz) - \cos(\omega t - kz - \pi/2)\right) - \hat{\mathbf{y}}\left(\cos(\omega t - kz) + \cos(\omega t - kz - \pi/2)\right) = \\ &= \hat{\mathbf{x}}\left(\cos(\omega t - kz) + \cos(\omega t - kz + \pi/2)\right) - \hat{\mathbf{y}}\left(\cos(\omega t - kz) + \cos(\omega t - kz - \pi/2)\right) \\ &= 2\hat{\mathbf{x}}\left(\cos(\omega t - kz + \pi/4)\cos(-\pi/4)\right) - 2\hat{\mathbf{y}}\left(\cos(\omega t - kz - \pi/4)\cos(\pi/4)\right) = \\ &= \sqrt{2}\hat{\mathbf{x}}\cos(\omega t - kz + \pi/4) - \sqrt{2}\hat{\mathbf{y}}\cos(\omega t - kz - \pi/4)\end{aligned} \qquad (2.11.5)$$

The electric field components are:

$$\begin{aligned}E_x(z,t) &= \sqrt{2}\cos(\omega t - kz + \pi/4) \\ E_y(z,t) &= -\sqrt{2}\cos(\omega t - kz - \pi/4) = \sqrt{2}\cos(\omega t - kz + 3\pi/4)\end{aligned}$$

and it is easy to note that the relative phase $\phi = \phi_x - \phi_y = -\pi/2$, so according to Fig. 2.8.2 and Fig. 2.8.3, the sense of rotation is clockwise. The field is forward–moving and, using the IEEE convention, the field is left–circular polarized ($|E_x| = |E_y|$).

In similar way, the exercise can be solved writing the complex amplitude of each component in the form:

$$a + jb = Me^{j\varphi} \text{ where } \begin{cases} M = \sqrt{a^2 + b^2} \\ \varphi = \arctan b/a \end{cases} \qquad (2.11.6)$$

Using (2.11.6), we obtain:

$$\begin{aligned}1 + j &= \sqrt{2}e^{j\pi/4} \\ -(1 - j) &= \sqrt{2}e^{j3\pi/4}\end{aligned}$$

and, consequently

$$\begin{aligned}\mathbf{E}(z,t) &= \Re e\left[\left(\sqrt{2}e^{j\pi/4}\hat{\mathbf{x}} + \sqrt{2}e^{j3\pi/4}\hat{\mathbf{y}}\right)e^{j(\omega t - kz)}\right] = \\ &= \sqrt{2}\Re e\left[\left(e^{j\pi/4}\hat{\mathbf{x}} + e^{j3\pi/4}\hat{\mathbf{y}}\right)e^{j(\omega t - kz)}\right] = \\ &= \sqrt{2}\hat{\mathbf{x}}\cos(\omega t - kz + \pi/4) + \sqrt{2}\hat{\mathbf{y}}\cos(\omega t - kz + 3\pi/4)\end{aligned}$$

2.12 ExerciseEquation Section (Next)

A uniform plane wave propagating in the z–direction ha the following real–valued electric field:

$$\mathbf{E}(t,z) = \hat{\mathbf{x}}\cos(\omega t - kz - \pi/4) + \hat{\mathbf{y}}\sin(\omega t - kz + \pi/4) \tag{2.12.1}$$

1. Determine the complex–phasor form of this electric field.
2. Determine the corresponding magnetic field $\mathbf{H}(t,z)$ given in its real–valued form.
3. Determine the polarization type (left, right, linear, etc.) of this wave.

Solution

- Question n°1

First of all, we rewrite (2.12.1) as follow:

$$\begin{aligned} E_x(z,t) &= \cos(\omega t - kz - \pi/4) \\ E_y(z,t) &= \cos(\omega t - kz - \pi/4) \end{aligned} \tag{2.12.2}$$

from which, we can obtain the complex form as eq. (2.8.2)

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}e^{j(\omega t - kz - \pi/4)} + \hat{\mathbf{y}}e^{j(\omega t - kz - \pi/4)} = \left(\hat{\mathbf{x}}e^{-j\pi/4} + \hat{\mathbf{y}}e^{-j\pi/4}\right)e^{j\omega t - jkz} =$$

$$= \frac{1}{\sqrt{2}}(1-j)(\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{j(\omega t - kz)}$$

- Question n°2

Using the relation

$$\mathbf{E} = Z_0(\mathbf{H} \times \hat{\mathbf{z}})$$

where Z_0 is the characteristic impedance of vacuum, we find:

$$\mathbf{H}(t,z) = \frac{1}{Z_0}\hat{\mathbf{z}} \times \mathbf{E}(t,z) = \frac{1}{Z_0}\hat{\mathbf{z}} \times \left[\hat{\mathbf{x}}\cos(\omega t - kz - \pi/4) + \hat{\mathbf{y}}\sin(\omega t - kz + \pi/4)\right] =$$

$$= \frac{1}{Z_0}\left[\hat{\mathbf{y}}\cos(\omega t - kz - \pi/4) - \hat{\mathbf{x}}\sin(\omega t - kz + \pi/4)\right]$$

- Question n°3

In (2.12.2) it is easy to note that the relative phase $\phi = \phi_x - \phi_y = 0$, so the wave is linear polarized, tilted by 45° with respect to the x–axis.

2.13 ExerciseEquation Section (Next)

Determine the polarization type (left, right, linear, etc.) and the direction of propagation of the following electric fields given in their phasor form:

a) $\mathbf{E}(z)=\left[\left(1+j\sqrt{3}\right)\hat{\mathbf{x}}+2\hat{\mathbf{y}}\right]e^{jkz}$

b) $\mathbf{E}(z)=\left[(1+j)\hat{\mathbf{x}}-(1-j)\hat{\mathbf{y}}\right]e^{-jkz}$

c) $\mathbf{E}(z)=\left[\hat{\mathbf{x}}-\hat{\mathbf{z}}+j\sqrt{2}\hat{\mathbf{y}}\right]e^{-jk(x+z)/\sqrt{2}}$

Solution

- Case (a)

We have to writing the complex amplitude of each component E_x, E_y in the form $Ae^{j\varphi}$, using (2.11.6):

$$\left(1+j\sqrt{3}\right)=\sqrt{(1)^2+\left(\sqrt{3}\right)^2}\,e^{j\arctan\left(\sqrt{3}/1\right)}=2e^{j\pi/3}$$

so:

$$\mathbf{E}(z)=2\left[\hat{\mathbf{x}}e^{j\pi/3}+\hat{\mathbf{y}}\right]e^{jkz}$$

The relative phase $\phi=\phi_x-\phi_y=\pi/3$, so it is inside the interval $[0,\pi/2]$ and, according to Fig. 2.8.2 and Fig. 2.8.3, the sense of rotation is counter–clockwise. The field is backward–moving and, using the IEEE convention, the field is left–circular polarized, being $|E_x|=|E_y|$.

- Case (b)

See case (b) of exercise n° 2.11.

- Case (c)

In this case the electric field doesn't propagate along the z direction, but it is tilted with respect to the z axis and lies on the z-x plane. So we have to identify a new coordinate system in order to apply the well–known steps to solve the problem.

Express the electric field in the real–valued form:

$$\begin{aligned}\mathbf{E}(t,z)&=\Re e\left[\left(\hat{\mathbf{x}}-\hat{\mathbf{z}}+j\sqrt{2}\hat{\mathbf{y}}\right)e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}\right]=\\&=\Re e\left[\left(\hat{\mathbf{x}}-\hat{\mathbf{z}}+j\sqrt{2}\hat{\mathbf{y}}\right)\left(\cos\left(\omega t-k(x+z)/\sqrt{2}\right)+j\sin\left(\omega t-k(x+z)/\sqrt{2}\right)\right)\right]=\\&=(\hat{\mathbf{x}}-\hat{\mathbf{z}})\cos\left(\omega t-k(x+z)/\sqrt{2}\right)-\sqrt{2}\hat{\mathbf{y}}\sin\left(\omega t-k(x+z)/\sqrt{2}\right)\end{aligned}$$

This wave does not propagate along the z–direction, and consequently the plane with constant phase are not identified for any constant value of z. So we have to apply a change of coordinate system $(x,y,z) \rightarrow (x',y,z')$ where the y axis is the same because $\mathbf{E}(t,z)$ has constant phase for any y.

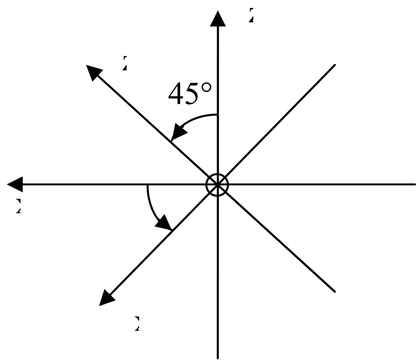

Fig. 2.13.1: Rotation of the coordinate system.

The expression of x' and z' are given by:

$$\begin{cases} \dfrac{x+z}{\sqrt{2}} = z' \\ \dfrac{x-z}{\sqrt{2}} = x' \end{cases} \Rightarrow \begin{cases} z = \dfrac{z'-x'}{\sqrt{2}} \\ x = \dfrac{x'+z'}{\sqrt{2}} \end{cases}$$

and we can rewrite the electric field in the new coordinate system:

$$\begin{aligned} \mathbf{E}(t,z) &= \sqrt{2}\hat{\mathbf{x}}'\cos(\omega t - kz') - \sqrt{2}\hat{\mathbf{y}}\sin(\omega t - kz') = \\ &= \sqrt{2}\hat{\mathbf{x}}'\cos(\omega t - kz') - \sqrt{2}\hat{\mathbf{y}}\cos(\omega t - kz' - \pi/2) = \\ &= \sqrt{2}\hat{\mathbf{x}}'\cos(\omega t - kz') + \sqrt{2}\hat{\mathbf{y}}\cos(\omega t - kz' + \pi/2) = \end{aligned}$$

The relative phase $\phi = \phi_x - \phi_y = -\pi/2$, so according to Fig. 2.8.2 and Fig. 2.8.3, the sense of rotation is clockwise. The field is forward–moving and , using the IEEE convention, the field is left–circular polarized , being $|E_x| = |E_y|$.

2.14 Exercise Equation Section (Next)

Consider a forward–moving wave in its real–valued form:

$$\mathbf{E}(t,z) = A\hat{\mathbf{x}}\cos(\omega t - kz + \phi_a) + B\hat{\mathbf{y}}\cos(\omega t - kz + \phi_b) \tag{2.14.1}$$

Show that:

$$\mathbf{E}(t+\Delta t, z+\Delta z)\times\mathbf{E}(t,z) = AB\hat{\mathbf{z}}\sin(\phi_a - \phi_b)\sin(\omega\Delta t - k\Delta z) \tag{2.14.2}$$

Solution

The cross product of two vectors $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ is defined as the determinant of the following matrix:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{2.14.3}$$

Using (2.14.3), we can write:

$$\mathbf{E}(t+\Delta t, z+\Delta z)\times\mathbf{E}(t,z) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A\cos\vartheta^{\Delta a} & B\cos\vartheta^{\Delta b} & 0 \\ A\cos\vartheta^{a} & B\cos\vartheta^{b} & 0 \end{vmatrix} = \mathrm{Det}(\underline{\mathbf{M}})$$

where $\vartheta^{\Delta i} = \omega(t+\Delta t) - k(z+\Delta z) + \phi_i$ and $\vartheta^{i} = \omega t - kz + \phi_i$ with $i = a, b$, and we have:

$$\begin{aligned} \mathrm{Det}(\underline{\mathbf{M}}) &= \hat{\mathbf{z}}\left[AB\cos\vartheta^{\Delta b}\cos\vartheta^{a} - AB\cos\vartheta^{\Delta a}\cos\vartheta^{b}\right] = \\ &= \hat{\mathbf{z}}AB\left[\cos\vartheta^{\Delta b}\cos\vartheta^{a} - \cos\vartheta^{\Delta a}\cos\vartheta^{b}\right] \end{aligned} \tag{2.14.4}$$

The expression inside the brackets can be simplified using the product–to–sum identity for cosine:

$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi) + \cos(\theta+\varphi)}{2} \tag{2.14.5}$$

and, consequently, (2.14.4) can be written as:

$$\mathrm{Det}(\underline{\mathbf{M}}) = \hat{\mathbf{z}}\mathrm{AB}\left[\cos\vartheta^{\Delta b}\cos\vartheta^{a} - \cos\vartheta^{\Delta a}\cos\vartheta^{b}\right] =$$

$$= \hat{\mathbf{z}}\mathrm{AB}\begin{bmatrix} +\frac{1}{2}\Big(\cos(\omega\Delta t - k\Delta z + \phi_b - \phi_a) + \cancel{\cos(2\omega t - 2kz + \omega\Delta t - k\Delta z + \phi_b + \phi_a)}\Big) + \\ -\frac{1}{2}\Big(\cos(\omega\Delta t - k\Delta z + \phi_a - \phi_b) + \cancel{\cos(2\omega t - 2kz + \omega\Delta t - k\Delta z + \phi_b + \phi_a)}\Big) \end{bmatrix} =$$

$$= \hat{\mathbf{z}}\frac{1}{2}\mathrm{AB}\left[\cos(\omega\Delta t - k\Delta z + \phi_b - \phi_a) - \cos(\omega\Delta t - k\Delta z + \phi_a - \phi_b)\right]$$

The expression inside the brackets can be still simplified using now the product–to– sum identity for sine:

$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi) - \cos(\theta+\varphi)}{2} \quad (2.14.6)$$

and, consequently, we have:

$$\begin{aligned} \mathrm{Det}(\underline{\mathbf{M}}) &= \hat{\mathbf{z}}\frac{1}{2}\mathrm{AB}\left[\cos(\omega\Delta t - k\Delta z + \phi_b - \phi_a) - \cos(\omega\Delta t - k\Delta z + \phi_a - \phi_b)\right] = \\ &= \hat{\mathbf{z}}\frac{1}{2}\mathrm{AB}\Big[\cos\big(\omega\Delta t - k\Delta z - (\phi_a - \phi_b)\big) - \cos\big(\omega\Delta t - k\Delta z + (\phi_a - \phi_b)\big)\Big] = \quad (2.14.7) \\ &= \hat{\mathbf{z}}\mathrm{AB}\sin(\omega\Delta t - k\Delta z)\sin(\phi_a - \phi_b) \end{aligned}$$

2.15 Exercise Equation Section (Next)

Show that in order for the polarization ellipse

$$\frac{E_x^2}{A^2}+\frac{E_y^2}{B^2}-2\frac{E_xE_y}{AB}\cos\phi=\sin^2\phi \tag{2.15.1}$$

to be equivalent to the rotated one with components

$$\begin{aligned} E'_x &= E_x\cos\theta+E_y\sin\theta \\ E'_y &= E_y\cos\theta-E_x\sin\theta \end{aligned} \tag{2.15.2}$$

