

Lecture Notes on Elliptic Filter Design

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1. Introduction

Elliptic filters [1-11], also known as Cauwer or Zolotarev filters, achieve the smallest filter order for the same specifications, or, the narrowest transition width for the same filter order, as compared to other filter types. On the negative side, they have the most nonlinear phase response over their passband. The following table compares the basic filter types with regard to filter order and phase response:

smaller order, or narrower transition	↓	Bessel Butterworth Chebyshev Elliptic	↑	more linear phase over the passband
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In these notes, we are primarily concerned with elliptic filters. But we will also discuss briefly the design of Butterworth, Chebyshev-1, and Chebyshev-2 filters and present a unified method of designing all cases. We also discuss the design of digital IIR filters using the bilinear transformation method.

The typical “brick wall” specifications for an analog lowpass filter are shown in Fig. 1 for the case of a monotonically decreasing Butterworth filter, normalized to unity gain at DC.

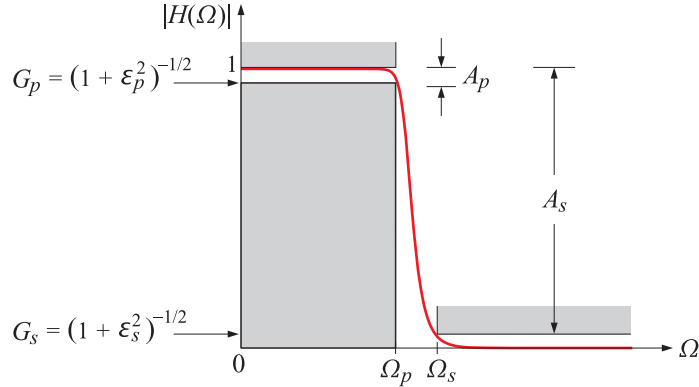


Fig. 1 Brick wall specifications for a Butterworth filter.

The passband and stopband gains G_p, G_s and the corresponding attenuations in dB are defined in terms of the “ripple” parameters ϵ_p, ϵ_s as follows:

$$G_p = \frac{1}{\sqrt{1 + \epsilon_p^2}} = 10^{-A_p/20}, \quad G_s = \frac{1}{\sqrt{1 + \epsilon_s^2}} = 10^{-A_s/20} \quad (1)$$

which can be inverted to give:

$$\begin{aligned} A_p = -20 \log_{10} G_p = 10 \log_{10} (1 + \epsilon_p^2) & \Rightarrow \epsilon_p = \sqrt{G_p^{-2} - 1} = \sqrt{10^{A_p/10} - 1} \\ A_s = -20 \log_{10} G_s = 10 \log_{10} (1 + \epsilon_s^2) & \Rightarrow \epsilon_s = \sqrt{G_s^{-2} - 1} = \sqrt{10^{A_s/10} - 1} \end{aligned} \quad (2)$$

Associated with these specifications, we define the following design parameters k, k_1 :

$$\boxed{k = \frac{\Omega_p}{\Omega_s}, \quad k_1 = \frac{\epsilon_p}{\epsilon_s}} \quad (3)$$

where k, k_1 are known as the selectivity and discrimination parameters, respectively. Both are less than unity. A narrow transition width would imply that $k \lesssim 1$, whereas a deep stopband

or a flat passband would imply that $k_1 \ll 1$. Thus, for most practical desired specifications, we will have $k_1 \ll k \lesssim 1$.

The magnitude responses of the analog lowpass Butterworth, Chebyshev, and elliptic filters are given as functions of the analog frequency Ω by:[†]

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_p^2 F_N^2(w)}, \quad w = \frac{\Omega}{\Omega_p} \quad (4)$$

where N is the filter order and $F_N(w)$ is a function of the normalized frequency w given by:

$$F_N(w) = \begin{cases} w^N, & \text{Butterworth} \\ C_N(w), & \text{Chebyshev, type-1} \\ [k_1 C_N(k^{-1}w^{-1})]^{-1}, & \text{Chebyshev, type-2} \\ \text{cd}(NuK_1, k_1), \quad w = \text{cd}(uK, k), & \text{Elliptic} \end{cases} \quad (5)$$

where $C_N(x)$ is the order- N Chebyshev polynomial, that is, $C_N(x) = \cos(N \cos^{-1} x)$, and $\text{cd}(x, k)$ denotes the Jacobian elliptic function cd with modulus k and real quarter-period K .

The Chebyshev-2 definition looks a little peculiar, but it is equivalent to that given in [12]. Indeed, noting that $k^{-1}w^{-1} = (\Omega_s/\Omega_p)(\Omega_p/\Omega) = \Omega_s/\Omega$ and that $\varepsilon_s = \varepsilon_p k_1^{-1}$, we have:

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_p^2 k_1^{-2} / C_N^2(k^{-1}w^{-1})} = \frac{1}{1 + \varepsilon_s^2 / C_N^2(\Omega_s/\Omega)} \quad (6)$$

The normalized frequency $w = 1$ corresponds to the passband edge frequency $\Omega = \Omega_p$, whereas the value $w = \Omega_s/\Omega_p = 1/k$ corresponds to the stopband edge $\Omega = \Omega_s$. The requirement that the passband and stopband specifications are met at the corners $\Omega = \Omega_p$ or $w = 1$ and $\Omega = \Omega_s$ or $w = k^{-1}$ gives rise to the following conditions:

$$\begin{aligned} |H(\Omega_p)|^2 &= \frac{1}{1 + \varepsilon_p^2 F_N^2(1)} = \frac{1}{1 + \varepsilon_p^2} \Rightarrow F_N(1) = 1 \\ |H(\Omega_s)|^2 &= \frac{1}{1 + \varepsilon_p^2 F_N^2(k^{-1})} = \frac{1}{1 + \varepsilon_s^2} \Rightarrow F_N(k^{-1}) = \frac{\varepsilon_s}{\varepsilon_p} = \frac{1}{k_1} \end{aligned} \quad (7)$$

Thus, in all four cases, the function $F_N(w)$ is normalized such that $F_N(1) = 1$ and must satisfy the following “degree equation” that relates the three design parameters N, k, k_1 :[†]

$$\boxed{F_N(k^{-1}) = k_1^{-1}} \quad (\text{degree equation}) \quad (8)$$

In particular, we find that the degree equation takes the following forms in the Butterworth and both Chebyshev cases:

$$\begin{aligned} k^{-N} = k_1^{-1} &\Rightarrow N = \frac{\ln(k_1^{-1})}{\ln(k^{-1})} = \frac{\ln(\varepsilon_s/\varepsilon_p)}{\ln(\Omega_s/\Omega_p)} \\ C_N(k^{-1}) = k_1^{-1} &\Rightarrow N = \frac{\text{acosh}(k_1^{-1})}{\text{acosh}(k^{-1})} = \frac{\text{acosh}(\varepsilon_s/\varepsilon_p)}{\text{acosh}(\Omega_s/\Omega_p)} \end{aligned} \quad (9)$$

These equations may be solved for any one of the three parameters N, k, k_1 in terms of the other two. Often, in practice, one specifies Ω_p, Ω_s and $\varepsilon_p, \varepsilon_s$, which fix the values of k, k_1 . Then,

[†] Ω is in units of radians per second and is related to the frequency f in Hz by $\Omega = 2\pi f$. For digital filter design, Ω is related to the physical digital frequency $\omega = 2\pi f/f_s$ via the appropriate bilinear transformation, e.g., for a lowpass design, $\Omega = \tan(\omega/2)$.

[†]For Chebyshev-2, it is $F_N(1) = 1$ that provides the desired relationship among N, k, k_1 .