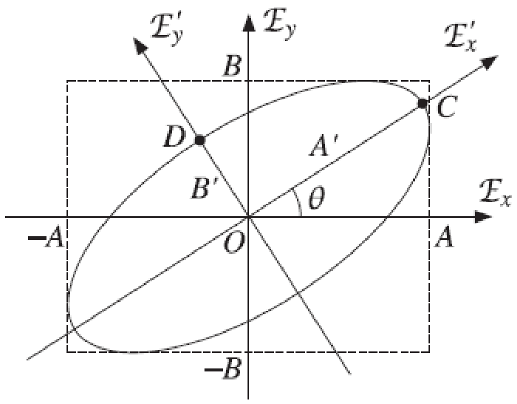

Fig. 2.15.1: General polarization ellipse.

one must determine the tilt angle θ such that the following matrix condition is satisfied:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{A^2} & -\frac{\cos\phi}{AB}\\ -\frac{\cos\phi}{AB} & \frac{1}{B^2}\end{bmatrix}\cdot\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\sin^2\phi\begin{bmatrix}\frac{1}{A'^2} & 0\\ 0 & \frac{1}{B'^2}\end{bmatrix} \tag{2.15.3}$$

Show that the required angle θ is given by

$$\tan 2\theta=\frac{2AB}{A^2-B^2}\cos\phi \tag{2.15.4}$$

Then show that the following condition is satisfied, where $\tau=\tan\theta$:

$$\frac{\left(A^2-B^2\tau^2\right)\left(B^2-A^2\tau^2\right)}{\left(1-\tau^2\right)^2}=A^2B^2\sin^2\phi \tag{2.15.5}$$

Using this property, show that the semi–axes A', B' are given by the equations:

$$A'^2=\frac{A^2-B^2\tau^2}{1-\tau^2},\qquad B'^2=\frac{B^2-A^2\tau^2}{1-\tau^2} \tag{2.15.6}$$

Then, transform these equations into the form:

$$A' = \sqrt{\frac{1}{2}\left(A^2 + B^2\right) + \frac{s}{2}\sqrt{\left(A^2 + B^2\right)^2 + 4A^2B^2\cos^2\phi}}$$
$$B' = \sqrt{\frac{1}{2}\left(A^2 + B^2\right) - \frac{s}{2}\sqrt{\left(A^2 + B^2\right)^2 + 4A^2B^2\cos^2\phi}} \quad (2.15.7)$$

where $s = \text{sign}(A - B)$.Finally, show that A', B' satisfy the relationships:

$$A'^2 + B'^2 = A^2 + B^2, \qquad A'B' = AB\left|\sin\phi\right| \quad (2.15.8)$$

Solution

The polarization ellipse in eq. (2.15.1) can be written in matrix form as follow:

$$\begin{bmatrix} E_x & E_y \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{A^2} & -\dfrac{\cos\phi}{AB} \\ -\dfrac{\cos\phi}{AB} & \dfrac{1}{B^2} \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \sin^2\phi \quad (2.15.9)$$

and, from the matrix form of (2.15.2), it is possible to obtain its inverse:

$$\begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} E_x \\ E_y \end{bmatrix} \Rightarrow \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} E'_x \\ E'_y \end{bmatrix} \quad (2.15.10)$$

Noting that the first vector in (2.15.9) is the transposed of the one in (2.15.10), we can substitute (2.15.10) into (2.15.9):

$$\begin{bmatrix} E'_x & E'_y \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{A^2} & -\dfrac{\cos\phi}{AB} \\ -\dfrac{\cos\phi}{AB} & \dfrac{1}{B^2} \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = \sin^2\phi \quad (2.15.11)$$

Eq. (2.15.11) represents the tilted ellipse shown in Fig. 2.15.1. The ellipse is not rotated with respect to the axes E'_x, E'_y and it is possible to define new values of the minor and major axis in this rotated coordinate system. As suggested from Fig. 2.15.1 the minor axis is B' and the major axis is A' and the equation of the tilted ellipse can be rewritten as:

$$\begin{bmatrix} E'_x & E'_y \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{A'^2} & 0 \\ 0 & \dfrac{1}{B'^2} \end{bmatrix} \cdot \begin{bmatrix} E'_x \\ E'_y \end{bmatrix} = 1 \quad (2.15.12)$$

Multiplying left and right side of (2.15.12) by $\sin^2\phi$

$$\sin^2\phi\begin{bmatrix}E'_x & E'_y\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{A'^2} & 0\\ 0 & \frac{1}{B'^2}\end{bmatrix}\cdot\begin{bmatrix}E'_x\\ E'_y\end{bmatrix}=\sin^2\phi \quad (2.15.13)$$

and comparing eq. (2.15.11) and eq. (2.15.13), we note they are equal if and only if:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{A^2} & -\frac{\cos\phi}{AB}\\ -\frac{\cos\phi}{AB} & \frac{1}{B^2}\end{bmatrix}\cdot\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\sin^2\phi\begin{bmatrix}\frac{1}{A'^2} & 0\\ 0 & \frac{1}{B'^2}\end{bmatrix} \quad (2.15.14)$$

The relationship (2.15.14) is fundamental to solve the whole exercise. Firstly, it can be manipulated in order to demonstrate eq. (2.15.4). Left multiplying both side by:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}^{-1}$$

we obtain:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}^{-1}\cdot\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{A^2} & -\frac{\cos\phi}{AB}\\ -\frac{\cos\phi}{AB} & \frac{1}{B^2}\end{bmatrix}\cdot\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=$$

$$=\sin^2\phi\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}^{-1}\cdot\begin{bmatrix}\frac{1}{A'^2} & 0\\ 0 & \frac{1}{B'^2}\end{bmatrix} \quad (2.15.15)$$

Since:

$$\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}^{-1}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

eq. (2.15.15) can be written as:

$$\begin{bmatrix}\frac{1}{A^2} & -\frac{\cos\phi}{AB}\\ -\frac{\cos\phi}{AB} & \frac{1}{B^2}\end{bmatrix}\cdot\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\sin^2\phi\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{A'^2} & 0\\ 0 & \frac{1}{B'^2}\end{bmatrix} \quad (2.15.16)$$

Let us now divide both side of (2.15.16) by $\cos\theta$ and, defining $\tau=\tan\theta$, we get:

$$\begin{bmatrix}\frac{1}{A^2} & -\frac{\cos\phi}{AB}\\ -\frac{\cos\phi}{AB} & \frac{1}{B^2}\end{bmatrix}\cdot\begin{bmatrix}1 & -\tau\\ \tau & 1\end{bmatrix}=\sin^2\phi\begin{bmatrix}1 & -\tau\\ \tau & 1\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{A'^2} & 0\\ 0 & \frac{1}{B'^2}\end{bmatrix} \quad (2.15.17)$$

Eq. (2.15.17) represents the following linear system:

$$\begin{cases} \dfrac{1}{A^2} - \tau\dfrac{\cos\phi}{AB} = \dfrac{\sin^2\phi}{A'^2} & \text{(a)} \\ \dfrac{1}{B^2} + \tau\dfrac{\cos\phi}{AB} = \dfrac{\sin^2\phi}{B'^2} & \text{(b)} \\ -\dfrac{\cos\phi}{AB} + \dfrac{\tau}{B^2} = \tau\dfrac{\sin^2\phi}{A'^2} & \text{(c)} \\ -\dfrac{\cos\phi}{AB} - \dfrac{\tau}{A^2} = \tau\dfrac{\sin^2\phi}{B'^2} & \text{(d)} \end{cases} \qquad (2.15.18)$$

Substituting (2.15.18)(a) and (2.15.18)(b) in (2.15.18)(c) and (2.15.18)(d) leads to:

$$\begin{cases} \dfrac{1}{A^2} - \tau\dfrac{\cos\phi}{AB} = \dfrac{\sin^2\phi}{A'^2} & \text{(a)} \\ \dfrac{1}{B^2} + \tau\dfrac{\cos\phi}{AB} = \dfrac{\sin^2\phi}{B'^2} & \text{(b)} \\ -\dfrac{\cos\phi}{AB} + \dfrac{\tau}{B^2} = \tau\left(\dfrac{1}{A^2} - \tau\dfrac{\cos\phi}{AB}\right) & \text{(c)} \\ -\dfrac{\cos\phi}{AB} - \dfrac{\tau}{A^2} = \tau\left(\dfrac{1}{B^2} + \tau\dfrac{\cos\phi}{AB}\right) & \text{(d)} \end{cases} \qquad (2.15.19)$$

It is easy to show that (2.15.19) (c) and (2.15.19) (d) are equivalent. In fact:

$$\text{(c)} \quad -\frac{\cos\phi}{AB} + \frac{\tau}{B^2} = \tau\left(\frac{1}{A^2} - \tau\frac{\cos\phi}{AB}\right) \Rightarrow$$

$$-\frac{\cos\phi}{AB} + \frac{\tau}{B^2} = \frac{\tau}{A^2} - \tau^2\frac{\cos\phi}{AB} \Rightarrow$$

$$-\frac{\cos\phi}{AB} - \frac{\tau}{A^2} = -\frac{\tau}{B^2} - \tau^2\frac{\cos\phi}{AB} \Rightarrow$$

$$-\frac{\cos\phi}{AB} - \frac{\tau}{A^2} = -\tau\left(\frac{1}{B^2} + \tau\frac{\cos\phi}{AB}\right) \quad \text{(d)}$$

Starting from (2.15.19) (c), or (d):

$$-\frac{\cos\phi}{AB} + \frac{\tau}{B^2} = \tau\left(\frac{1}{A^2} - \tau\frac{\cos\phi}{AB}\right) \Rightarrow$$

$$-\frac{\cos\phi}{AB} + \frac{\tau}{B^2} = \frac{\tau}{A^2} - \tau^2\frac{\cos\phi}{AB} \Rightarrow$$

$$\frac{\tau}{B^2} - \frac{\tau}{A^2} = -\tau^2\frac{\cos\phi}{AB} + \frac{\cos\phi}{AB} \Rightarrow$$

$$\tau\left(\frac{A^2 - B^2}{A^2B^2}\right) = \frac{\cos\phi}{AB}\left(1 - \tau^2\right)$$

$$\frac{\tau}{1-\tau^2} = \left(\frac{A^{\cancel{2}}B^{\cancel{2}}}{A^2-B^2}\right)\frac{\cos\phi}{\cancel{AB}} = \frac{AB}{A^2-B^2}\cos\phi \tag{2.15.20}$$

It is known that:

$$\tan 2\theta = 2\frac{\tan\theta}{1-\tan^2\theta} = 2\frac{\tau}{1-\tau^2}$$

so (2.15.20) can be written as

$$\tan 2\theta = \frac{2\tau}{1-\tau^2} = \frac{2AB}{A^2-B^2}\cos\phi \tag{2.15.21}$$

which is the same of eq. (2.15.4).

Eq. (2.15.19)(c), or (d), can be viewed as a quadratic equation in τ:

$$\tau^2\frac{\cos\phi}{AB} + \tau\left(\frac{A^2-B^2}{A^2B^2}\right) - \frac{\cos\phi}{AB} = 0 \tag{2.15.22}$$

with its solution given by:

$$\begin{aligned}\tau_{1,2} &= \frac{-\left(\frac{A^2-B^2}{A^2B^2}\right) \pm \sqrt{\left(\frac{A^2-B^2}{A^2B^2}\right)^2 + 4\frac{\cos^2\phi}{A^2B^2}}}{2\frac{\cos\phi}{AB}} = \\ &= \frac{-\left(\frac{A^2-B^2}{A^2B^2}\right) \pm \frac{1}{A^2B^2}\sqrt{\left(A^2-B^2\right)^2 + 4A^2B^2\cos^2\phi}}{2\frac{\cos\phi}{AB}} = \\ &= \frac{B^2-A^2 \pm \sqrt{\left(A^2-B^2\right)^2 + 4A^2B^2\cos^2\phi}}{2AB\cos\phi} = \frac{B^2-A^2+sD}{2AB\cos\phi} = \tau_s\end{aligned} \tag{2.15.23}$$

where:

- $s = \pm 1$;
- $D = \sqrt{\left(A^2-B^2\right)^2 + 4A^2B^2\cos^2\phi} = \sqrt{\left(A^2+B^2\right)^2 - 4A^2B^2\sin^2\phi}$;
- τ_s identifies the two solution for $s = \pm 1$.

The choice between $s = 1$ and $-s = 1$ does not matter because both are solution of eq. (2.15.22). So, for simplicity, we can set the value of s as:

$$s = \text{sign}(A-B) \tag{2.15.24}$$

which is the same value set in the text of the exercise. This choice is according to the fact that in general one defines the major axis with A and the minor axis with B and we have that $s = 1$.

The values $\tau_{\rm s}$, $\tau_{-\rm s}$ satisfy the identities (2.15.19) (c) and (2.15.19) (d) and so, if we substitute them inside (2.15.19) (a) or (b), we are able to eliminate ϕ in the definition of $\rm A'$, $\rm B'$ and A, B. We want to remark that the direct substitution is not a good choice because it leads to an expression difficult to be managed. So it is better to follow a longer way to solve the problem, but easier to be understood.