Eqs. (9) may be solved for N , which must be rounded up to the next integer value. Since N is slightly increased, Eqs. (9) may be used to recompute either k in terms of N, k_1 , or alternatively, k_1 in terms of N, k . Because k is an increasing function of N , and k_1 , a decreasing one, it follows that in either case, the final design will have slightly improved specifications, either by making the transition width narrower, or by increasing the stopband or decreasing the passband attenuations. Fig. 2 shows an example designed with Butterworth, Chebyshev types-1&2, and elliptic filters. Fig. 3 shows the corresponding phase responses (their piece-wise nature arises because they are always wrapped modulo 2π to lie within $[-\pi, \pi]$.) The specifications were as follows:

$$\begin{aligned} f_p &= 4, & G_p &= 0.95, & A_p &= -20 \log_{10} G_p = 0.4455 \text{ dB} \\ f_s &= 4.5, & G_s &= 0.05, & A_s &= -20 \log_{10} G_s = 26.0206 \text{ dB} \end{aligned} \quad (10)$$

where the radian frequencies were computed as $\Omega_p = 2\pi f_p, \Omega_s = 2\pi f_s$. The design parameters k, k_1 were computed to be:

$$k = \frac{\Omega_p}{\Omega_s} = \frac{f_p}{f_s} = 0.8889, \quad k_1 = \frac{\varepsilon_p}{\varepsilon_s} = \frac{\sqrt{10^{A_p/10} - 1}}{\sqrt{10^{A_s/10} - 1}} = \frac{\sqrt{10^{0.4455} - 1}}{\sqrt{10^{26.0206} - 1}} = 0.0165 \quad (11)$$

We note that the elliptic design has the smallest filter order N , and the Butterworth, the largest. The difference between the Chebyshev designs is that type-1 is equiripple in the passband, whereas type-2 is equiripple in the stopband. It follows from Eq. (5) that the replacement

$$C_N(w) \longrightarrow \frac{1}{k_1 C_N(k^{-1}w^{-1})}$$

causes the type-1 case to be replaced by the type-2 case, and the equal ripples in the passband to become equal ripples in the stopband.

In the elliptic case, we want a filter that is equiripple in both the passband and the stopband, as shown in Fig. 2. This will be accomplished if we can find a filter function $F_N(w)$ that is equiripple in the passband and satisfies the identity:

$$F_N(w) = \frac{1}{k_1 F_N(k^{-1}w^{-1})} \quad (12)$$

which is equivalent to $\varepsilon_p F_N(\Omega/\Omega_p) = \varepsilon_s / F_N(\Omega_s/\Omega)$, so that in this case the magnitude response can be written as follows and will have ripples in both the passband and stopband:

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_p^2 F_N^2(\Omega/\Omega_p)} = \frac{1}{1 + \varepsilon_s^2 / F_N^2(\Omega_s/\Omega)} \quad (13)$$

We note that the Butterworth filter also satisfies Eq. (12), because of the degree equation $k_1 = k^N$, but in this case $F_N(w)$ is monotonic in both the passband and the stopband.

2. Jacobian Elliptic Functions

Jacobian elliptic functions are a fascinating subject with many applications [13-20]. Here, we give some definitions and discuss some of the properties that are relevant in filter design [8]. The elliptic function $w = \text{sn}(z, k)$ is defined indirectly through the elliptic integral:

$$z = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \int_0^w \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}, \quad w = \sin \phi \quad (14)$$

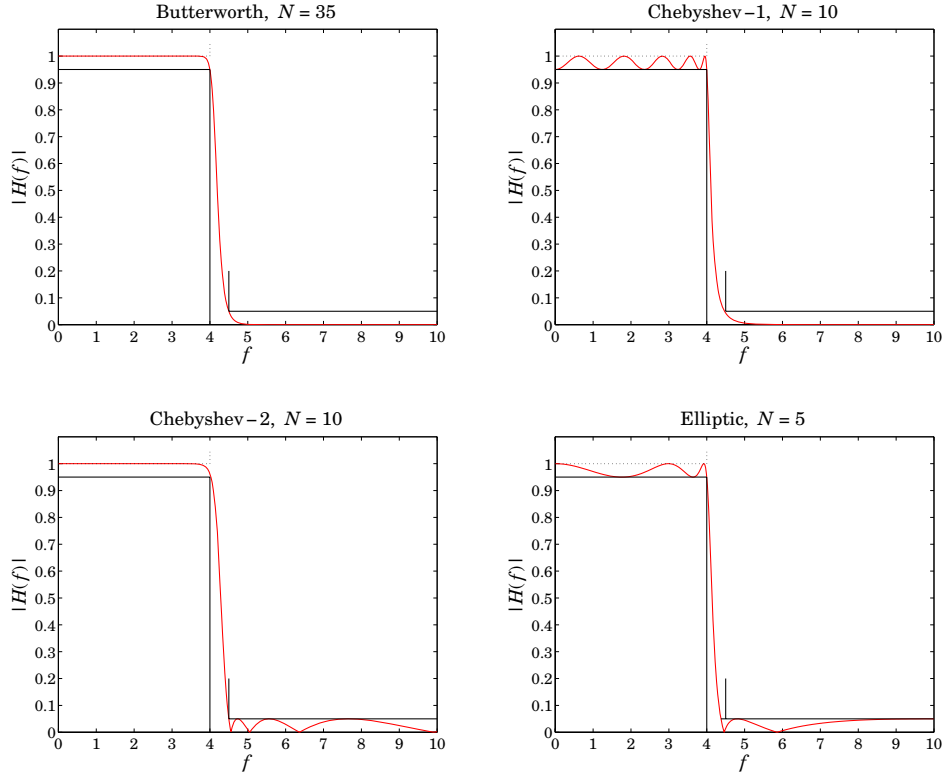


Fig. 2 Butterworth, Chebyshev, and elliptic design examples.

where the second integral was obtained from the first by the change of variables $t = \sin \theta$ and $w = \sin \phi$. The parameter k is called the *elliptic modulus*[†] and is assumed to be a real number in the interval $0 \leq k \leq 1$. Thus, writing $\phi = \phi(z, k)$, the function sn is defined as:

$$w = \text{sn}(z, k) = \sin \phi(z, k) \quad (15)$$

The three related elliptic functions, cn, dn, cd, are defined by:

$$\begin{aligned} w &= \text{cn}(z, k) = \cos \phi(z, k) \\ w &= \text{dn}(z, k) = \sqrt{1 - k^2 \text{sn}^2(z, k)} \\ w &= \text{cd}(z, k) = \frac{\text{cn}(z, k)}{\text{dn}(z, k)} \end{aligned} \quad (16)$$

In filter design, only the functions sn and cd are needed. In the limits $k = 0$ and $k = 1$, we obtain the trigonometric and hyperbolic functions, respectively:

$$\begin{aligned} \text{sn}(z, 0) &= \sin z, & \text{sn}(z, 1) &= \tanh z \\ \text{cn}(z, 0) &= \cos z, & \text{cn}(z, 1) &= \text{sech } z \\ \text{dn}(z, 0) &= 1, & \text{dn}(z, 1) &= \text{sech } z \\ \text{cd}(z, 0) &= \cos z, & \text{cd}(z, 1) &= 1 \end{aligned} \quad (17)$$

[†]In some discussions, as well as in MATLAB, the parameter $m = k^2$ is used instead of k .

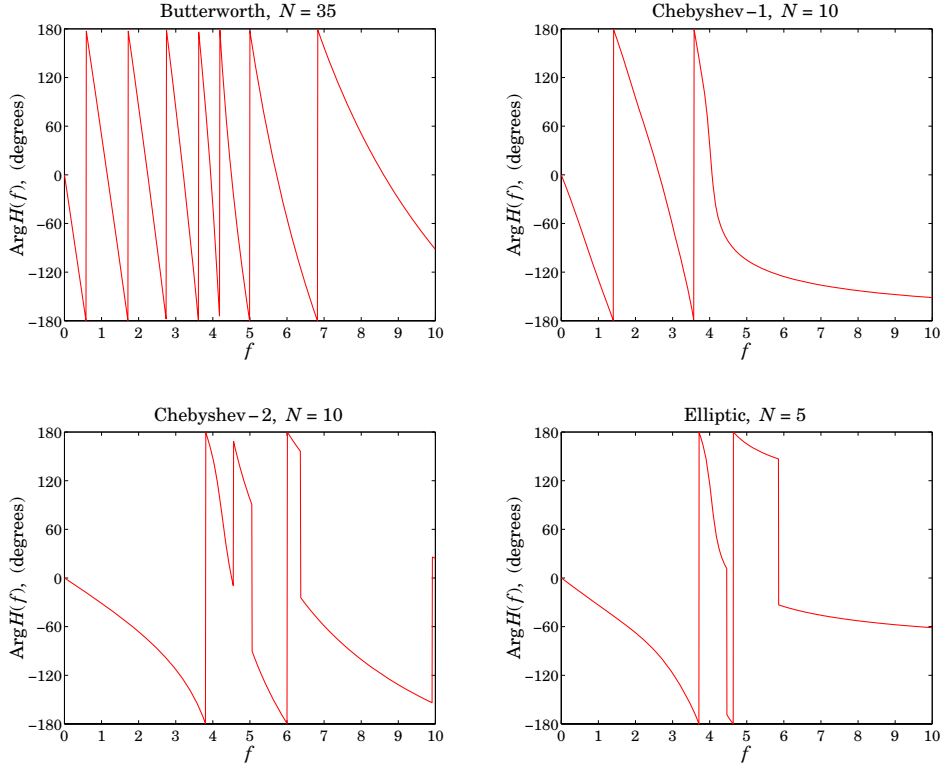


Fig. 3 Phase responses.

The functions sn , cn , dn , cd satisfy the properties, where $k' = (1 - k^2)^{1/2}$:

$$\begin{aligned}
 \text{sn}^2(z, k) + \text{cn}^2(z, k) &= 1 \\
 k^2 \text{sn}^2(z, k) + \text{dn}^2(z, k) &= 1 \\
 \text{dn}^2(z, k) - k^2 \text{cn}^2(z, k) &= k'^2 \\
 k'^2 \text{sn}^2(z, k) + \text{cn}^2(z, k) &= \text{dn}^2(z, k) \\
 \text{sn}^2(z, k) &= \frac{1 - \text{cd}^2(z, k)}{1 - k^2 \text{cd}^2(z, k)}
 \end{aligned} \tag{18}$$

The value of z at $\phi = \pi/2$ in Eq. (14) defines the so-called *complete elliptic integral* of the first kind, which is denoted by $K(k)$ or simply K :

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (\text{complete elliptic integral}) \tag{19}$$

It follows from the definitions (15) and (16) that $\text{sn}(K, k) = 1$ and $\text{cd}(K, k) = 0$. Associated with an elliptic modulus k , we may define the *complementary modulus* $k' = (1 - k^2)^{1/2}$ and its associated complete elliptic integral $K(k')$ denoted by $K'(k)$ or simply K' :

$$K' = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - k^2) \sin^2 \theta}}, \quad k' = \sqrt{1 - k^2} \tag{20}$$

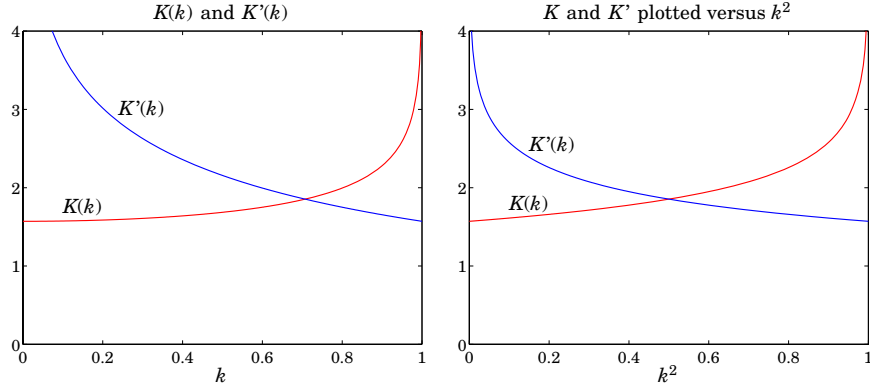


Fig. 4 Complete elliptic integrals $K(k)$ and $K'(k)$, where $K(0) = K'(1) = \pi/2$.

The quantities K, K' are referred to as *quarter periods*. At the end-point $k = 0$, we have $K = \pi/2, K' = \infty$; at the other end $k = 1$, we have $K = \infty, K' = \pi/2$. Fig. 4 shows a plot of K, K' versus k . The curves intersect at $k = 1/\sqrt{2}$ and are symmetric if plotted versus k^2 .

The significance of the quarter periods K, K' is that sn and cd are *doubly-periodic functions* in the z -plane with a real period $4K$ and a complex period $2jK'$. Fig. 5 shows the graphs of $w = \text{sn}(uK, k)$ and $w = \text{cd}(uK, k)$ plotted over two real periods. The argument of the functions is $z = uK$, where u is in units of the quarter period K , so that the range $-4 \leq u \leq 4$ is equivalent to two real periods $-4K \leq z \leq 4K$. For $k \leq 0.5$, $\text{sn}(uK, k)$ and $\text{cd}(uK, k)$ are almost identical to the trigonometric functions $\sin(u\pi/2)$ and $\cos(u\pi/2)$, that is, to the limiting case $k = 0$.

We note that $\text{sn}(z, k)$ is an odd function of z , and $\text{cd}(z, k)$, an even function. Moreover, by analogy with the property that a cosine and sine are shifted relative to each other by a quarter period $2\pi/4 = \pi/2$, that is, $\cos z = \sin(z + \pi/2) = \sin(\pi/2 - z)$, the functions cd and sn are shifted by a quarter period K , satisfying the following identity, which is valid for all complex values of z and can be used as an alternative definition of the function cd :

$$\text{cd}(z, k) = \text{sn}(z + K, k) = \text{sn}(K - z, k) \quad (21)$$

This property is evident in Fig. 5. The functions $\text{dn}(uK, k)$ and $\text{dn}(uK', k')$ are plotted in Fig. 6 for the values $k = 0.8$ and $k' = \sqrt{1 - k^2} = 0.6$. Because $\text{dn}(uK, k) = \sqrt{1 - k^2} \text{sn}^2(uK, k)$, we have the range of variation $k' \leq \text{dn}(uK, k) \leq 1$, for real u , and similarly, $k \leq \text{dn}(uK', k') \leq 1$.