Let us now stop to consider the linear system (2.15.19) and let us diagonalize the matrix:

$$\begin{bmatrix} \dfrac{1}{\rm A^2} & -\dfrac{\cos\phi}{\rm AB} \\ -\dfrac{\cos\phi}{\rm AB} & \dfrac{1}{\rm B^2} \end{bmatrix} \Rightarrow \text{ diagonalize } \Rightarrow \begin{bmatrix} \dfrac{1}{\rm A'^2} & 0 \\ 0 & \dfrac{1}{\rm B'^2} \end{bmatrix}$$

The eigenvalues are the roots of the characteristic polynomial:

$$\operatorname{Det}\left(\begin{bmatrix} \dfrac{1}{\rm A^2} & -\dfrac{\cos\phi}{\rm AB} \\ -\dfrac{\cos\phi}{\rm AB} & \dfrac{1}{\rm B^2} \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

that is

$$\begin{aligned}
&\left(\frac{1}{\rm A^2}-\lambda\right)\left(\frac{1}{\rm B^2}-\lambda\right)-\frac{\cos^2\phi}{\rm A^2B^2}=0 && \Rightarrow \\
&\frac{1}{\rm A^2B^2}-\frac{\lambda}{\rm A^2}-\frac{\lambda}{\rm B^2}+\lambda^2-\frac{\cos^2\phi}{\rm A^2B^2}=0 && \Rightarrow \\
&\lambda^2-\lambda\left(\frac{\rm A^2+B^2}{\rm A^2B^2}\right)+\frac{1}{\rm A^2B^2}\left(1-\cos^2\phi\right)=0 && \Rightarrow \qquad (2.15.25) \\
&\lambda^2\left(\rm A^2B^2\right)-\lambda\left(\rm A^2+B^2\right)+\left(1-\cos^2\phi\right)=0 && \Rightarrow \\
&\lambda^2\left(\rm A^2B^2\right)-\lambda\left(\rm A^2+B^2\right)+\sin^2\phi=0
\end{aligned}$$

Finally we get:

$$\lambda_{\rm s}=\frac{\left(\rm A^2+B^2\right)+s\sqrt{\left(A^2+B^2\right)^2-4A^2B^2\sin^2\phi}}{\rm 2A^2B^2}=\frac{\rm A^2+B^2+sD}{\rm 2A^2B^2}, \quad \text{with} \quad \rm s=\pm1 \quad (2.15.26)$$

$\lambda_{\pm\rm s}$ are the eigenvalues, useful to simplify the expressions when substituting $\tau_{\rm s}$ in eq. (2.15.18) (a) and (b). In fact:

$$\frac{1}{A^2}-\tau_s\frac{\cos\phi}{AB}=\frac{1}{A^2}-\frac{B^2-A^2+sD}{2AB\cancel{\cos\phi}}\frac{\cancel{\cos\phi}}{AB}=\frac{1}{A^2}-\frac{B^2-A^2+sD}{2A^2B^2}=\frac{A^2+B^2-sD}{2A^2B^2}=\lambda_{-s}$$
$$\frac{1}{B^2}-\tau_s\frac{\cos\phi}{AB}=\frac{1}{B^2}-\frac{B^2-A^2+sD}{2AB\cancel{\cos\phi}}\frac{\cancel{\cos\phi}}{AB}=\frac{1}{B^2}-\frac{B^2-A^2+sD}{2A^2B^2}=\frac{A^2+B^2+sD}{2A^2B^2}=\lambda_s \qquad (2.15.27)$$

or in a more compact form:

$$\begin{cases}\dfrac{1}{A^2}-\tau_s\dfrac{\cos\phi}{AB}=\lambda_{-s}\\[2mm] \dfrac{1}{B^2}-\tau_s\dfrac{\cos\phi}{AB}=\lambda_s\end{cases} \qquad (2.15.28)$$

Comparing (2.15.28) with (2.15.19) (a) and (2.15.19) (b), we can write:

$$\begin{cases}\dfrac{\sin^2\phi}{A'^2}=\lambda_{-s}\\[2mm] \dfrac{\sin^2\phi}{B'^2}=\lambda_s\end{cases} \qquad (2.15.29)$$

and consequently

$$\begin{cases}A'^2=\dfrac{\sin^2\phi}{\lambda_{-s}}=\dfrac{\sin^2\phi}{\lambda_s\lambda_{-s}}\lambda_s\\[2mm] B'^2=\dfrac{\sin^2\phi}{\lambda_s}=\dfrac{\sin^2\phi}{\lambda_s\lambda_{-s}}\lambda_{-s}\end{cases} \qquad (2.15.30)$$

The product $\lambda_s\lambda_{-s}$ is given by:

$$\begin{aligned}\lambda_s\lambda_{-s}&=\frac{A^2+B^2+sD}{2A^2B^2}\cdot\frac{A^2+B^2-sD}{2A^2B^2}=\\ &=\frac{\left(A^2+B^2\right)^2-D^2}{4A^4B^4}=\\ &=\frac{\cancel{\left(A^2+B^2\right)^2}-\cancel{\left(A^2+B^2\right)^2}+\cancel{4}A^2B^2\sin^2\phi}{\cancel{4}A^4B^4}=\frac{\sin^2\phi}{A^2B^2}\end{aligned} \qquad (2.15.31)$$

and consequently the equations (2.15.30) became:

$$A'^2=\frac{\sin^2\phi}{\lambda_s\lambda_{-s}}\lambda_s=\frac{\sin^2\phi}{\dfrac{\sin^2\phi}{A^2B^2}}\lambda_s=A^2B^2\lambda_s=\frac{1}{2}\left[A^2+B^2+sD\right]$$
$$B'^2=\frac{\sin^2\phi}{\lambda_s\lambda_{-s}}\lambda_{-s}=\frac{\sin^2\phi}{\dfrac{\sin^2\phi}{A^2B^2}}\lambda_{-s}=A^2B^2\lambda_{-s}=\frac{1}{2}\left[A^2+B^2-sD\right] \qquad (2.15.32)$$

giving the expressions for A' and B' as required by the exercise [see eq. (2.15.7)].

In the same way, starting from (2.15.32):

$$A'^2 = A^2B^2\lambda_s$$
$$B'^2 = A^2B^2\lambda_{-s}$$

we can substitute the other value of $\lambda_{\pm s}$ as in eq.(2.15.28), that is:

$$\begin{cases} \lambda_{-s} = \dfrac{1}{A^2} - \tau_s \dfrac{\cos\phi}{AB} \\ \lambda_s = \dfrac{1}{B^2} + \tau_s \dfrac{\cos\phi}{AB} \end{cases}$$

and we have:

$$\begin{aligned} A'^2 &= A^2B^2\left(\frac{1}{B^2} + \tau_s \frac{\cos\phi}{AB}\right) = A^2 + \tau_s AB\cos\phi \\ B'^2 &= A^2B^2\left(\frac{1}{A^2} - \tau_s \frac{\cos\phi}{AB}\right) = B^2 - \tau_s AB\cos\phi \end{aligned} \tag{2.15.33}$$

From eq. (2.15.21) we have also:

$$\frac{2\tau_s}{1-\tau_s^2} = \frac{2AB}{A^2 - B^2}\cos\phi \quad \Rightarrow \quad AB\cos\phi = \frac{\left(A^2 - B^2\right)\tau_s}{1-\tau_s^2} \tag{2.15.34}$$

which can be substituted in (2.15.33):

$$\begin{aligned} A'^2 &= A^2 + \tau_s AB\cos\phi = A^2 + \tau_s \frac{\left(A^2 - B^2\right)\tau_s}{1-\tau_s^2} = \frac{A^2 - B^2\tau_s^2}{1-\tau_s^2} \\ B'^2 &= B^2 - \tau_s AB\cos\phi = B^2 - \tau_s AB \frac{\left(A^2 - B^2\right)\tau_s}{1-\tau_s^2} = \frac{B^2 - A^2\tau_s^2}{1-\tau_s^2} \end{aligned} \tag{2.15.35}$$

Eq. (2.15.35) gives the expression of A' and B' as required by the exercise [see eq.(2.15.6)].

Adding A' and B', we obtain:

$$\begin{aligned} A'^2 + B'^2 &= \frac{A^2 - B^2\tau_s^2}{1-\tau_s^2} + \frac{B^2 - A^2\tau_s^2}{1-\tau_s^2} = \\ &= \frac{A^2 - B^2\tau_s^2 + B^2 - A^2\tau_s^2}{1-\tau_s^2} = \frac{\left(A^2 + B^2\right)\cancel{\left(1-\tau_s^2\right)}}{\cancel{1-\tau_s^2}} = A^2 + B^2 \end{aligned} \tag{2.15.36}$$

and, multiplying A' and B', we obtain:

$$A'^2B'^2 = \left(A^2B^2\right)^2 \lambda_s\lambda_{-s} = \left(A^2B^2\right)^{\cancel{2}} \frac{\sin^2\phi}{\cancel{A^2B^2}} = A^2B^2\sin^2\phi \quad \Rightarrow \quad A'B' = AB\left|\sin\phi\right| \tag{2.15.37}$$

It is easy to note that eq. (2.15.36) and (2.15.37) are equal to eq. (2.15.8) in the text of the exercise.

2.16 ExerciseEquation Section (Next)

Considering the electric field $\mathbf{E}(t)=\hat{\mathbf{x}}A\cos(\omega t+\phi_a)+\hat{\mathbf{y}}B\cos(\omega t+\phi_b)$, show the cross–product equation:

$$\mathbf{E}(0)\times\mathbf{E}(t)=\hat{\mathbf{z}}AB\sin\phi\sin\omega t \tag{2.16.1}$$

where $\phi=\phi_a-\phi_b$. Then prove the more general relationship:

$$\mathbf{E}(t_1)\times\mathbf{E}(t_2)=\hat{\mathbf{z}}AB\sin\phi\sin\left(\omega(t_2-t_1)\right) \tag{2.16.2}$$

Discuss how linear polarization can be explained with the help of this result.

Solution

Using (2.14.3), we can write:

$$\mathbf{E}(0)\times\mathbf{E}(t)=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A\cos\phi_a & B\cos\phi_b & 0\\ A\cos(\omega t+\phi_a) & B\cos(\omega t+\phi_b) & 0\end{vmatrix} \tag{2.16.3}$$

and we have:

$$\begin{aligned}\mathbf{E}(0)\times\mathbf{E}(t)&=\hat{\mathbf{z}}\left[AB\cos\phi_a\cos(\omega t+\phi_b)-AB\cos\phi_b\cos(\omega t+\phi_a)\right]=\\ &=\hat{\mathbf{z}}AB\left[\cos\phi_a\cos(\omega t+\phi_b)-\cos\phi_b\cos(\omega t+\phi_a)\right]\end{aligned} \tag{2.16.4}$$

The expression inside the brackets can be simplified using the product–to–sum identity for cosine:

$$\cos\theta\cos\varphi=\frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2} \tag{2.16.5}$$

and, consequently, (2.16.4) can be written as:

$$\begin{aligned}\mathbf{E}(0)\times\mathbf{E}(t)&=\hat{\mathbf{z}}AB\left[\cos\phi_a\cos(\omega t+\phi_b)-\cos\phi_b\cos(\omega t+\phi_a)\right]\\ &=\hat{\mathbf{z}}AB\begin{bmatrix}+\frac{1}{2}\left(\cos(\phi_a-\omega t-\phi_b)+\cancel{\cos(\phi_a+\omega t+\phi_b)}\right)+\\ -\frac{1}{2}\left(\cos(\phi_b-\omega t-\phi_a)+\cancel{\cos(\phi_b+\omega t+\phi_a)}\right)\end{bmatrix}=\\ &=\hat{\mathbf{z}}\frac{AB}{2}\left[\cos(\phi_a-\omega t-\phi_b)-\cos(\phi_b-\omega t-\phi_a)\right]=\\ &=\hat{\mathbf{z}}\frac{AB}{2}\left[\cos(-\omega t+\phi)-\cos(-\omega t-\phi)\right]\end{aligned}$$

The cosine is an even function, i.e. $\cos(\alpha)=\cos(-\alpha)$, so:

$$\mathbf{E}(0)\times\mathbf{E}(t)=\hat{\mathbf{z}}\frac{AB}{2}\left[\cos(\omega t-\phi)-\cos(\omega t+\phi)\right]$$

The expression inside the brackets can be still simplified using now the product–to– sum identity for sine:

$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2} \tag{2.16.6}$$

and, consequently, we have:

$$\begin{aligned}\mathbf{E}(0)\times\mathbf{E}(t) &= \hat{\mathbf{z}}\frac{AB}{2}\left[\cos(\omega t-\phi)-\cos(\omega t+\phi)\right]=\\ &= \hat{\mathbf{z}}AB\sin\phi\sin\omega t\end{aligned} \tag{2.16.7}$$

The more general relationship (2.16.2) can be proven in the same way of eq. (2.16.1):

$$\begin{aligned}\mathbf{E}(t_1)\times\mathbf{E}(t_2) &= \mathrm{Det}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A\cos(\omega t_1+\phi_a) & B\cos(\omega t_1+\phi_b) & 0\\ A\cos(\omega t_2+\phi_a) & B\cos(\omega t_2+\phi_b) & 0\end{vmatrix}=\\ &= \hat{\mathbf{z}}AB\left[\cos(\omega t_1+\phi_a)\cos(\omega t_2+\phi_b)-\cos(\omega t_1+\phi_b)\cos(\omega t_2+\phi_a)\right]=\\ &= \hat{\mathbf{z}}\frac{AB}{2}\begin{bmatrix}\cos(\omega t_1+\phi_a-\omega t_2-\phi_b)+\cancel{\cos(\omega t_1+\phi_a+\omega t_2+\phi_b)}+\\ -\cos(\omega t_1+\phi_b-\omega t_2-\phi_a)-\cancel{\cos(\omega t_1+\phi_b+\omega t_2+\phi_a)}\end{bmatrix}=\\ &= \hat{\mathbf{z}}\frac{AB}{2}\left[\cos(\omega(t_1-t_2)+\phi)-\cos(\omega(t_1-t_2)-\phi)\right]=\\ &= \hat{\mathbf{z}}\frac{AB}{2}\left[\cos(\omega(t_2-t_1)-\phi)-\cos(\omega(t_2-t_1)+\phi)\right]=\\ &= \hat{\mathbf{z}}AB\sin\phi\sin(\omega(t_2-t_1))\end{aligned} \tag{2.16.8}$$

When the electric field is linear polarized, the electric field vector, sampled in any t, is always along a fixed direction, so the angle between the two vectors $\mathbf{E}(t_1)$ and $\mathbf{E}(t_2)$, represented by the relative phase $\phi=\phi_a-\phi_b$, is always zero and the cross–product is null at any $\Delta t = t_2 - t_1$.

2.17 ExerciseEquation Section (Next)

Using the properties $k_c\eta_c = \omega\mu$ and $k_c^2 = \omega^2\mu\varepsilon_c$ for the complex–valued quantities k_c, η_c of equation:

$$k_c = \omega\sqrt{\mu\varepsilon_c}, \qquad \eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \tag{2.17.1}$$

where $\varepsilon_c = \varepsilon' - j\varepsilon''$ is the complex value of permittivity of a lossy media and $k_c = \beta - j\alpha$, show the following relationships:

$$\Re e\left[\eta_c^{-1}\right] = \frac{\omega\varepsilon''}{2\alpha} = \frac{\beta}{\omega\mu} \tag{2.17.2}$$

Solution

From the first property, let us express the characteristic impedance as:

$$\eta_c = \frac{\omega\mu}{k_c} \tag{2.17.3}$$

and invert it:

$$\eta_c^{-1} = \frac{1}{\eta_c} = \frac{k_c}{\omega\mu} \tag{2.17.4}$$

Now it is possible to substitute $k_c \to \beta - j\alpha$ in eq. (2.17.4) and extract the real part:

$$\Re e\left[\eta_c^{-1}\right] = \Re e\left[\frac{\beta - j\alpha}{\omega\mu}\right] = \frac{\beta}{\omega\mu}$$

2.18 ExerciseEquation Section (Next)

Show that for a lossy medium the complex–valued quantities $\mathrm{k_c}$ and η_c may be expressed as follows, in terms of the loss angle θ defined in:

$$\tau = \tan\theta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma + \omega\varepsilon''_d}{\omega\varepsilon'_d} \tag{2.18.1}$$

$$\begin{aligned} \mathrm{k_c} &= \beta - \mathrm{j}\alpha = \omega\sqrt{\mu\varepsilon'_d}\left(\cos\frac{\theta}{2} - \mathrm{j}\sin\frac{\theta}{2}\right)(\cos\theta)^{-1/2} \\ \eta_c &= \eta' - \mathrm{j}\eta'' = \sqrt{\frac{\mu_d}{\varepsilon'}}\left(\cos\frac{\theta}{2} + \mathrm{j}\sin\frac{\theta}{2}\right)(\cos\theta)^{1/2} \end{aligned} \tag{2.18.2}$$

Solution

Using the definition of $\mathrm{k_c}$ in (2.17.1) and the relationship (2.18.1), we can write:

$$\begin{aligned} \mathrm{k_c} &= \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu(\varepsilon' - \mathrm{j}\varepsilon'')} = \\ &= \omega\sqrt{\mu\varepsilon'(1 - \mathrm{j}\tan\theta)} = \omega\sqrt{\mu\varepsilon'}\,(1 - \mathrm{j}\tan\theta)^{1/2} \end{aligned} \tag{2.18.3}$$