Four additional properties, which will prove useful in filter design, are:

$$\text{cd}(z + (2i - 1)K, k) = (-1)^i \text{sn}(z, k), \quad \text{for any integer } i \quad (22)$$

$$\text{cd}(z + 2iK, k) = (-1)^i \text{cd}(z, k), \quad \text{for any integer } i \quad (23)$$

$$\text{cd}(z + jK', k) = \frac{1}{k \text{cd}(z, k)} \quad (24)$$

$$\text{cd}(jz, k) = \frac{1}{\text{dn}(z, k')}, \quad \text{for real } z \quad (25)$$

In particular, setting $z = 0$ in (24), or, $z = K'$ in (25), we obtain:

$$\text{cd}(jK', k) = \frac{1}{k} \quad (26)$$

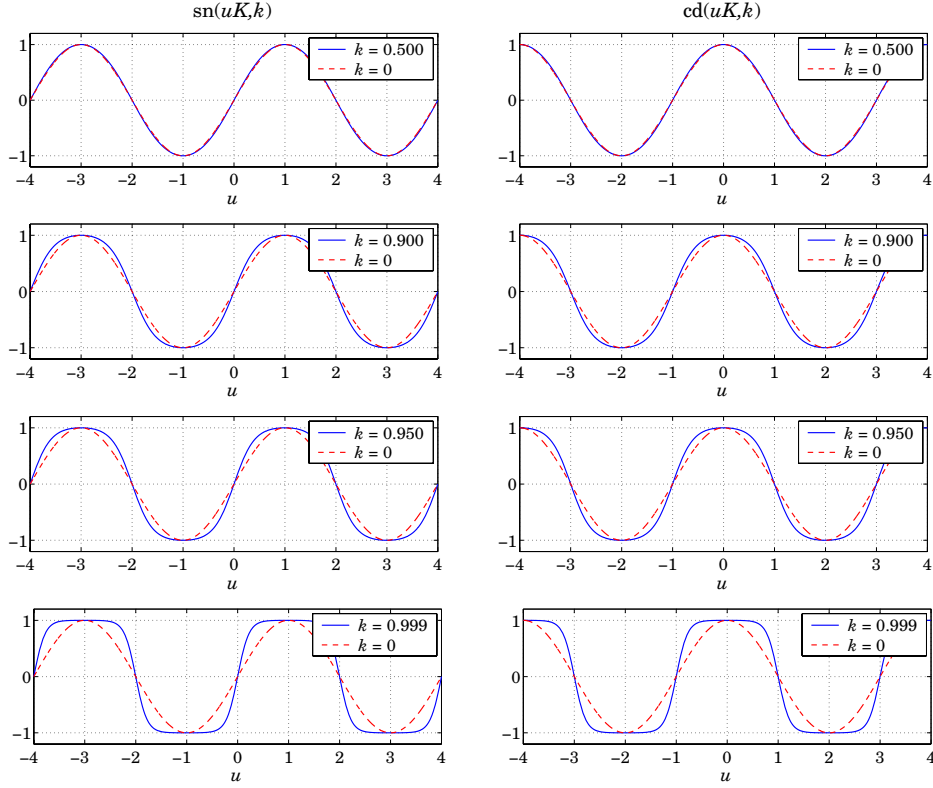


Fig. 5 Elliptic functions sn and cd.

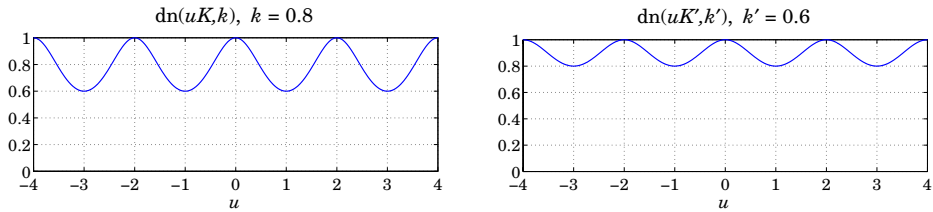


Fig. 6 The function dn with complementary moduli $k = 0.8$ and $k' = 0.6$.

The naming convention of the Jacobian elliptic functions may be understood with reference to the so-called *fundamental rectangle* on the complex z -plane with corners at $\{0, K, jK', K + jK'\}$, as shown in Fig. 7, where these corners are labeled with the letters S, C, N, D .

An elliptic function $pq(z, k)$ is named such that the first letter p can be any of the four letters $\{s, c, d, n\}$, and the second letter q , any of the remaining three letters. Thus, there are $4 \times 3 = 12$ Jacobian elliptic functions, namely, sn, sd, sc, cn, cd, cs, dn, dc, ds, ns, nd, nc.

Each function $pq(z, k)$ has a *simple zero* at corner p and a *simple pole* at corner q of the fundamental rectangle. For example, sn(z, k) has a zero at the point $S, z = 0$, and a pole at the point $N, z = jK'$. Similarly, cd(z, k) has a zero at the point $C, z = K$, and a pole at the point $D, z = K + jK'$. Moreover, the following relationships hold:

$$pq(z, k) = \frac{1}{qp(z, k)}, \quad pq(z, k) = \frac{pr(z, k)}{qr(z, k)} \quad (27)$$

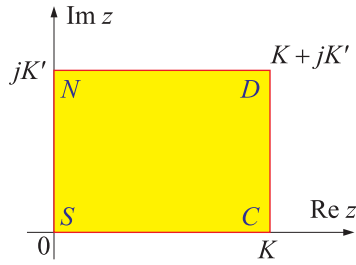


Fig. 7 The fundamental rectangle.

where r is any one of the letters $\{s, c, d, n\}$ distinct from p and q , for example, as we saw in Eq. (16), $cd(z, k) = cn(z, k) / dn(z, k)$.

The zeros and poles of the function pq are congruent modulo $2K$ and $2jK'$ to those at the corners p and q of the fundamental rectangle. In particular, the zeros and poles of $cd(z, k)$, shown in Fig. 8, are given follows, where n, m are arbitrary integers (positive, negative, or zero):

$$\begin{aligned} \text{zeros: } z &= K + 2mK + 2njK' = (2m + 1)K + 2njK' \\ \text{poles: } z &= K + jK' + 2mK + 2njK' = (2m + 1)K + (2n + 1)jK' \end{aligned} \quad (28)$$

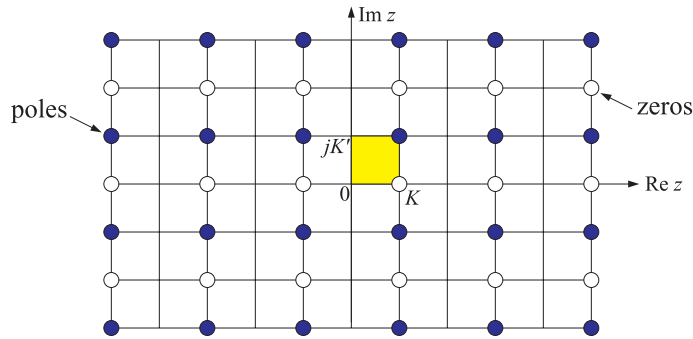


Fig. 8 Pole and zero patterns of the function $cd(z, k)$.

The functions $w = cd(z, k)$ and $w = sn(z, k)$ map the z -plane conformally onto the w -plane. The smallest region of the z -plane that gets mapped onto the whole of the w -plane is called a *fundamental region*. For each function $pq(z, k)$, such a region is centered at the zero point p and surrounded by four fundamental rectangles, each rectangle being mapped onto a particular quadrant of the w -plane [8]. For example, the fundamental regions of the $cd(z, k)$ and $sn(z, k)$ functions are centered at the points C and S , respectively, and are defined by:

$$\begin{aligned} cd(z, k): \quad & 0 \leq \text{Re } z \leq 2K, \quad -K' \leq \text{Im } z \leq K' \\ sn(z, k): \quad & -K \leq \text{Re } z \leq K, \quad -K' \leq \text{Im } z \leq K' \end{aligned} \quad (\text{fundamental regions}) \quad (29)$$

These are shown in Figs. 9 and 10. The w -plane quadrants to which the z -plane quadrants map have been labeled by the quadrant numbers $\{1, 2, 3, 4\}$. In particular, we note in Fig. 9 that the bottom two z -plane quadrants are mapped onto the first and second w -plane quadrants, that is, $z = z_1 - jz_2$ with $0 \leq z_1 \leq 2K$ and $0 < z_2 < K'$ gets mapped onto $w = w_1 + jw_2$ with $w_2 > 0$. Because the s -plane is related to the frequency plane by $s = jw$, it follows that the first and second w -plane quadrants will get mapped onto the left-hand s -plane, indeed,

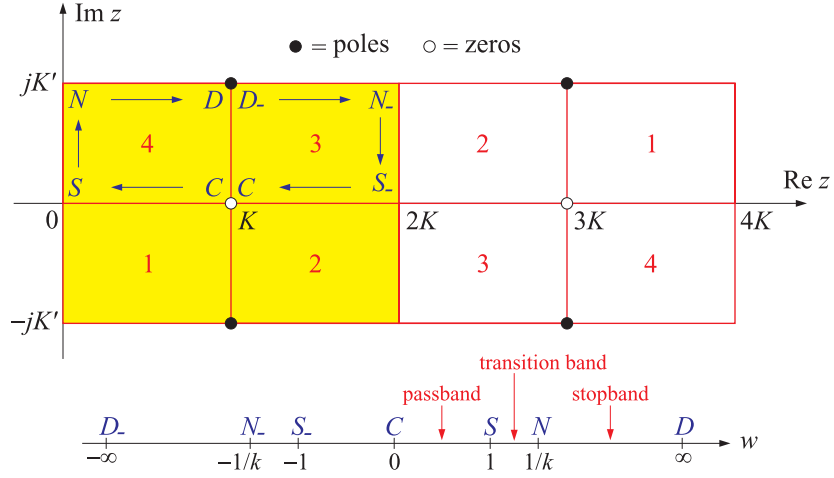


Fig. 9 Fundamental region, quadrant mappings, and period rectangle of the function $w = cd(z, k)$.

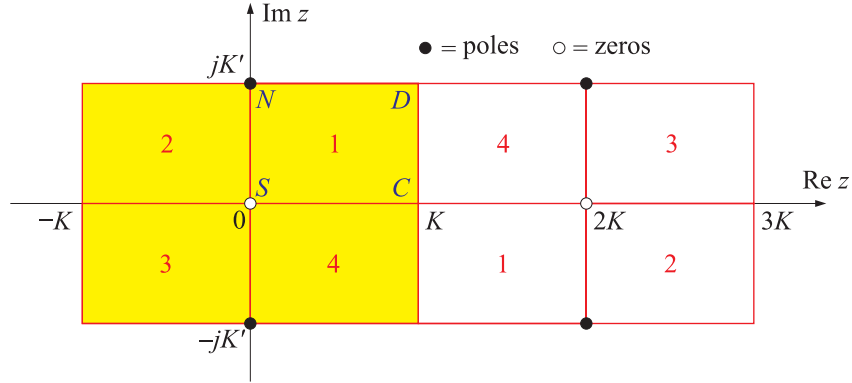


Fig. 10 Fundamental region, quadrant mappings, and period rectangle of the function $w = sn(z, k)$.

$s = j(w_1 + jw_2) = -w_2 + jw_1$. This property will be used in the construction of the analog filter's left-hand s -plane poles.

Recalling that the periods of cd and sn are $4K$ and $2jK'$, we have doubled-up the fundamental regions in Figs. 9 and 10 to cover one complete *period rectangle*, that is,

$$\begin{aligned} cd(z, k): \quad & 0 \leq \text{Re } z \leq 4K, \quad -K' \leq \text{Im } z \leq K' \\ sn(z, k): \quad & -K \leq \text{Re } z \leq 3K, \quad -K' \leq \text{Im } z \leq K' \end{aligned} \quad (\text{period rectangles}) \quad (30)$$

Of particular interest to filter design is the property that for the function $w = cd(z, k)$, the path around the fundamental rectangle $C \rightarrow S \rightarrow N \rightarrow D$ shown in Fig. 9, from the zero C to the pole D , gets mapped onto the positive real w -axis, such that the individual path segments, parametrized with the real parameter $0 \leq u \leq 1$, get mapped as follows:

$$\begin{aligned} \text{path } C \rightarrow S, \quad & 0 \leq u \leq 1, \quad z = K - Ku \quad \Rightarrow \quad 0 \leq w \leq 1, \quad \text{passband} \\ \text{path } S \rightarrow N, \quad & 0 \leq u \leq 1, \quad z = jK'u \quad \Rightarrow \quad 1 \leq w \leq 1/k, \quad \text{transition region} \\ \text{path } N \rightarrow D, \quad & 0 \leq u \leq 1, \quad z = Ku + jK' \quad \Rightarrow \quad 1/k \leq w \leq \infty, \quad \text{stopband} \end{aligned} \quad (31)$$

Because of the filter definition, Eq. (5), the above intervals of the $w = \Omega/\Omega_p$ axis will correspond to the passband, transition region, and stopband. Similarly, the continuation of the

path to $D \rightarrow N_- \rightarrow S_- \rightarrow C$ covers the negative w -axis. To verify these properties, we note that for the first segment $C \rightarrow S$, the argument $z = K - Ku$ is real and the values of $w = \text{cd}(K - Ku, k) = \text{sn}(Ku, k)$ will vary over the interval $0 \leq w \leq 1$, as seen in Fig. 5. For the segment $S \rightarrow N$, using property (25), we have $w = \text{cd}(juK', k) = 1/\text{dn}(uK', k')$, which increases from $w = 1$ at $u = 0$ to the value $w = \text{cd}(jK', k) = 1/\text{dn}(K', k') = 1/k$ at $u = 1$. Finally, for the segment $N \rightarrow D$, we use the property (24) to get:

$$w = \text{cd}(Ku + jK', k) = \frac{1}{k \text{cd}(Ku, k)}$$

with a starting value of $k \text{cd}(0, k) = k$ in the denominator or $w = 1/k$, and an ending value of $k \text{cd}(K, k) = 0$ or $w = \infty$.

In filter design, it is also required to be able to invert the functions $w = \text{cd}(z, k)$ and $w = \text{sn}(z, k)$, that is, to determine the value of z corresponding to a given complex-valued w . The resulting z is not unique. However, z becomes unique if it is restricted to lie within the fundamental region, that is, satisfying Eqs. (29). We will denote such an inverse by $z = \text{cd}^{-1}(w, k)$ or $z = \text{acd}(w, k)$. We note that within a period rectangle there are *two* values of z , the one in the fundamental region, the other in the adjacent region.

For example, if $z = \text{cd}^{-1}(w, k)$ lies in the fundamental region, then $z_1 = 4K - z$ lies in the adjacent region and both satisfy $w = \text{cd}(z, k) = \text{cd}(z_1, k)$. Similarly, for the sn function the inverses are z and $z_1 = 2K - z$, with z satisfying $-K \leq \text{Re } z \leq K$ and $K \leq \text{Re } z_1 \leq 3K$, and $w = \text{sn}(z, k) = \text{sn}(z_1, k)$.

The MATLAB functions `acde` and `asne` mentioned in Sect. 4 allow the computation of the inverse functions. Because $\text{sn}(z, k) = \text{cd}(K - z, k)$, the inverse of the sn function may be computed from the inverse of cd by $z = K - \text{cd}^{-1}(w, k)$.