The complex–valued permittivity ε_c is also defined as

$$\varepsilon_c = \varepsilon' - \mathrm{j}\varepsilon'' = \varepsilon'_d - \mathrm{j}\left(\varepsilon''_d + \frac{\sigma}{\omega}\right) \tag{2.18.4}$$

where $\varepsilon_d = \varepsilon'_d - \mathrm{j}\varepsilon''_d$ is the permittivity of dielectric and σ its conductivity. So in (2.18.3) we can substitute $\varepsilon' \to \varepsilon'_d$ and $\tan\theta \to \sin\theta/\cos\theta$ to obtain:

$$\begin{aligned} \mathrm{k_c} &= \omega\sqrt{\mu\varepsilon'_d}\left(1 - \mathrm{j}\frac{\sin\theta}{\cos\theta}\right)^{1/2} = \\ &= \omega\sqrt{\mu\varepsilon'_d}\left(\frac{\cos\theta - \mathrm{j}\sin\theta}{\cos\theta}\right)^{1/2} = \\ &= \omega\sqrt{\mu\varepsilon'_d}\,(\cos\theta - \mathrm{j}\sin\theta)^{1/2}\left(\frac{1}{\cos\theta}\right)^{1/2} = \\ &= \omega\sqrt{\mu\varepsilon'_d}\left(\mathrm{e}^{-\mathrm{j}\theta}\right)^{1/2}(\cos\theta)^{-1/2} = \\ &= \omega\sqrt{\mu\varepsilon'_d}\left(\mathrm{e}^{-\mathrm{j}\frac{\theta}{2}}\right)(\cos\theta)^{-1/2} = \omega\sqrt{\mu\varepsilon'_d}\left(\cos\frac{\theta}{2} - \mathrm{j}\sin\frac{\theta}{2}\right)(\cos\theta)^{-1/2} \end{aligned} \tag{2.18.5}$$

In the same way it is possible to express η_c as in (2.18.2), starting from its definition in (2.17.1):

$$
\begin{aligned}
\eta_c &= \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon' - j\varepsilon''}} = \sqrt{\frac{\mu}{\varepsilon' - j\varepsilon' \tan\theta}} = \\
&= \sqrt{\frac{\mu}{\varepsilon'_d}} \left(\frac{1}{1 - j\tan\theta} \right)^{1/2} = \sqrt{\frac{\mu}{\varepsilon'_d}} \left(\frac{\cos\theta}{\cos\theta - j\sin\theta} \right)^{1/2} \\
&= \sqrt{\frac{\mu}{\varepsilon'_d}} \left(e^{-j\theta} \right)^{-1/2} \left(\cos\theta \right)^{1/2} = \sqrt{\frac{\mu}{\varepsilon'_d}} \left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2} \right) \left(\cos\theta \right)^{1/2}
\end{aligned}
\tag{2.18.6}
$$

2.19 ExerciseEquation Section (Next)

It is desired to reheat frozen mesh potatoes and frozen cooked carrots in a microwave oven operating at 2.45 GHz. Determine the penetration depth and assess effectiveness of this heating method. Moreover, determine the attenuation of the electric field (in dB and absolute units) at a depth of 1 cm from the surface of the food. The complex dielectric constants of the mashed potatoes and carrots are $\varepsilon_c = (65 - j25)\varepsilon_0$ and $\varepsilon_c = (75 - j25)\varepsilon_0$, respectively.

Solution

First of all we have to express the complex–values of the permittivity as follow:

$$a + jb = Me^{j\varphi} \quad \text{where} \begin{cases} M = \sqrt{a^2 + b^2} \\ \varphi = \arctan b/a \end{cases} \tag{2.19.1}$$

so, using the superscripts 1 and 2 to indicate the permittivity of potatoes and carrots, respectively, we have:

$$\varepsilon_c^1 = (65 - j25)\varepsilon_0 = \sqrt{65^2 + 25^2} e^{j\mathrm{Arc}\tan(-25/65)} \simeq 69.64e^{-j0.367}$$
$$\varepsilon_c^2 = (75 - j25)\varepsilon_0 = \sqrt{75^2 + 25^2} e^{j\mathrm{Arc}\tan(-25/75)} \simeq 79.06e^{-j0.322}$$

The free–space wave number of a microwave at 2.45 GHz is:

$$k_0 = \omega\sqrt{\mu_0\varepsilon_0} = \frac{2\pi f}{c_0} = \frac{2\pi \times 2.45 \times 10^9}{3 \times 10^8} = 51.31 \frac{\text{rad}}{\text{m}}$$

Using $k_c = \omega\sqrt{\mu_0\varepsilon_c} = \omega\sqrt{\mu_0\varepsilon_0(\varepsilon_c/\varepsilon_0)} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{\varepsilon_c/\varepsilon_0} = k_0\sqrt{\varepsilon_c/\varepsilon_0}$, we calculate the wavenumbers:

$$\begin{aligned} k_c^1 &= \beta - j\alpha = 51.31\sqrt{65 - j25} \simeq 51.31\sqrt{69.64} e^{-j\frac{0.367}{2}} = \\ &\simeq 428.18e^{-j0.1835} = 428.18(\cos(0.1835) - j\sin(0.1835)) = 421 - j78.13 \text{ m}^{-1} \end{aligned} \tag{2.19.2}$$

$$\begin{aligned} k_c^2 &= \beta - j\alpha = 51.31\sqrt{75 - j25} \simeq 51.31\sqrt{79.06} e^{-j\frac{0.322}{2}} = \\ &\simeq 456.23e^{-j0.161} = 456.23(\cos(0.161) - j\sin(0.161)) = 450 - j73.14 \text{ m}^{-1} \end{aligned} \tag{2.19.3}$$

The corresponding attenuation constants and penetration depths are:

$$\alpha^1 = 78.13 \text{ nepers/m}, \qquad \delta^1 = 1/\alpha^1 = 12.8 \text{ cm}$$
$$\alpha^2 = 73.14 \text{ nepers/m}, \qquad \delta^2 = 1/\alpha^2 = 13.7 \text{ cm}$$

This heating method is effective because the penetration depths are bigger than the dimension of mesh potatoes and carrots. The energy in the electromagnetic waves reheat successfully the foods.

The attenuation of the electric field (in dB and absolute units) at a depth of 1 cm from the surface of the food is:

$$\begin{aligned} A^1_{dB}(z=1\text{ cm}) &= 8.686\,z/\delta^1 = 8.686/12.8 = 0.68\text{ dB} \\ A^2_{dB}(z=1\text{ cm}) &= 8.686\,z/\delta^2 = 8.686/13.7 = 0.63\text{ dB} \end{aligned} \tag{2.19.4}$$

$$\begin{aligned} A^1 &= 10^{-\frac{A^1_{dB}}{20}} = 0.925 \\ A^1 &= 10^{-\frac{A^2_{dB}}{20}} = 0.930 \end{aligned} \tag{2.19.5}$$

Thus, the fields at a depth of 1 cm are 92.5% and 93% of their values at the surface.

2.20 ExerciseEquation Section (Next)

We wish to shield a piece of equipment from RF interference over the frequency range from 10 kHz to 1 GHz by enclosing it in a copper enclosure. The RF interference inside the enclosure is required to be at least 50 dB down compared to its value outside. What is the minimum thickness of the copper shield in mm?

Solution

The parameters β, α and δ in a good conductor are:

$$\beta = \alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma} \tag{2.20.1}$$

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu\sigma}} \tag{2.20.2}$$

The conductibility of the copper is 5.8×10^7 Siemens/m, so the skin depth at frequency f is:

$$\delta = \frac{1}{\sqrt{\pi f \mu\sigma}} = \frac{1}{\sqrt{\pi\times4\pi\times10^{-7}\times5.8\times10^{7}}} f^{-1/2} = 0.0661\times f^{-1/2} \tag{2.20.3}$$

where the frequency f is expressed in Hertz.

The attenuation in dB is:

$$A_{dB}(z) = 8.686\, z/\delta \tag{2.20.4}$$

and its minimum value over the frequency range of interest is at least 50 dB. So inverting the eq. (2.20.4) with the assumption that $A_{dB}(z) \geq 50\text{dB}$, we have:

$$A_{dB}(z) = 8.686\, z/\delta \geq 50\text{dB} \quad \Rightarrow \quad z \geq \frac{50}{8.686}\delta = \frac{50}{8.686}0.0661\times f^{-1/2} = 0.3805\times f^{-1/2} \tag{2.20.5}$$

The inequality (2.20.5) can be plotted as function of frequency in the range 10 kHz–1 GHz:

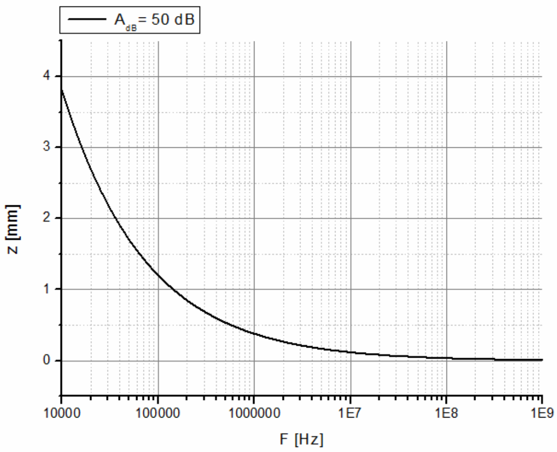

Fig. 2.20.1: Thickness of copper shield in mm for 50dB of attenuation.

The high frequency interference is attenuated of 50dB using a copper shield with thickness very low, exactly, at 1 GHz, 0.012 mm of copper are sufficient. On the contrary at low frequencies the thickness is more, exactly, at 10 kHz, z =3.8 mm, that represents the minimum thickness of the shield in order to satisfy the attenuation limit.

2.21 ExerciseEquation Section (Next)

In order to protect a piece of equipment from RF interference, we construct an enclosure made of aluminium foil (you may assume a reasonable value for its thickness). The conductivity of aluminium is $3.5\times10^{7}\,\mathrm{S/m}$. Over what frequency range can this shield protect our equipment assuming the same 50dB attenuation requirement of the previous problem?

Solution

First of all we have to evaluate the skin depth as function of the frequency f:

$$\delta=\frac{1}{\sqrt{\pi f\mu\sigma}}=\frac{1}{\sqrt{\pi\times4\pi\times10^{-7}\times3.5\times10^{7}}}f^{-1/2}\simeq0.0851\times f^{-1/2} \qquad (2.21.1)$$

The attenuation in dB is:

$$A_{dB}(z)=8.686\,z/\delta \qquad (2.21.2)$$

and its minimum value over the frequency range of interest is at least 50 dB. So inverting the eq. (2.21.2) with the assumption that $A_{dB}(z)\geq50\mathrm{dB}$, we have:

$$A_{dB}(z)=8.686\,z/\delta\geq50\mathrm{dB}\quad\Rightarrow\quad z\geq\frac{50}{8.686}\delta=\frac{50}{8.686}0.0661\times f^{-1/2}=0.3805\times f^{-1/2} \qquad (2.21.3)$$

The inequality (2.21.3) can be plotted as function of frequency in the range 10 kHz–1 GHz:

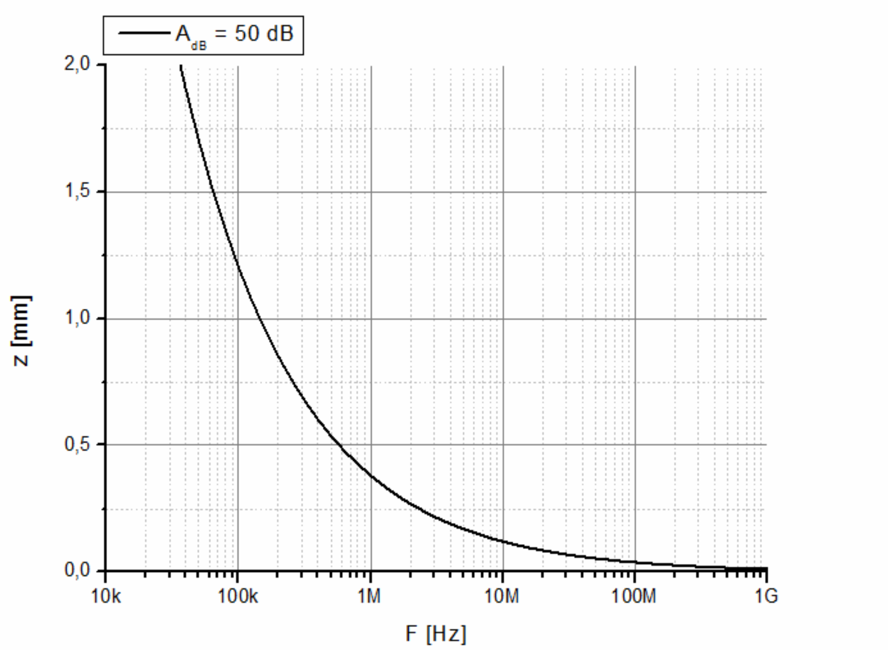

Fig. 2.21.1: Thickness of aluminium shield in mm for 50dB of attenuation.

A typical thickness of aluminium is about 1 mm, and this shield reduces by 50 dB only electric fields with frequency greater than 150 kHz.

2.22 ExerciseEquation Section (Next)

A uniform plane wave propagating towards the positive z–direction in empty space has an electric field at $z=0$ that is a linear superposition of two components of frequencies ω_1 and ω_2:

$$\mathbf{E}(0,t)=\hat{\mathbf{x}}\left(E_1 e^{j\omega_1 t}+E_2 e^{j\omega_2 t}\right) \tag{2.22.1}$$

Determine the fields $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ at any point z.

Solution

For a forward–moving wave, we have $\mathbf{E}(z,t)=\mathbf{F}(z-ct)=\mathbf{F}(0-c(t-z/c))$, which implies that $\mathbf{E}(z,t)$ is completely determined by $\mathbf{E}(z,0)$ or, alternatively, by $\mathbf{E}(0,t)$:

$$\mathbf{E}(z,t)=\mathbf{E}(z-ct,0)=\mathbf{E}(0,t-z/c)$$

Using this property , we find for the electric and magnetic fields:

$$\begin{aligned}\mathbf{E}(z,t)&=\mathbf{E}(0,t-z/c)=\hat{\mathbf{x}}\left(E_1 e^{j\omega_1(t+z/c)}+E_2 e^{j\omega_2(t+z/c)}\right)=\\&=\hat{\mathbf{x}}\left(E_1 e^{j\omega_1 t}e^{jk_1 z}+E_2 e^{j\omega_1 t}e^{jk_2 z}\right)\end{aligned} \tag{2.22.2}$$

where $k_i=\omega_i/c$ with $i=1,2$, and the magnetic field is:

$$\mathbf{H}(z,t)=\frac{1}{Z_0}\hat{\mathbf{z}}\times\mathbf{E}(z,t)=\hat{\mathbf{y}}\left(H_1 e^{j\omega_1 t}e^{jk_1 z}+H_2 e^{j\omega_1 t}e^{jk_2 z}\right) \tag{2.22.3}$$

where $H_i=E_i/Z_0$ with $i=1,2$.

2.23 ExerciseEquation Section (Next)

An electromagnetic wave propagating in a lossless dielectric is described by the electric and magnetic fields, $\mathbf{E}(z)=\hat{\mathbf{x}}E(z)$ and $\mathbf{H}(z)=\hat{\mathbf{y}}H(z)$, consisting of the forward and backward components:

$$\begin{aligned} E(z) &= E_+e^{-jkz}+E_-e^{jkz} \\ H(z) &= \frac{1}{\eta}\left(E_+e^{-jkz}-E_-e^{jkz}\right) \end{aligned} \tag{2.23.1}$$

1) Verify that these expressions satisfy all of Maxwell's equations.
2) Show that the time–averaged energy flux in the z–direction is independent of z and is given by:

$$P_z=\frac{1}{2}\Re e\left\{E(z)H^*(z)\right\}=\frac{1}{2\eta}\left(|E_+|^2+|E_-|^2\right) \tag{2.23.2}$$

3) Assuming $\mu=\mu_0$ and $\varepsilon=n^2\varepsilon_0$, so that n is the refractive index of the dielectric, show that the fields at two different z–locations, say at $z=z_1$ and $z=z_2$ are related by the matrix equation:

$$\begin{bmatrix} E(z_1) \\ \eta_0 H(z_1) \end{bmatrix}=\begin{bmatrix} \cos k\ell & j\eta^{-1}\sin k\ell \\ j\eta\sin k\ell & \cos k\ell \end{bmatrix}\begin{bmatrix} E(z_2) \\ \eta_0 H(z_2) \end{bmatrix} \tag{2.23.3}$$

where $\ell=z_2-z_1$, and we multiplied the magnetic field by $\eta_0=\sqrt{\mu_0/\varepsilon_0}$ in order to give it the same dimensions as the electric field.

4) Let $Z(z)=\dfrac{E(z)}{\eta_0 H(z)}$ and $Y(z)=\dfrac{1}{Z(z)}$ be the normalized wave impedance and admittance at location z. Show the relationships at the location z_1 and z_2:

$$Z(z_1)=\frac{Z(z_2)+j\eta^{-1}\tan k\ell}{1+j\eta Z(z_2)\tan k\ell}, \qquad Y(z_1)=\frac{Y(z_2)+j\eta\tan k\ell}{1+j\eta^{-1}Y(z_2)\tan k\ell} \tag{2.23.4}$$

What would be these relationships if had we normalized to the medium impedance, that is, $Z(z)=E(z)/\eta H(z)$?

Solution

- Question n° 1

Assuming an harmonic time dependence $e^{j\omega t}$, the Maxwell's equations can be written as follow:

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mathbf{B} \\ \nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J} \\ \nabla \cdot \mathbf{D} = \rho \\ \nabla \cdot \mathbf{B} = 0 \end{cases} \tag{2.23.5}$$

In a source less, linear, isotropic and uniform medium, the quantities ρ and $\mathbf{J}$ are zero and the constitutive relations $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \varepsilon\mathbf{E}$ are valid. So the set (2.23.5) becomes:

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} \\ \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases} \tag{2.23.6}$$

Now it is possible to verify the first Maxwell's equation in set (2.23.6):

$$\begin{aligned} \nabla \times \hat{\mathbf{x}}E(z) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ E(z) & 0 & 0 \end{vmatrix} = \hat{\mathbf{y}}\partial_z E(z) - \cancel{\hat{\mathbf{z}}\partial_y E(z)} = \\ &= \hat{\mathbf{y}}\partial_z\left(E_+e^{-jkz} + E_-e^{jkz}\right) = \hat{\mathbf{y}}\left(-jkE_+e^{-jkz} + jkE_-e^{jkz}\right) = \\ &= -\hat{\mathbf{y}}jk\left(E_+e^{-jkz} - E_-e^{jkz}\right) \end{aligned} \tag{2.23.7}$$

From exercise 2.7 we know the $k-\omega$ relationship and as consequence also the expression of the characteristic impedance η:

$$k = \omega\sqrt{\mu\varepsilon}, \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{2.23.8}$$

where $\mu = \mu_0\mu_r$, $\varepsilon = \varepsilon_0\varepsilon_r$. So inserting (2.23.8) in (2.23.7), we obtain:

$$\begin{aligned} \nabla \times \hat{\mathbf{x}}E(z) &= -\hat{\mathbf{y}}jkE(z) = -j\hat{\mathbf{y}}\omega\sqrt{\mu\varepsilon}E(z) = \\ &= -j\omega\sqrt{\frac{\mu^2\varepsilon}{\mu}}\hat{\mathbf{y}}E(z) = -j\omega\mu\hat{\mathbf{y}}\sqrt{\frac{\varepsilon}{\mu}}E(z) = \\ &= -j\omega\mu\hat{\mathbf{y}}\frac{E(z)}{\eta} = -j\omega\mu H(z) \end{aligned} \tag{2.23.9}$$

It is very simple to verify the second Maxwell's equation in the same way. The third and fourth equation are the divergence of the electric and magnetic field respectively:

$$\nabla \cdot \mathbf{E} = \nabla \cdot \hat{\mathbf{x}}E(z) = \partial_x E(z) = 0 \tag{2.23.10}$$

$$\nabla \cdot \mathbf{H} = \nabla \cdot \hat{\mathbf{y}}H(z) = \partial_y H(z) = 0 \tag{2.23.11}$$

- Question n° 2

The energy flux can be evaluated as dot product of the Poynting vector and the unit vector along the z–direction:

$$P_z = \mathbf{P}\cdot\hat{\mathbf{z}} = \frac{1}{2}\Re e\left[\mathbf{E}\times\mathbf{H}^*\right]\cdot\hat{\mathbf{z}} \tag{2.23.12}$$

Substituting (2.23.1) in eq. (2.23.12), we have:

$$\begin{aligned} P_z &= \frac{1}{2}\Re e\left[\left(E_+e^{-jkz}+E_-e^{jkz}\right)\hat{\mathbf{x}}\times\frac{\left(E_+^*e^{+jkz}-E_-^*e^{-jkz}\right)}{\eta}\hat{\mathbf{y}}\right]\cdot\hat{\mathbf{z}} = \\ &= \frac{1}{2\eta}\Re e\left[\left(E_+e^{-jkz}+E_-e^{jkz}\right)\left(E_+^*e^{+jkz}-E_-^*e^{-jkz}\right)\hat{\mathbf{z}}\right]\cdot\hat{\mathbf{z}} = \\ &= \frac{1}{2\eta}\Re e\left[\left(|E_+|^2-|E_-|^2\right)\right] = \frac{1}{2\eta}\left(|E_+|^2-|E_-|^2\right) \end{aligned} \tag{2.23.13}$$

- Question n° 3

Consider the expression for $E(z_1)$ and multiply it by the neutral term $e^{jkz_2}e^{-jkz_2}$:

$$E(z_1) = E_+e^{-jkz_1}e^{jkz_2}e^{-jkz_2} + E_-e^{jkz_1}e^{jkz_2}e^{-jkz_2}$$

consequently,

$$E(z_1) = E_+e^{-jkz_2}e^{jk\ell} + E_-e^{jkz_2}e^{-jk\ell} \tag{2.23.14}$$

where $\ell = z_2 - z_1$. Using the Euler's formula $e^{jx} = \cos x + j\sin x$, eq. (2.23.14) can be written as:

$$\begin{aligned} E(z_1) &= E_+e^{-jkz_2}\left(\cos k\ell + j\sin k\ell\right) + E_-e^{jkz_2}\left(\cos k\ell - j\sin k\ell\right) = \\ &= \left(E_+e^{-jkz_2} + E_-e^{jkz_2}\right)\cos k\ell + j\left(E_+e^{-jkz_2} - E_-e^{jkz_2}\right)\sin k\ell \end{aligned} \tag{2.23.15}$$

The term that multiplies $\cos k\ell$ is simply $E(z_2)$ and the term that multiplies $\sin k\ell$ is simply $\eta H(z_2)$. Since the characteristic impedance of the medium η can be written as:

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0\varepsilon_r}} = \frac{\eta_0}{\sqrt{\varepsilon_r}} = \frac{\eta_0}{n}$$

where n is the refractive index, the eq. (2.23.15) becomes:

$$E(z_1) = E(z_2)\cos k\ell + j\frac{\eta_0 H(z_2)}{n}\sin k\ell \tag{2.23.16}$$

In the same way it is possible to write $H(z_1)$ as a function of $E(z_2)$ and $H(z_2)$:

$$\eta_0 H(z_1) = jnE(z_2)\sin k\ell + \eta_0 H(z_2)\cos k\ell \tag{2.23.17}$$

Now eq. (2.23.16) and eq. (2.23.17) can be written in the matrix form as (2.23.3).

- Question n° 4

The normalized wave impedance at location z_1 is:

$$Z(z_1)=\frac{E(z_1)}{\eta_0 H(z_1)}$$

and we can substitute the electric and magnetic field with their respective relationships (2.23.16) and (2.23.17). So:

$$Z(z_1)=\frac{E(z_2)\cos k\ell + j\dfrac{\eta_0 H(z_2)}{n}\sin k\ell}{jnE(z_2)\sin k\ell+\eta_0 H(z_2)\cos k\ell} \tag{2.23.18}$$

Dividing the numerator and denominator by $\eta_0 H(z_2)\cos k\ell$, we have:

$$Z(z_1)=\frac{\dfrac{E(z_2)}{\eta_0 H(z_2)}\dfrac{\cos k\ell}{\cos k\ell}+jn^{-1}\dfrac{\sin k\ell}{\cos k\ell}}{1+jn\dfrac{E(z_2)}{\eta_0 H(z_2)}\dfrac{\sin k\ell}{\cos k\ell}}=\frac{Z(z_2)+jn^{-1}\tan k\ell}{1+jnZ(z_2)\tan k\ell} \tag{2.23.19}$$

The admittance $Y(z_1)$ is the inverse of $Z(z_1)$:

$$\begin{aligned}Y(z_1)&=\frac{1}{Z(z_1)}=\frac{1+jnZ(z_2)\tan k\ell}{Z(z_2)+jn^{-1}\tan k\ell}=\\&=\frac{\dfrac{1}{Z(z_2)}+jn\tan k\ell}{1+j\dfrac{n^{-1}}{Z(z_2)}\tan k\ell}=\frac{Y(z_2)+jn\tan k\ell}{1+jn^{-1}Y(z_2)\tan k\ell}\end{aligned} \tag{2.23.20}$$

If we had normalized $Z(z)$ and $Y(z)$ to the medium impedance, simply we have to cancel the term n of refractive index inside eq. (2.23.19) and (2.23.20):

$$\begin{cases}Z(z_1)=\dfrac{Z(z_2)+j\tan k\ell}{1+jZ(z_2)\tan k\ell}\\[2ex] Y(z_1)=\dfrac{Y(z_2)+j\tan k\ell}{1+jY(z_2)\tan k\ell}\end{cases}$$

2.24 ExerciseEquation Section (Next)

Show that the time–averaged energy density and Poynting vector of the obliquely moving wave:

$$\mathbf{E}(\mathbf{r},t) = [\hat{\mathbf{x}}'A + \hat{\mathbf{y}}'B]e^{j\omega t - jkz'}$$
$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\eta}[\hat{\mathbf{y}}'A - \hat{\mathbf{x}}'B]e^{j\omega t - jkz'} \tag{2.24.1}$$

where (x', y', z') is a rotated coordinate system with respect to a fix coordinate system (x, y, z) as shown in

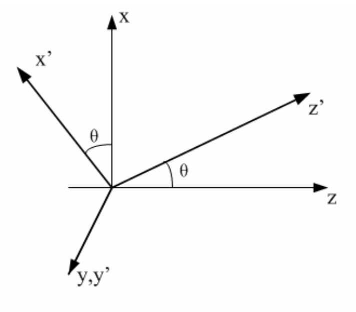

Fig. 2.24.1: Rotation of coordinate system.

are given by:

$$w = \frac{1}{2}\Re e\left[\frac{1}{2}\varepsilon\mathbf{E}\cdot\mathbf{E}^* + \frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}^*\right] = \frac{1}{2}\varepsilon\left(|A|^2 + |B|^2\right)$$
$$\mathbf{P} = \frac{1}{2}\Re e\left[\mathbf{E}\times\mathbf{H}^*\right] = \hat{\mathbf{z}}'\frac{1}{2\eta}\left(|A|^2 + |B|^2\right) = (\hat{\mathbf{z}}\cos\theta + \hat{\mathbf{x}}\sin\theta)\frac{1}{2\eta}\left(|A|^2 + |B|^2\right) \tag{2.24.2}$$

where $\hat{\mathbf{z}}' = (\hat{\mathbf{z}}\cos\theta + \hat{\mathbf{x}}\sin\theta)$ is the unit vector in the direction of propagation. Show that the energy transport velocity is $\mathbf{v} = \frac{\mathbf{P}}{w} = c\hat{\mathbf{z}}'$.

Solution

The dot products $\mathbf{E}\cdot\mathbf{E}^*$ and $\mathbf{H}\cdot\mathbf{H}^*$ can be evaluated as follow:

$$\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)^* = [\hat{\mathbf{x}}'A + \hat{\mathbf{y}}'B]\cdot\left[\hat{\mathbf{x}}'A^* + \hat{\mathbf{y}}'B^*\right] = AA^* + BB^* = |A|^2 + |B|^2$$
$$\mathbf{H}(\mathbf{r},t)\cdot\mathbf{H}(\mathbf{r},t)^* = \frac{1}{\eta^2}[\hat{\mathbf{y}}'A - \hat{\mathbf{x}}'B]\cdot\left[\hat{\mathbf{y}}'A^* - \hat{\mathbf{x}}'B^*\right] = \frac{1}{\eta^2}\left(AA^* + BB^*\right) = \frac{1}{\eta^2}\left(|A|^2 + |B|^2\right)$$

and now we can substitute them inside the definition of the time–averaged energy density:

$$w=\frac{1}{2}\Re e\left[\frac{1}{2}\varepsilon\mathbf{E}\cdot\mathbf{E}^*+\frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}^*\right]=\frac{1}{2}\Re e\left[\frac{1}{2}\varepsilon\left(|A|^2+|B|^2\right)+\frac{1}{2\eta^2}\mu\left(|A|^2+|B|^2\right)\right]=$$
$$=\frac{1}{2}\Re e\left[\frac{1}{2}\varepsilon\left(|A|^2+|B|^2\right)+\frac{1}{2}\varepsilon\left(|A|^2+|B|^2\right)\right]=\frac{1}{2}\varepsilon\left(|A|^2+|B|^2\right) \quad (2.24.3)$$

On the contrary the cross product $\mathbf{E}\times\mathbf{H}^*$ can be written as:

$$\mathbf{E}\times\mathbf{H}^*=\begin{vmatrix}\hat{\mathbf{x}}' & \hat{\mathbf{y}}' & \hat{\mathbf{z}}' \\ A & B & 0 \\ -B^*/\eta & A^*/\eta & 0\end{vmatrix}=\hat{\mathbf{z}}'\left(\frac{|A|^2}{\eta}+\frac{|B|^2}{\eta}\right)=\hat{\mathbf{z}}'\frac{1}{\eta}\left(|A|^2+|B|^2\right)$$

and, consequently,

$$\mathbf{P}=\frac{1}{2}\Re e\left[\mathbf{E}\times\mathbf{H}^*\right]=\hat{\mathbf{z}}'\frac{1}{2\eta}\left(|A|^2+|B|^2\right)=\left(\hat{\mathbf{z}}\cos\theta+\hat{\mathbf{x}}\sin\theta\right)\frac{1}{2\eta}\left(|A|^2+|B|^2\right) \quad (2.24.4)$$

Substituting eq. (2.24.3) and (2.24.4) in the definition of th energy transport velocity, we obtain:

$$\mathbf{v}=\frac{\mathbf{P}}{w}=\frac{\hat{\mathbf{z}}'\frac{1}{2}\frac{1}{\eta}\left(|A|^2+|B|^2\right)}{\frac{1}{2}\varepsilon\left(|A|^2+|B|^2\right)}=\hat{\mathbf{z}}'\frac{1}{\eta\varepsilon}=\hat{\mathbf{z}}'\frac{1}{\sqrt{\frac{\mu}{\varepsilon}}\varepsilon}=\hat{\mathbf{z}}'\frac{1}{\sqrt{\mu\varepsilon}}=c\hat{\mathbf{z}}'$$

where $c=\frac{1}{\sqrt{\mu\varepsilon}}$ is the speed of light in the free space.