3. Elliptic Rational Function and the Degree Equation

The analog filter characteristic function $F_N(w)$ was defined in the elliptic case by Eq. (5) in terms of the cd function [8]:

$$F_N(w) = \text{cd}(NuK_1, k_1), \quad w = \text{cd}(uK, k) \quad (32)$$

where $w = \text{cd}(uK, k)$ may be inverted to give u as a function of w , that is, $uK = \text{cd}^{-1}(w, k)$. This indirect way of writing the function $F_N(w)$ is analogous to the Chebyshev-1 case, which can be thought of as the limit $k = k_1 = 0$ of the elliptic case:

$$C_N(w) = \cos(Nu\pi/2), \quad w = \cos(u\pi/2) \quad (33)$$

where in this limit $K = K_1 = \pi/2$. In order for the function $F_N(w)$ to satisfy the identity of Eq. (12), the three parameters N, k, k_1 must satisfy the following constraint, which is known as the *degree equation* for elliptic filters:

$$\boxed{N \frac{K'}{K} = \frac{K'_1}{K_1}} \quad (\text{degree equation}) \quad (34)$$

where K, K_1 are the complete elliptic integrals (19) corresponding to the moduli k, k_1 , and K', K'_1 are the complete elliptic integrals corresponding to the complementary moduli $k' = (1 - k^2)^{1/2}$ and $k'_1 = (1 - k_1^2)^{1/2}$. To verify this constraint, we use the definition (32) and Eq. (24) to obtain:

$$\begin{aligned} k^{-1}w^{-1} &= \frac{1}{k \text{cd}(uK, k)} = \text{cd}(uK + jK', k) = \text{cd}\left(\left(u + \frac{jK'}{K}\right)K, k\right) \\ F_N(k^{-1}w^{-1}) &= \text{cd}\left(N\left(u + \frac{jK'}{K}\right)K_1, k_1\right) = \text{cd}\left(NuK_1 + \frac{jNK'K_1}{K}, k_1\right) \end{aligned} \quad (35)$$

and using Eq. (24) again, applied with respect to the modulus k_1 , we have:

$$\frac{1}{k_1 F_N(w)} = \frac{1}{k_1 \operatorname{cd}(NuK_1, k_1)} = \operatorname{cd}(NuK_1 + jK'_1, k_1) \quad (36)$$

Comparing Eqs. (35) and (36), we conclude that in order to satisfy $F_N(k^{-1}w^{-1}) = [k_1 F_N(w)]^{-1}$, the following identity must be satisfied for all u :

$$\operatorname{cd}\left(NuK_1 + \frac{jNK'_1K_1}{K}, k_1\right) = \operatorname{cd}(NuK_1 + jK'_1, k_1) \Rightarrow \frac{NK'_1K_1}{K} = K'_1 \quad (37)$$

from which Eq. (34) follows. We will see below that condition (8) that was obtained earlier,

$$F_N(k^{-1}) = k_1^{-1} \quad (38)$$

actually provides the *solution* of Eq. (34) for the parameter k_1 in terms of N, k , or for the parameter k in terms of N, k_1 . Using Eq. (34), we may also determine the values of $F_N(w)$ along the z -plane path $C \rightarrow S \rightarrow N \rightarrow D$. It follows from Eq. (31) and (32) that

$$\begin{aligned} C \rightarrow S, \quad w &= \operatorname{cd}(K - Ku, k), \quad F_N(w) = \operatorname{cd}(NK_1 - NK_1u, k_1) \\ S \rightarrow N, \quad w &= \operatorname{cd}(juK', k), \quad F_N(w) = \operatorname{cd}(juK'_1, k_1) = 1/\operatorname{dn}(uK'_1, k'_1) \\ N \rightarrow D, \quad w &= \operatorname{cd}(Ku + jK', k), \quad F_N(w) = \operatorname{cd}(NK_1u + jK'_1, k_1) = [k_1 \operatorname{cd}(NK_1u, k_1)]^{-1} \end{aligned} \quad (39)$$

For the path $C \rightarrow S$, $F_N(w)$ is equiripple and bounded by $|F_N(w)| \leq 1$. For the path $S \rightarrow N$, we have, using the degree equation (34):

$$w = \operatorname{cd}((juK'/K)K, k) \Rightarrow F_N(w) = \operatorname{cd}(jNK_1(juK'/K), k_1) = \operatorname{cd}(juK'_1, k_1) = \frac{1}{\operatorname{dn}(uK'_1, k'_1)}$$

and therefore, $F_N(w)$ is an increasing function taking the values $1 \leq |F_N(w)| \leq 1/k_1$. Finally, for $N \rightarrow D$, we have, using $w = \operatorname{cd}(Ku + jK', k) = \operatorname{cd}((u + jK'/K)K, k)$ and the degree equation:

$$F_N(w) = \operatorname{cd}(jNK_1(u + jK'/K), k_1) = \operatorname{cd}(NK_1u + jK'_1, k_1) = \frac{1}{k_1 \operatorname{cd}(NK_1u, k_1)}$$

Thus, the inverse $1/F_N(w) = k_1 \operatorname{cd}(NK_1u, k_1)$ is equiripple and remains bounded in the interval $|1/F_N(w)| \leq k_1$. These properties cause the magnitude response (4) to be equiripple in the passband and stopband, and monotonically decreasing in the transition band.

Next, we construct $F_N(w)$ as a rational function of w . In the same way that Eq. (33) implies that $C_N(w)$ is a polynomial of degree N , Eq. (32) implies that $F_N(w)$ will be a rational function of w of order N .

Let us look briefly at the construction of $C_N(w)$ in terms of its zeros. Then, we will use the same technique to construct $F_N(w)$. Setting $N = 2L + r$, where $r = 0$ if N is even, and $r = 1$ if N is odd, with L representing the number of second-order sections, we note that $C_N(w)$ is even in w if N is even, and odd if N is odd. Thus, $C_N(w)$ can be factored in the form $C_N(w) = [w]^r G(w^2)$, where $[w]^r$ means that the factor w is present if $r = 1$ and absent if $r = 0$, and $G(w^2)$ will be an L -th degree polynomial in w^2 . To construct it, we solve the equation $C_N(w) = 0$, or

$$\begin{aligned} \cos(Nu\pi/2) = 0 \quad \Rightarrow \quad Nu_i \frac{\pi}{2} &= (2i - 1) \frac{\pi}{2}, \quad \text{or,} \\ u_i &= \frac{2i - 1}{N}, \quad i = 1, 2, \dots, L \end{aligned} \quad (40)$$

with the zeros of $C_N(w)$ constructed by

$$\zeta_i = \cos(u_i\pi/2), \quad i = 1, 2, \dots, L \quad (41)$$

resulting in the N th degree polynomial $C_N(w)$:

$$C_N(w) = [w]^r \prod_{i=1}^L \frac{w^2 - \zeta_i^2}{1 - \zeta_i^2} \quad (42)$$

normalized such that $C_N(1) = 1$. Thus, Eq. (42) is the representation of the polynomial $C_N(w)$ in terms of its N zeros.