2.25 ExerciseEquation Section (Next)

A uniform plane wave propagating in empty space has electric field:

$$\mathbf{E}(x,y,z)=\hat{\mathbf{y}}E_0 e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}, \qquad k=\frac{\omega_0}{c_0} \tag{2.25.1}$$

1. Inserting $\mathbf{E}(x,y,z)$ into Maxwell's equations, work out an expression for the corresponding magnetic field $\mathbf{H}(x,y,z)$.
2. What is the direction of propagation and its unit vector $\hat{\mathbf{k}}$?
3. working with Maxwell's equations, determine the electric field $\mathbf{E}(x,y,z,t)$ and the propagation direction $\hat{\mathbf{k}}$, if we started with a magnetic field given by:

$$\mathbf{H}(x,z,t)=\hat{\mathbf{y}}H_0 e^{j\omega t}e^{-jk\left(\sqrt{3}z-x\right)/2} \tag{2.25.2}$$

Solution

- Question n° 1

From the first Maxwell's equation, we can find the magnetic field as follow:

$$\mathbf{H}(x,z,t)=\frac{1}{-j\omega\mu}\nabla\times\mathbf{E}(x,z,t) \tag{2.25.3}$$

The cross product has to be evaluated:

$$\nabla\times\mathbf{E}(x,z,t)=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \partial_x & \partial_y & \partial_z\\ 0 & E(x,z,t) & 0\end{vmatrix} \tag{2.25.4}$$

where $E(x,z,t)=E_0 e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}$. The determinant of matrix (2.25.4) is:

$$\begin{aligned}\nabla\times\mathbf{E}(x,z,t)&=-\hat{\mathbf{x}}\frac{\partial}{\partial_z}\left(E_0 e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}\right)+\hat{\mathbf{z}}\frac{\partial}{\partial_x}\left(E_0 e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}\right)=\\ &=-\hat{\mathbf{x}}\frac{-jk}{\sqrt{2}}\left(E_0 e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}\right)+\hat{\mathbf{z}}\frac{-jk}{\sqrt{2}}\left(E_0 e^{j\omega t}e^{-jk(x+z)/\sqrt{2}}\right)\end{aligned} \tag{2.25.5}$$

and consequently

$$\begin{aligned}\mathbf{H}(x,z,t) &= \frac{1}{-j\omega\mu}\nabla\times\mathbf{E}(x,z,t) = \\ &= \frac{1}{-j\omega\mu}\left[\hat{\mathbf{x}}\frac{jk}{\sqrt{2}}E(x,z,t) - \hat{\mathbf{z}}\frac{jk}{\sqrt{2}}E(x,z,t)\right] = \\ &= \frac{1}{-j\omega\mu}\left[\hat{\mathbf{x}}\frac{j\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}E(x,z,t) - \hat{\mathbf{z}}\frac{j\omega\sqrt{\mu\varepsilon}}{\sqrt{2}}E(x,z,t)\right] = \\ &= \hat{\mathbf{z}}\frac{E(x,z,t)}{\eta\sqrt{2}} - \hat{\mathbf{x}}\frac{E(x,z,t)}{\eta\sqrt{2}} = \frac{E(x,z,t)}{\eta}\hat{\mathbf{z}}'\end{aligned} \tag{2.25.6}$$

where $k = \omega\sqrt{\mu\varepsilon}$, $\eta=\sqrt{\mu/\varepsilon}$ and $\hat{\mathbf{z}}' = (\hat{\mathbf{z}} - \hat{\mathbf{x}})/\sqrt{2}$. We can assume a new coordinate system aligned with the components of the electromagnetic wave as depicted in Fig. 2.25.1.

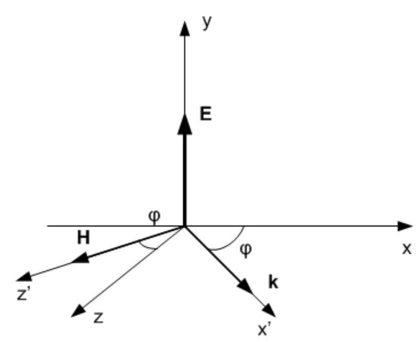

Fig. 2.25.1: Rotation of the coordinate system.

- Question n° 2

The direction of propagation can be found as cross product between the direction of oscillation of the electric and magnetic field. So:

$$\hat{\mathbf{k}} = \hat{\mathbf{y}}\times\hat{\mathbf{z}}' = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{vmatrix} = \frac{(\hat{\mathbf{x}}+\hat{\mathbf{z}})}{\sqrt{2}} = \hat{\mathbf{x}}' \tag{2.25.7}$$

- Question n° 3

Using the inverse form of the second Maxwell's equation, we have:

$$\mathbf{E}(x,z,t) = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H}(x,z,t) = \frac{1}{j\omega\varepsilon}\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ 0 & H(x,z,t) & 0\end{vmatrix} \tag{2.25.8}$$

where $H(x,z,t) = H_0 e^{j\omega t} e^{-jk(\sqrt{3}z - x)/2}$. So:

$$\begin{aligned}
\mathbf{E}(x,z,t) &= \frac{1}{j\omega\varepsilon}\left[-\hat{\mathbf{x}}\frac{\partial}{\partial_z}\left(H_0 e^{j\omega t} e^{-jk\left(\sqrt{3}z-x\right)/2}\right)+\hat{\mathbf{z}}\frac{\partial}{\partial_x}\left(H_0 e^{j\omega t} e^{-jk\left(\sqrt{3}z-x\right)/2}\right)\right]= \\
&= \frac{1}{j\omega\varepsilon}\left[-\hat{\mathbf{x}}\frac{-j\sqrt{3}k}{2}H(x,z,t)+\hat{\mathbf{z}}\frac{jk}{2}H(x,z,t)\right]= \\
&= \frac{1}{j\omega\varepsilon}\left[-\hat{\mathbf{x}}\frac{-j\omega\sqrt{3}\sqrt{\mu\varepsilon}}{2}H(x,z,t)+\hat{\mathbf{z}}\frac{j\omega\sqrt{\mu\varepsilon}}{2}H(x,z,t)\right]= \\
&= \frac{1}{2}\eta H(x,z,t)\left(\sqrt{3}\hat{\mathbf{x}}+\hat{\mathbf{z}}\right)=\frac{1}{2}\eta H(x,z,t)\hat{\mathbf{x}}'
\end{aligned} \tag{2.25.9}$$

where $k=\omega\sqrt{\mu\varepsilon}$, $\eta=\sqrt{\mu/\varepsilon}$ and $\hat{\mathbf{x}}'=\left(\sqrt{3}\hat{\mathbf{x}}+\hat{\mathbf{z}}\right)$. We can assume a new coordinate system aligned with the components of the electromagnetic wave as depicted in

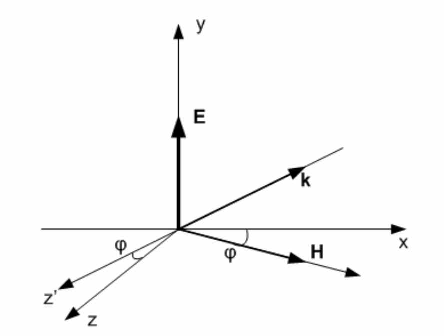

Fig. 2.25.2: Rotation of the coordinate system.

The direction of propagation can be found as cross product between the direction of oscillation of the electric and magnetic field. So:

$$\hat{\mathbf{k}}=\hat{\mathbf{y}}\times\hat{\mathbf{z}}'=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1\end{vmatrix}=\hat{\mathbf{x}}-\sqrt{3}\hat{\mathbf{z}}=-\hat{\mathbf{z}}'$$

2.26 ExerciseEquation Section (Next)

A linearly polarized light wave with electric field $\mathbf{E}_0$ at angle θ with respect to the x–axis is incident on a polarizing filter, follow by an identical polarizer (the analyzer) whose primary axes are rotated by an angle φ relative to the axes of the first polarizer, as shown in

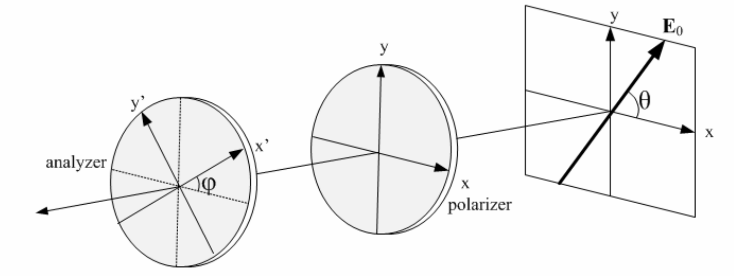

Fig. 2.26.1: Polarizer–analyzer filter combination.

Assume that the amplitude attenuation through the first polarizer are a_1, a_2 with respect to the x– and y–directions. The polarizer transmits primarily the x–polarization, so that $a_2 \ll a_1$. The analyzer is rotated by an angle φ so that the same gains a_1, a_2 now refer to the x'– and y'–directions.

1. Ignoring the phase retardance introduced by each polarizer, show that the polarization vectors at the input, and after the first and second polarizer, are:

$$\begin{aligned}
\mathbf{E}_0 &= \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta \\
\mathbf{E}_1 &= \hat{\mathbf{x}}a_1\cos\theta + \hat{\mathbf{y}}a_2\sin\theta \\
\mathbf{E}_2 &= \hat{\mathbf{x}}'\left(a_1^2\cos\varphi\cos\theta + a_1a_2\sin\varphi\sin\theta\right) + \hat{\mathbf{y}}'\left(a_2^2\cos\varphi\sin\theta - a_1a_2\sin\varphi\cos\theta\right)
\end{aligned} \qquad (2.26.1)$$

where $(\hat{\mathbf{x}}',\hat{\mathbf{y}}')$ are related to $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ as follow:

$$\begin{bmatrix}\hat{\mathbf{x}}' \\ \hat{\mathbf{y}}'\end{bmatrix} = \begin{bmatrix}\cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}} \\ \hat{\mathbf{y}}\end{bmatrix} \qquad (2.26.2)$$

2. Explain the meaning an usefulness of the matrix operations:

$$\begin{bmatrix}a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \qquad (2.26.3)$$

and

$$\begin{bmatrix}\cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}a_1 & 0 \\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \qquad (2.26.4)$$

3. Show that the input light intensity is proportional to the quantity:

$$\begin{aligned}\mathrm{I} = &\left(a_1^4\cos^2\theta + a_2^4\sin^2\theta\right)\cos^2\varphi + a_1^2 a_2^2\sin^2\varphi + \\ &+2a_1 a_2\left(a_1^2 - a_2^2\right)\cos\varphi\sin\varphi\cos\theta\sin\theta\end{aligned} \tag{2.26.5}$$

4. If the input light were unpolarized, that is incoherent, show that the average of the intensity of part (3) over all angles $0 \le \theta \le 2\pi$, will be given by the generalized Malus's law:

$$\overline{\mathrm{I}} = \frac{1}{2}\left(a_1^4 + a_2^4\right)\cos^2\varphi + a_1^2 a_2^2\sin^2\varphi \tag{2.26.6}$$

The case $a_2 = 0$ represents the usual Malus's law.

Solution

- Question n° 1

The electric field $\mathbf{E}_1$ after the polarizer is an attenuated form of the field $\mathbf{E}_0$, that is each component is attenuated by a factor a_1 or a_2, according to x– or y–directions respectively. So we can characterize the polarizer with a own attenuation matrix:

$$\underline{\mathbf{A}} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

and, consequently,

$$\mathbf{E}_1 = \underline{\mathbf{A}}\cdot\mathbf{E}_0 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} a_1\cos\theta \\ a_2\sin\theta \end{bmatrix} = \hat{\mathbf{x}}a_1\cos\theta + \hat{\mathbf{y}}a_2\sin\theta \tag{2.26.7}$$

The electric field $\mathbf{E}_1$ passes thought the second polarizer that is rotated of an angle by an angle φ so that the same gains a_1, a_2 now refer to the x'– and y'–directions. The rotation can be expressed by the matrix (2.26.2) and then the x'– and y'–components of the field $\mathbf{E}_2$ have to be attenuated by a factor a_1 or a_2, according to x– or y–directions respectively. So:

$$\mathbf{E}_2 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}\mathbf{E}_1 \tag{2.26.8}$$

from which:

$$\begin{aligned}\mathbf{E}_2 &= \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} a_1\cos\theta \\ a_2\sin\theta \end{bmatrix} = \\ &= \hat{\mathbf{x}}'\left(a_1^2\cos\varphi\cos\theta + a_1 a_2\sin\varphi\sin\theta\right) + \hat{\mathbf{y}}'\left(a_2^2\cos\varphi\sin\theta - a_1 a_2\sin\varphi\cos\theta\right)\end{aligned} \tag{2.26.9}$$

- Question n° 2

The matrix operation (2.26.3) is simply the cascade of the matrix that we used to solve the point (1). In fact the first two matrixes are necessary to rotate and attenuate the field $\mathbf{E}_1$ that passes through the analyzer, the third matrix represents the attenuation through the polarizer and the fourth represents the tilt of the electric field vector with respect to the x– and y–directions. Using these matrixes operation, it is a very simple model system.

In the matrix operation (2.26.4), shown here for simplicity, it is possible to note that there is a new matrix $\underline{\mathbf{M}}$ at the beginning of the expression:

$$\underbrace{\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}}_{\uparrow\ \underline{\mathbf{M}}}\begin{bmatrix}a_1 & 0\\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}a_1 & 0\\ 0 & a_2\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$

This has the same structure of a rotation matrix, but the angle is opposite, i.e. $-\varphi$. This suggest that (2.26.4) represents a system with another analyzer at the end tilted of an angle $-\varphi$, but without any attenuations.

- Question n° 3

The light intensity is the time–averaged energy density multiplied by the speed of light in the host medium, that is vacuum in this case.

From exercise 2.24, we have already demonstrated the expression of the time–averaged energy density w and so:

$$\mathrm{I}=\frac{1}{2}\mathrm{c}\varepsilon\left|\mathbf{E}\right|^2=\frac{1}{2}\mathrm{c}\varepsilon\left(\left|\mathrm{E}_{x'}\right|^2+\left|\mathrm{E}_{y'}\right|^2\right) \tag{2.26.10}$$

that is the light intensity is proportional to the sum of the square module of the components of electric field.

$$\begin{aligned}\mathrm{I}&=\left|\mathrm{E}_{x'}\right|^2+\left|\mathrm{E}_{y'}\right|^2=\left(a_1^2\cos\varphi\cos\theta+a_1a_2\sin\varphi\sin\theta\right)^2+\left(a_2^2\cos\varphi\sin\theta-a_1a_2\sin\varphi\cos\theta\right)^2=\\&=\left(a_1^4\cos^2\varphi\cos^2\theta+a_1^2a_2^2\sin^2\varphi\sin^2\theta+2a_1^3a_2\sin\varphi\sin\theta\cos\varphi\cos\theta\right)+\\&\quad+\left(a_2^4\cos^2\varphi\sin^2\theta+a_1^2a_2^2\sin^2\varphi\cos^2\theta-2a_1a_2^3\sin\varphi\sin\theta\cos\varphi\cos\theta\right)\\&=\left(a_1^4\cos^2\theta+a_2^4\sin^2\theta\right)\cos^2\varphi+a_1^2a_2^2\sin^2\varphi+2a_1a_2\left(a_1^2-a_2^2\right)\sin\varphi\sin\theta\cos\varphi\cos\theta\end{aligned} \tag{2.26.11}$$

- Question n° 4

The average of the intensity over all angles $0\le\theta\le 2\pi$ can be evaluated integrating the eq. (2.26.11) and dividing it by 2π as follow:

$$\bar{\text{I}} = \frac{1}{2\pi}\int_0^{2\pi} \text{I}\ \text{d}\theta = \frac{a_1^4\cos^2\varphi}{2\pi}\int_0^{2\pi}\cos^2\theta\ \text{d}\theta + \frac{a_2^4\cos^2\varphi}{2\pi}\int_0^{2\pi}\sin^2\theta\ \text{d}\theta + \frac{a_1^2 a_2^2\sin^2\varphi}{2\pi}\int_0^{2\pi}\text{d}\theta + \\ + \frac{1}{2\pi}2a_1a_2\left(a_1^2 - a_2^2\right)\sin\varphi\cos\varphi\int_0^{2\pi}\cos\theta\sin\theta\ \text{d}\theta \tag{2.26.12}$$

We can solve each integral separately as follow:

$$\begin{cases} \displaystyle\int_0^{2\pi}\cos^2\theta\ \text{d}\theta = \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\ \text{d}\theta = \frac{1}{2}\int_0^{2\pi}\text{d}\theta + \frac{1}{2}\int_0^{2\pi}\cos 2\theta \text{d}\theta = \pi \\ \displaystyle\int_0^{2\pi}\sin^2\theta\ \text{d}\theta = \int_0^{2\pi}\frac{1-\cos 2\theta}{2}\ \text{d}\theta = \frac{1}{2}\int_0^{2\pi}\text{d}\theta - \frac{1}{2}\int_0^{2\pi}\cos 2\theta \text{d}\theta = \pi \\ \displaystyle\int_0^{2\pi}\cos\theta\sin\theta\ \text{d}\theta = \frac{1}{2}\int_0^{2\pi}\sin 2\theta\ \text{d}\theta = 0 \end{cases} \tag{2.26.13}$$

and, substituting (2.26.13) in (2.26.12), we obtain:

$$\bar{\text{I}} = \frac{1}{2\pi}\int_0^{2\pi}\text{I}\ \text{d}\theta = \frac{1}{2}\left[a_1^4 + a_2^4\right]\cos^2\varphi + a_1^2a_2^2\sin^2\varphi$$

2.27 Exercise Equation Section (Next)

Consider an uniform plane wave propagating in vacuum as viewed from the vintage point of two coordinate frames: a fixed frame S and a frame S' moving towards the z–direction with velocity v. We assume that the wave–vector $\mathbf{k}$ in S lies in the xz–plane and forma an angle θ with the z–axis as shown in Fig. 2.27.1.

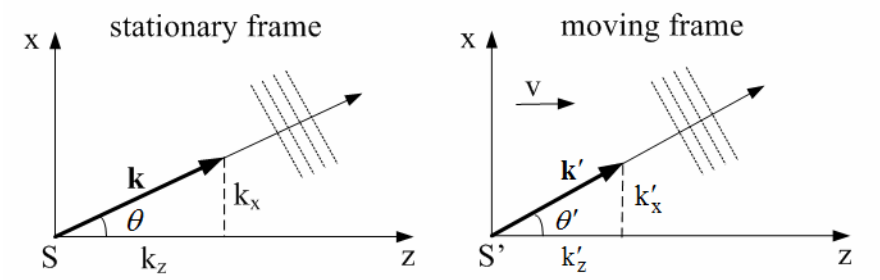

Fig. 2.27.1: Plane wave viewed from stationary and moving frames.

First, prove the equivalence of the three relationships given by:

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta} \quad \Leftrightarrow \quad \sin\theta' = \frac{\sin\theta}{\gamma\left(1-\beta\cos\theta\right)} \quad \Leftrightarrow \quad \frac{\tan\left(\theta'/2\right)}{\tan\left(\theta/2\right)} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{2.27.1}$$

where $\beta = \mathrm{v}/\mathrm{c}$ and $\gamma = 1\big/\sqrt{1-\beta^2}$.\

Then, prove the following identity between the angle $\theta,\ \theta'$:

$$\left(1-\beta\cos\theta\right)\left(1+\beta\cos\theta'\right) = \left(1+\beta\cos\theta\right)\left(1-\beta\cos\theta'\right) = 1-\beta^2 \tag{2.27.2}$$

Using this identity, prove the alternative Doppler formulas:

$$\mathrm{f}' = \mathrm{f}\gamma\left(1-\beta\cos\theta\right) = \frac{\mathrm{f}}{\gamma\left(1+\beta\cos\theta'\right)} = \mathrm{f}\sqrt{\frac{1-\beta\cos\theta}{1+\beta\cos\theta'}} \tag{2.27.3}$$

Solution

The three relationships in (2.27.1) relate the apparent propagation angles θ, θ' in the two frames that are different because of the aberration of light due to the motion. They are a consequence of the Lorentz transformation of the frequency–wavenumber four–vector $\left(\omega/\mathrm{c},\mathbf{k}\right)$:

$$\begin{aligned} \omega' &= \gamma\left(\omega - \beta \mathrm{c}\mathrm{k}_z\right) \\ \mathrm{k}'_z &= \gamma\left(\mathrm{k}_z - \frac{\beta}{\mathrm{c}}\omega\right) \\ \mathrm{k}'_x &= \mathrm{k}_x \end{aligned} \tag{2.27.4}$$

where $\beta = \mathrm{v/c}$ and $\gamma = 1\big/\sqrt{1-\beta^2}$. Setting $\mathrm{k_z} = \mathrm{k}\cos\theta$, $\mathrm{k_x} = \mathrm{k}\sin\theta$, with $\mathrm{k} = \omega/\mathrm{c}$, and similarly in the frame S', $\mathrm{k'_z} = \mathrm{k'}\cos\theta'$, $\mathrm{k'_x} = \mathrm{k'}\sin\theta'$, with $\mathrm{k'} = \omega'/\mathrm{c}$, the Eqs. (2.27.4) may be rewritten in the form:

$$\begin{aligned} \omega' &= \omega\gamma\left(1-\beta\cos\theta\right) \\ \omega'\cos\theta' &= \omega\gamma\left(\cos\theta-\beta\right) \\ \omega'\sin\theta' &= \omega\sin\theta \end{aligned} \tag{2.27.5}$$

The three equations are equivalent to evaluate the angular frequency ω' in the moving frame S' and they relate the different angular θ, θ' regardless of the frequency. From the first and second equation we can obtain the expression for $\cos\theta'$:

$$\begin{cases} \omega\gamma = \dfrac{\omega'}{\left(1-\beta\cos\theta\right)} \\ \omega'\cos\theta' = \omega\gamma\left(\cos\theta-\beta\right) \end{cases} \Rightarrow \quad \cancel{\omega'}\cos\theta' = \frac{\cancel{\omega'}\left(\cos\theta-\beta\right)}{\left(1-\beta\cos\theta\right)} \tag{2.27.6}$$

From the first and third equation, we have:

$$\begin{cases} \omega = \dfrac{\omega'}{\gamma\left(1-\beta\cos\theta\right)} \\ \omega'\sin\theta' = \omega\sin\theta \end{cases} \Rightarrow \quad \cancel{\omega'}\sin\theta' = \frac{\cancel{\omega'}\sin\theta}{\gamma\left(1-\beta\cos\theta\right)} \tag{2.27.7}$$

The half–angle formula for the tangent is in general:

$$\tan\frac{\mathrm{x}}{2} = \frac{\sin\mathrm{x}}{1+\cos\mathrm{x}} = \frac{1-\cos\mathrm{x}}{\sin\mathrm{x}} \tag{2.27.8}$$

and using it, we can obtain the third relationship of (2.27.1):

$$\begin{aligned} \tan\theta'/2 &= \frac{\sin\theta'}{1+\cos\theta'} = \frac{\sin\theta}{\gamma\left(1-\beta\cos\theta\right)\left(1+\cos\theta'\right)} = \\ &= \frac{\sin\theta}{\gamma\left(1-\beta\cos\theta\right)\left(1+\left(\dfrac{\cos\theta-\beta}{1-\beta\cos\theta}\right)\right)} = \\ &= \frac{\sin\theta}{\cancel{\gamma\left(1-\beta\cos\theta\right)}\left(\dfrac{1-\beta\cos\theta+\cos\theta-\beta}{\cancel{1-\beta\cos\theta}}\right)} = \\ &= \frac{\sin\theta}{\gamma\left(1-\beta\right)\left(1+\cos\theta\right)} = \frac{1}{\gamma\left(1-\beta\right)}\tan\theta/2 \end{aligned} \tag{2.27.9}$$

So:

$$\frac{\tan\theta'/2}{\tan\theta/2} = \frac{1}{\gamma\left(1-\beta\right)} = \frac{\sqrt{1-\beta^2}}{1-\beta} = \frac{\sqrt{1-\beta}\sqrt{1+\beta}}{1-\beta} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{2.27.10}$$

In the identity of eq. (2.27.2) between the angle $\theta,\ \theta'$ we note that the last term is equal to $1/\gamma^2$, that can be expressed using the second identity in eq. (2.27.1):

$$\sin\theta' = \frac{\sin\theta}{\gamma(1-\beta\cos\theta)} \quad\Rightarrow\quad \frac{1}{\gamma^2} = 1-\beta^2 = \frac{\sin^2\theta'}{\sin^2\theta}(1-\beta\cos\theta)^2 \tag{2.27.11}$$

Consequently since

$$\frac{\sin^2\theta'}{\sin^2\theta}(1-\beta\cos\theta)^2 = 1-\beta^2 \underset{\substack{\uparrow\\(2.27.2)}}{=} (1-\beta\cos\theta)(1+\beta\cos\theta')$$

we have to prove that:

$$\frac{\sin^2\theta'}{\sin^2\theta}(1-\beta\cos\theta) = 1+\beta\cos\theta' \tag{2.27.12}$$

So:

$$\begin{aligned}\frac{\sin^2\theta'}{\sin^2\theta}(1-\beta\cos\theta) &= \frac{1}{\gamma^2(1-\beta\cos\theta)^{\not 2}}\cancel{(1-\beta\cos\theta)} = \\ &= \frac{1-\beta^2}{(1-\beta\cos\theta)} = \frac{1-\beta^2+\beta\cos\theta-\beta\cos\theta}{(1-\beta\cos\theta)} = \\ &= \frac{(1-\beta\cos\theta)+\beta(\cos\theta-\beta)}{(1-\beta\cos\theta)} = 1+\beta\frac{(\cos\theta-\beta)}{(1-\beta\cos\theta)} = \\ &= 1+\beta\cos\theta'\end{aligned} \tag{2.27.13}$$

Using (2.27.11) and (2.27.13) we can write that:

$$1-\beta^2 = \frac{\sin^2\theta'}{\sin^2\theta}(1-\beta\cos\theta)^2 = (1+\beta\cos\theta')(1-\beta\cos\theta) \tag{2.27.14}$$

The second identity in eq. (2.27.2), i.e. $(1+\beta\cos\theta)(1-\beta\cos\theta') = 1-\beta^2$, is formally identical to the first identity, but the angles $\theta,\ \theta'$ are inverted. This is reasonable if the velocity vector $\mathbf{v}$ is directed in the negative z–direction, that is $\mathbf{v}\to-\mathbf{v}$. Since the term β is defined as $\mathrm{v/c}$ and it is the square, the sign of v is negligible and the identity is valid.

The alternative Doppler formulas in eqs. (2.27.3) are obtained applying (2.27.14) to the relativistic Doppler formula, relating the frequency of the wave as measured by an observer in the moving frame S' to the frequency of a source in the fixed frame S:

$$\mathrm{f}' = \mathrm{f}\gamma(1-\beta\cos\theta) = \mathrm{f}\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} \tag{2.27.15}$$

From eq. (2.27.14), we can write two identity:

$$\begin{cases} \left(1-\beta\cos\theta\right) = \dfrac{1-\beta^2}{\left(1+\beta\cos\theta'\right)} = \dfrac{1}{\gamma^2\left(1+\beta\cos\theta'\right)} \\ \left(\sqrt{1-\beta^2}\right)^{-1} = \gamma = \dfrac{1}{\sqrt{\left(1+\beta\cos\theta'\right)\left(1-\beta\cos\theta\right)}} \end{cases}$$

and then substitute them inside (2.27.15):

$$\begin{aligned} f' &= f\gamma\left(1-\beta\cos\theta\right) = \frac{f\,\cancel{\gamma}}{\gamma^{\cancel{2}}\left(1+\beta\cos\theta'\right)} = \frac{f}{\gamma\left(1+\beta\cos\theta'\right)} \\ f' &= f\,\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} = f\,\frac{1-\beta\cos\theta}{\sqrt{\left(1+\beta\cos\theta'\right)\left(1-\beta\cos\theta\right)}} = f\sqrt{\frac{1-\beta\cos\theta}{1+\beta\cos\theta'}} \end{aligned} \tag{2.27.16}$$

2.28 ExerciseEquation Section (Next)

We consider two reference frame S_a, S_b moving along the z–direction with velocities v_a, v_b with respect to our fixed frame S, and we assume that $\theta = 0°$ in the frame S. Let f_a and f_b be the frequencies of the wave as measured in the frames S_a, S_b.