Next, we construct the function $F_N(w)$. It follows from the definition (32) that $F_N(w)$ will be an even (odd) function of w if N is even (odd). Indeed, applying Eq. (23) with $i = 1$ and $i = N$:

$$\begin{aligned} -w &= -\text{cd}(uK, k) = \text{cd}(uK + 2K, k) = \text{cd}((u + 2)K, k) \\ F_N(-w) &= \text{cd}(N(u + 2)K_1, k_1) = \text{cd}(NuK_1 + 2NK_1, k_1) \\ &= (-1)^N \text{cd}(NuK_1, k_1) = (-1)^N F_N(w) \end{aligned}$$

The zeros of $F_N(w)$ are obtained by solving $\text{cd}(NuK_1, k_1) = 0$. It follows from Eq. (28) that:

$$\begin{aligned} \text{cd}(NuK_1, k_1) = 0 &\Rightarrow Nu_iK_1 = (2i - 1)K_1 \quad \text{or,} \\ u_i &= \frac{2i - 1}{N}, \quad i = 1, 2, \dots, L \end{aligned} \quad (43)$$

so that the u_i are the same as those in Eq. (40). Thus, the corresponding zeros of $F_N(w)$ will be at the frequencies $w_i = \text{cd}(u_iK, k)$, and we denote them by:

$$\boxed{\zeta_i = \text{cd}(u_iK, k), \quad i = 1, 2, \dots, L} \quad (44)$$

Because of the relationship $F_N(k^{-1}w^{-1}) = [k_1F_N(w)]^{-1}$, the frequencies $w_i = (k\zeta_i)^{-1}$ will be the poles of $F_N(w)$. Thus, we may construct $F_N(w)$ as a rational function from its poles and zeros, and normalize it such that $F_N(1) = 1$:

$$\boxed{F_N(w) = [w]^r \prod_{i=1}^L \left[\left(\frac{w^2 - \zeta_i^2}{1 - w^2k^2\zeta_i^2} \right) \left(\frac{1 - k^2\zeta_i^2}{1 - \zeta_i^2} \right) \right]} \quad (45)$$

Eq. (45) is known as an *elliptic rational function*, or a *Chebyshev rational function*. We note that in the limit $k = 0$, Eq. (44) reduces to (41), and Eq. (45) reduces to (42).

Next, we obtain the solution of the degree equation (34). Using the condition (38) and setting $w = 1/k$ and $F_N(w) = 1/k_1$ in Eq. (45), we obtain the following formula for k_1 in terms of N, k :

$$k_1^{-1} = [k^{-1}]^r \prod_{i=1}^L \left[\left(\frac{k^{-2} - \zeta_i^2}{1 - \zeta_i^2} \right) \left(\frac{1 - k^2\zeta_i^2}{1 - \zeta_i^2} \right) \right] = [k^{-1}]^{2L+r} \prod_{i=1}^L \left(\frac{1 - k^2\zeta_i^2}{1 - \zeta_i^2} \right)^2$$

Noting that $N = 2L + r$, this can be rearranged into:

$$\boxed{k_1 = k^N \prod_{i=1}^L \text{sn}^4(u_iK, k)} \quad (\text{degree equation}) \quad (46)$$

where we used the property $(1 - \zeta_i^2)/(1 - k^2\zeta_i^2) = \text{sn}^2(u_iK, k)$, which follows from the last of Eqs. (18). Noting the invariance [11] of the degree equation (34) under the substitutions

$k \rightarrow k'_1$ and $k_1 \rightarrow k'$, we also obtain the exact solution for k in terms of N, k_1 , expressed via the complementary moduli k', k'_1 :

$$\boxed{k' = (k'_1)^N \prod_{i=1}^L \operatorname{sn}^4(u_i K'_1, k'_1)} \quad (\text{degree equation}) \quad (47)$$

Eqs. (46) and (47)—known as the *modular equations*—were derived first by Jacobi in his original treatise on elliptic functions [13] and have been used since in the context of elliptic filter design [6,10,11].

The degree equation can also be solved approximately, and accurately, by working with the nomes q, q_1 corresponding to the moduli k, k_1 . Exponentiating Eq. (34), we have:

$$q_1 = q^N \Leftrightarrow q = q_1^{1/N} \quad (48)$$

where the nomes are defined by $q = e^{-\pi K'/K}$ and $q_1 = e^{-\pi K'_1/K_1}$. Once q has been calculated from N and q_1 , the modulus k can be determined from the series expansion [17]:

$$k = 4\sqrt{q} \left(\frac{\sum_{m=0}^{\infty} q^{m(m+1)}}{1 + 2 \sum_{m=1}^{\infty} q^{m^2}} \right)^2 \quad (49)$$

which converges very fast. For example, keeping only the terms up to $m = 7$, gives a very accurate approximation.

4. Landen Transformations

The key tool for the evaluation of the elliptic functions $w = \operatorname{cd}(z, k)$ and $w = \operatorname{sn}(z, k)$ at any complex-valued argument z is the Landen transformation [8,18], which starts with a given elliptic modulus k and generates a sequence of *decreasing* moduli k_n via the following recursion, initialized at $k_0 = k$:

$$\boxed{k_n = \left(\frac{k_{n-1}}{1 + k'_{n-1}} \right)^2, \quad n = 1, 2, \dots, M} \quad (50)$$

where $k'_{n-1} = (1 - k_{n-1}^2)^{1/2}$. The moduli k_n decrease rapidly to zero. The recursion is stopped at $n = M$ when k_M has become smaller than a specified tolerance level, for example, smaller than the machine epsilon.[†] For all practical values of k , such as those in the range $0 \leq k \leq 0.999$, the recursion may be stopped at $M = 5$, with all subsequent k_n being smaller than 10^{-15} , while for $k \leq 0.99$, the subsequent k_n remain smaller than 10^{-20} . The recursion (50) may also be written in the equivalent form:

$$k_n = \frac{1 - k'_{n-1}}{1 + k'_{n-1}}$$

The inverse of the recursion (50) is:

$$k_{n-1} = \frac{2\sqrt{k_n}}{1 + k_n}, \quad n = M, M-1, \dots, 1 \quad (51)$$

The Landen recursions (50) imply the following recursions [18] for the complete elliptic integral $K_n = K(k_n)$ corresponding to the modulus k_n :

[†]The machine epsilon for MATLAB is $\epsilon = 2^{-52} = 2.2204 \times 10^{-16}$.

$$K_{n-1} = (1 + k_n)K_n \quad (52)$$

The recursion (52) can be repeated to compute the elliptic integral $K = K(k)$ at the initial modulus k , that is, $K = K_0 = (1 + k_1)K_1 = (1 + k_1)(1 + k_2)K_2$, and so on, yielding:

$$K = (1 + k_1)(1 + k_2) \cdots (1 + k_M)K_M, \quad K_M = \frac{\pi}{2} \quad (53)$$

Because k_M is almost zero, its elliptic integral will be essentially equal to $K_M = \pi/2$. The elliptic integral K' can be computed in the same way by applying the Landen recursion to k' . Floating point accuracy limits the applicability of Eq. (53) to roughly the range $0 \leq k \leq k_{\max}$, where $k_{\max} = (1 - k_{\min}^2)^{1/2}$, with $k_{\min} = 10^{-6}$. For k in the range $k_{\max} < k \leq 1 - \epsilon$, where ϵ is the machine epsilon, one may use the expansion:

$$K = L + (L - 1) \frac{k'^2}{2}, \quad L = -\ln\left(\frac{k'}{4}\right), \quad k' = (1 - k^2)^{1/2}$$

The Landen transformations allow also the efficient evaluation of the elliptic functions cd and sn via the following backward recursion, known as the *Gauss transformation* [18], and written in the notation of [8]:

$$\frac{1}{\text{cd}(uK_{n-1}, k_{n-1})} = \frac{1}{1 + k_n} \left[\frac{1}{\text{cd}(uK_n, k_n)} + k_n \text{cd}(uK_n, k_n) \right] \quad (54)$$

for $n = M, M-1, \dots, 1$. The recursion is initialized at $n = M$ where k_M is so small that the cd function is indistinguishable from a cosine, that is, $\text{cd}(uK_M, k_M) \simeq \cos(u\pi/2)$. Thus, the computation of $w = \text{cd}(uK, k)$, at any complex value of u , proceeds by calculating the quantities $w_n = \text{cd}(uK_n, k_n)$, initialized at $w_M = \cos(u\pi/2)$, and ending with $w_0 = w = \text{cd}(uK, k)$:

$$w_{n-1}^{-1} = \frac{1}{1 + k_n} [w_n^{-1} + k_n w_n], \quad n = M, M-1, \dots, 1 \quad (55)$$