In proving the relativistic Doppler formula:

$$f_a = f\sqrt{\frac{1-\beta_a}{1+\beta_a}},\; f_b = f\sqrt{\frac{1-\beta_b}{1+\beta_b}} \quad\Rightarrow\quad f_b = f_a\sqrt{\frac{1-\beta_b}{1+\beta_b}\frac{1+\beta_a}{1-\beta_a}} \tag{2.28.1}$$

it was assumed that the plane wave was propagating in the z–direction in all three reference frames S, S_a, S_b.

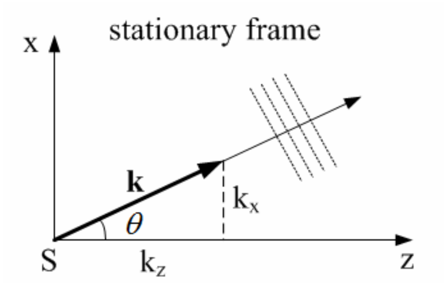

Fig. 2.28.1: Propagating plane wave along the θ–direction.

If in the frame S the wave is propagating along the θ–direction shown in Fig. 2.28.1, show that the Doppler formula may be written in the following equivalent forms:

$$f_b = f_a\frac{\gamma_b(1-\beta_b\cos\theta)}{\gamma_a(1-\beta_a\cos\theta)} = f_a\gamma(1-\beta\cos\theta_a) = \frac{f_a}{\gamma(1-\beta\cos\theta_b)} = f_a\sqrt{\frac{1-\beta\cos\theta_a}{1-\beta\cos\theta_b}} \tag{2.28.2}$$

where

$$\beta_a = \frac{v_a}{c},\; \beta_b = \frac{v_b}{c},\; \beta = \frac{v}{c},\; \gamma_a = \frac{1}{\sqrt{1-\beta_a^2}},\; \gamma_b = \frac{1}{\sqrt{1-\beta_b^2}},\; \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.28.3}$$

and v is the relative velocity of the observer and source given by

$$v = \frac{v_b - v_a}{1 - v_b v_a/c^2} \tag{2.28.4}$$

and θ_a, θ_b are the propagation directions in the frame S_a, S_b. Moreover, show the following relations among these angles:

$$\cos\theta_a = \frac{\cos\theta-\beta_a}{1-\beta_a\cos\theta},\quad \cos\theta_b = \frac{\cos\theta-\beta_b}{1-\beta_b\cos\theta},\quad \cos\theta_b = \frac{\cos\theta_a-\beta}{1-\beta\cos\theta_a} \tag{2.28.5}$$

Solution

The relativistic Doppler formula relates the frequency of the wave as measured by an observer in the moving frame S' to the frequency of a source in the fixed frame S:

$$f' = f\gamma\left(1-\beta\cos\theta\right) = f\frac{1-\beta\cos\theta}{\sqrt{1-\beta^2}} \tag{2.28.6}$$

If we consider separately the frame S_a and the frame S_b, we can write for each frame a relativistic Doppler formula:

$$\begin{aligned} f_a &= f\gamma_a\left(1-\beta_a\cos\theta\right) = f\frac{1-\beta_a\cos\theta}{\sqrt{1-\beta_a^2}} \\ f_b &= f\gamma_b\left(1-\beta_b\cos\theta\right) = f\frac{1-\beta_b\cos\theta}{\sqrt{1-\beta_b^2}} \end{aligned} \tag{2.28.7}$$

From the first equation in (2.28.7), we can write f as a function of f_a, and substitute it inside the second equation, in order to obtain:

$$f_b = f_a\frac{\gamma_b\left(1-\beta_b\cos\theta\right)}{\gamma_a\left(1-\beta_a\cos\theta\right)} \tag{2.28.8}$$

If the observer moves with the same velocity of the frame S_a, he will have the sensation that the frame S_a is fixed and the frame S_b is moving. So it is possible to use the relationships (2.27.16):

$$f' = f\gamma\left(1-\beta\cos\theta\right) = \frac{f}{\gamma\left(1+\beta\cos\theta'\right)} = f\sqrt{\frac{1-\beta\cos\theta}{1+\beta\cos\theta'}} \tag{2.28.9}$$

where

$$\begin{cases} f\to f_a, & f'\to f_b \\ \theta\to\theta_a, & \theta'\to\theta_b \end{cases}$$

and γ, β are expressed in (2.28.3). So:

$$f_b = f_a\gamma\left(1-\beta\cos\theta_a\right) = \frac{f_a}{\gamma\left(1+\beta\cos\theta_b\right)} = f_a\sqrt{\frac{1-\beta\cos\theta_a}{1+\beta\cos\theta_b}} \tag{2.28.10}$$

The relationships between the angles θ_a, θ_b expressed in (2.28.5) can be obtained from the analogue expression for the angles θ, θ':

$$\cos\theta' = \frac{\cos\theta-\beta}{1-\beta\cos\theta} \tag{2.28.11}$$

where θ' is the apparent angle along which the wave propagates in the moving frame, θ is the angle along which the wave propagates in the stationary frame and β is the ratio between the velocity v of the moving frame with respect to the fixed frame and the speed of light in vacuum. So if we consider first the frame S_a and then the frame S_b with respect to the fixed frame S, we can write the two following relationships:

$$\cos\theta_a = \frac{\cos\theta - \beta_a}{1-\beta_a\cos\theta}, \qquad \cos\theta_b = \frac{\cos\theta - \beta_b}{1-\beta_b\cos\theta} \tag{2.28.12}$$

from which we can obtain the expression of $\cos\theta_b$ as a function of $\cos\theta_a$. From the first identity of (2.28.12) we can write:

$$\cos\theta = \frac{\beta_a + \cos\theta_a}{1+\beta_a\cos\theta_a}$$

and we can substitute it in the second identity:

$$\cos\theta_b = \frac{\cos\theta - \beta_b}{1-\beta_b\cos\theta} = \frac{\dfrac{\beta_a + \cos\theta_a}{1+\beta_a\cos\theta_a} - \beta_b}{1-\beta_b\dfrac{\beta_a + \cos\theta_a}{1+\beta_a\cos\theta_a}} =$$

$$= \frac{\beta_a + \cos\theta_a - \beta_b\left(1+\beta_a\cos\theta_a\right)}{1+\beta_a\cos\theta_a - \beta_b\beta_a + \beta_b\cos\theta_a} =$$

$$= \frac{\beta_a + \cos\theta_a - \beta_b - \beta_b\beta_a\cos\theta_a}{1+\beta_a\cos\theta_a - \beta_b\beta_a - \beta_b\cos\theta_a} =$$

$$= \frac{\left(1-\beta_b\beta_a\right)\cos\theta_a - \left(\beta_b - \beta_a\right)}{\left(1-\beta_b\beta_a\right) - \left(\beta_b - \beta_a\right)\cos\theta_a} =$$

$$= \frac{\cos\theta_a - \left(\dfrac{\beta_b - \beta_a}{1-\beta_b\beta_a}\right)}{1-\left(\dfrac{\beta_b - \beta_a}{1-\beta_b\beta_a}\right)\cos\theta_a} = \frac{\cos\theta_a - \beta}{1-\beta\cos\theta_a}$$

2.29 ExerciseEquation Section (Next)

Ground–penetrating radar operating at 900 MHz is used to detect underground objects, as shown in Fig. 2.29.1 for a buried–pipe. Assume that the earth has conductivity $\sigma = 10^{-3}\ \text{S/m}$, permittivity $\varepsilon = 9\varepsilon_0$, and permeability $\mu = \mu_0$. You may use the "weakly lossy dielectric" approximation.

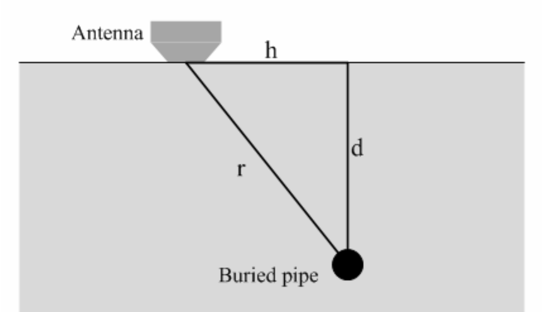

Fig. 2.29.1: Section of the ground with an underground object.

1. Determine the numerical value of the wavenumber $k = \beta - j\alpha$ in meters^{-1}, and the penetration depth $\delta = 1/\alpha$ in meters.
2. Determine the value of the complex refractive index $n_c = n_r - jn_i$ of the ground at 900 MHz.
3. With reference to the above figure, explain why the electric field returning back to the radar antenna after getting reflected by the buried–pipe is given by:

$$\left|\frac{E_{ret}}{E_0}\right|^2 = \exp\left[-\frac{4\sqrt{h^2+d^2}}{\delta}\right] \tag{2.29.1}$$

   where E_0 is the transmitted signal, d is the depth of the pipe, and h is the horizontal displacement of the antenna from the pipe. You may ignore the angular response of the radar antenna and assume it emits isotropically in all directions into the ground.
4. The depth d may be determined by measuring the roundtrip time $t(h)$ of the transmitted signal at successive horizontal distances h. Show that $t(h)$ is given by:

$$t(h) = \frac{2n_r}{c_0}\sqrt{d^2+h^2} \tag{2.29.2}$$

   where n_r is the real part of the complex refractive index n_c.
5. Suppose $t(h)$ is measured over the range $-2 \le h \le 2$ meters over the pipe and its minimum recorded value is $t_{min} = 0.2\ \mu s$. What is the depth d in meters?

Solution

- Question n° 1

In the weakly lossy case, the propagation parameter k becomes:

$$k = \beta - j\alpha = \omega\sqrt{\mu\varepsilon'_d}\left(1 - j\frac{\tau}{2}\right) = \omega\sqrt{\mu\varepsilon'_d}\left(1 - j\frac{\sigma + \omega\varepsilon''_d}{2\omega\varepsilon'_d}\right) \qquad (2.29.3)$$

where ε'_d and ε''_d are the real and imaginary part of the dielectric constant ε, i.e. $\varepsilon = \varepsilon'_d + j\varepsilon''_d$, and $\tau = \tan\theta = \varepsilon''_d / \varepsilon'_d$ is the loss tangent that is a convenient way to quantify the losses. In this case $\varepsilon'_d = 9\varepsilon_0$ and $\varepsilon''_d = \sigma/\omega$. So we can evaluate (2.29.3):

$$k = \beta - j\alpha = \omega\sqrt{9\mu_0\varepsilon_0}\left(1 - j\frac{\sigma}{2\omega 9\varepsilon_0}\right) = \omega\sqrt{9\mu_0\varepsilon_0} - j\frac{\sigma}{2}\sqrt{\frac{\mu_0}{9\varepsilon_0}}$$

$$\Downarrow$$

$$\begin{cases} \beta = 3\omega\sqrt{\mu_0\varepsilon_0} = 6\pi \times 900 \times 10^6 \times \sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}} = 56.57 \text{ rad/m} \\ \alpha = \frac{\sigma}{2}\sqrt{\frac{\mu_0}{9\varepsilon_0}} = \frac{10^{-3}}{2}\sqrt{\frac{4\pi \times 10^{-7}}{9 \times 8.85 \times 10^{-12}}} = 0.063 \text{ rad/m} \end{cases}$$

The corresponding penetration depth $\delta = 1/\alpha = 15.87$ meters.

- Question n° 2

The definition of the refractive index is:

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \qquad (2.29.4)$$

and in this case the relative permittivity is complex because the material is lossy, i.e. the permittivity is complex.

The complex value of the permittivity of the ground at 900 MHz is:

$$\varepsilon = 9\varepsilon_0 - j\frac{\sigma}{\omega} = 9 \times 8.85 \times 10^{-12} - j\frac{10^{-3}}{900 \times 10^6} = (79.65 - j1.11) \times 10^{-12} =$$
$$= 10^{-12}\sqrt{79.65^2 + 1.11^2}\, e^{j\tan^{1.11}/_{79.65}} = 10^{-12}\sqrt{79.65^2 + 1.11^2}\, e^{j\tan^{1.11}/_{79.65}} =$$
$$= 79.66 \times 10^{-12} e^{j0.0045\pi}$$

and consequently, using (2.29.4), the complex refractive index of the ground is:

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{\frac{79.66 \times \cancel{10^{-12}}\, e^{j0.0045\pi}}{8.85 \times \cancel{10^{-12}}}} = 3e^{j0.0022\pi} =$$
$$= 3(\cos 0.0022\pi + j\sin 0.0022\pi) = 2.99 + j0.02 \qquad (2.29.5)$$

- Question n° 3

Ignoring the angular response of the radar antenna and assuming it emits isotropically in all directions into the ground, the electric field into the ground is:

$$E_{ground} = E_0 e^{-jkr} \underset{k=\beta-j\alpha}{\overset{}{\underset{\uparrow}{=}}} E_0 e^{-\alpha r} e^{-j\beta r} \tag{2.29.6}$$

The distance r between the antenna and the buried–pipe can be evaluated using the Pythagoras' theorem:

$$r = \sqrt{h^2 + d^2} \tag{2.29.7}$$

The transmitted signal reaches the object and returns back to the radar antenna after getting reflected by the buried–pipe. So the round trip is two times long and the backward signal, using (2.29.6) and (2.29.7), can be expressed as:

$$E_{ret} = E_0 e^{-2jkr} = E_0 e^{-2\alpha\sqrt{h^2+d^2}} e^{-2j\beta\sqrt{h^2+d^2}} \tag{2.29.8}$$

The module of eq. (2.29.8) is:

$$|E_{ret}| = |E_0| e^{-2\alpha\sqrt{h^2+d^2}}$$

from which

$$\left|\frac{E_{ret}}{E_0}\right|^2 = e^{-4\alpha\sqrt{h^2+d^2}} \underset{\alpha=1/\delta}{\underset{\uparrow}{=}} e^{-4\frac{\sqrt{h^2+d^2}}{\delta}} \tag{2.29.9}$$

- Question n° 4

The time is the ratio of distance divided by speed that explains the amount of distance covered in a given time:

$$s = v \cdot t \tag{2.29.10}$$

where s is the distance in meter, v is the constant speed in meter per second and t is the time in second.

The wave propagates into the ground with velocity $c_g = c_0/n_r$, where n_r is the real part of the refraction index of the ground, and the round trip of the wave from the antenna to the buried–pipe is 2r long. This values can be substituted inside (2.29.10) and, using (2.29.7), we obtain:

$$2r = \frac{c_0}{n_r} t(h) \quad \Rightarrow \quad t(h) = \frac{2n_r}{c_0} r = \frac{2n_r}{c_0}\sqrt{d^2 + h^2} \tag{2.29.11}$$

- Question n° 5

It is possible to note from (2.29.11) that the minimum roundtrip time $t(h)$ is when the antenna is aligned with the pipe, that is the value of h is zero. Using (2.29.11), we can evaluate the depth d in meters:

$$d = \frac{c_0}{2n_r} t(h) = \frac{3\times 10^8}{2\times 3} 0.2\times 10^{-6} = 10 \text{ m} \tag{2.29.12}$$