The function $w = \text{sn}(uK, k)$ can be evaluated by the same recursion, initialized at $w_M = \sin(u\pi/2)$. The recursion (55) can also be used to calculate the *inverse* cd and sn functions by inverting it to proceed forward from $n = 1$ to $n = M$:

$$w_n = \frac{2w_{n-1}}{(1 + k_n) \left(1 + \sqrt{1 - k_{n-1}^2 w_{n-1}^2}\right)}, \quad n = 1, 2, \dots, M \quad (56)$$

Starting with a given complex value $w = \text{cd}(uK, k)$, and setting $w_0 = w$, the recursion will end at $w_M = \cos(u\pi/2)$, which may be inverted to yield $u = (2/\pi)\text{acos}(w_M)$. Because u is not unique, it may be reduced to lie within its fundamental region, $0 \leq \text{Re}(u) \leq 2$ and $0 \leq |\text{Im}(u)| \leq K'/K$. The inverse of $w = \text{sn}(uK, k)$ is obtained from the same recursion, but with $u = (2/\pi)\text{asin}(w_M)$, and reduced to lie in $-1 \leq \text{Re}(u) \leq 1$ and $0 \leq |\text{Im}(u)| \leq K'/K$.

All elliptic function computations described above can be carried out by the following set of MATLAB functions [30,31]:

<code>landen</code>	Landen transformation, Eq. (50)
<code>cde, acde</code>	cd elliptic function and its inverse, Eqs. (55) and (56)
<code>sne, asne</code>	sn elliptic function and its inverse, Eqs. (55) and (56)
<code>cne, dne</code>	cn and dn elliptic functions (for real arguments)
<code>ellipk</code>	complete elliptic integral $K(k)$, Eq. (53)
<code>ellipdeg</code>	exact solution of degree equation (k from N, k_1), Eq. (47)
<code>ellipdeg1</code>	exact solution of degree equation (k_1 from N, k), Eq. (46)
<code>ellipdeg2</code>	solution of degree equation using nomex, Eq. (49)
<code>elliprf</code>	elliptic rational function, Eq. (45)

5. Analog Elliptic Filter Design

The transfer function of an elliptic (as well as Butterworth and Chebyshev) lowpass analog filter is constructed from its zeros and poles $\{z_{ai}, p_{ai}\}$ in the second-order factored form:[†]

$$H_a(s) = H_0 \left[\frac{1}{1 - s/p_{a0}} \right]^r \prod_{i=1}^L \left[\frac{(1 - s/z_{ai})(1 - s/z_{ai}^*)}{(1 - s/p_{ai})(1 - s/p_{ai}^*)} \right] \quad (57)$$

where L is the number of analog second-order sections, related to the filter order by $N = 2L + r$. Again, the notation $[F]^r$ means that the factor F is present if $r = 1$ and absent if $r = 0$. The quantity H_0 is the gain at $\Omega = 0$ and is given as follows:

$$H_0 = \begin{cases} 1, & \text{Butterworth and Chebyshev-2} \\ G_p^{1-r}, & \text{Chebyshev-1 and Elliptic} \end{cases} \quad (58)$$

where $G_p = (1 + \varepsilon_p^2)^{-1/2}$ is the passband gain. The variable s must be replaced by $s = j\Omega = j2\pi f$ to get the filter's frequency response. Multiplying the second-order factors, we may write the transfer function in the form:

$$H(s) = H_0 \left[\frac{1}{1 + A_{01}s} \right]^r \prod_{i=1}^L \left[\frac{1 + B_{i1}s + B_{i2}s^2}{1 + A_{i1}s + A_{i2}s^2} \right] \quad (59)$$

where the numerator and denominator coefficients are given by

$$\begin{aligned} [1, B_{i1}, B_{i2}] &= [1, -2 \operatorname{Re}\left(\frac{1}{z_{ai}}\right), \frac{1}{|z_{ai}|^2}] \\ [1, A_{i1}, A_{i2}] &= [1, -2 \operatorname{Re}\left(\frac{1}{p_{ai}}\right), \frac{1}{|p_{ai}|^2}] \\ [1, A_{01}] &= [1, -\frac{1}{p_{a0}}] \end{aligned} \quad (60)$$

Because the magnitude response corresponding to Eq. (57) is given by

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_p^2 F_N^2(w)}, \quad w = \frac{\Omega}{\Omega_p}, \quad (61)$$

it follows that the zeros z_{ai} will arise from the poles of $F_N(w)$, and the poles p_{ai} will arise from the zeros of the denominator, that is, $1 + \varepsilon_p^2 F_N^2(w) = 0$. We saw in the previous section that the poles of $F_N(w)$ occur at the normalized frequencies $w_i = (k\zeta_i)^{-1}$. Therefore, taking into account the normalization factor Ω_p , the denormalized s -plane zeros $z_{ai} = \Omega_p j w_i$ will be:

$$\boxed{z_{ai} = \Omega_p j (k\zeta_i)^{-1}, \quad i = 1, 2, \dots, L} \quad (\text{s-plane zeros}) \quad (62)$$

The poles p_{ai} are found by solving the equation: $1 + \varepsilon_p^2 F_N^2(w) = 0$, or,

$$F_N(w) = \pm j \frac{1}{\varepsilon_p} \quad (63)$$

The complex-frequency solutions w_i of (63) determine the denormalized poles by setting $p_{ai} = \Omega_p j w_i$. The resulting left-hand s -plane poles p_{ai} are found to be:

$$\boxed{p_{ai} = \Omega_p j \operatorname{cd}((u_i - jv_0)K, k), \quad i = 1, 2, \dots, L} \quad (\text{left-hand s-plane poles}) \quad (64)$$

[†]The Butterworth and Chebyshev-1 cases do not have any zero factors.

where the u_i are the same as in Eq. (43), and v_0 is the real-valued solution of the equation:

$$\boxed{\operatorname{sn}(jv_0NK_1, k_1) = j\frac{1}{\varepsilon_p}} \Rightarrow v_0 = -\frac{j}{NK_1} \operatorname{sn}^{-1}\left(\frac{j}{\varepsilon_p}, k_1\right) \quad (65)$$

As noted earlier in Fig. 9, the bottom two quadrants of the fundamental rectangle on the z - or u -plane get mapped onto the left-hand s -plane. If N is odd, there is an additional real-valued left-hand s -plane pole p_{a0} obtained from Eq. (64) by setting $u_i = 1$ (which corresponds to the index $i = L + 1$):

$$p_{a0} = \Omega_p j \operatorname{cd}((1 - jv_0)K, k) = \Omega_p j \operatorname{sn}(jv_0K, k) \quad (66)$$

To verify that $w_i = \operatorname{cd}((u_i - jv_0)K, k)$ is a solution of Eq. (63), we use the definition (32), property (22), and condition (65) to obtain:

$$\begin{aligned} F_N(w_i) &= \operatorname{cd}((u_i - jv_0)NK_1, k_1) = \operatorname{cd}(u_iNK_1 - jv_0NK_1, k_1) = \operatorname{cd}((2i - 1)K_1 - jv_0NK_1, k_1) \\ &= (-1)^i \operatorname{sn}(-jv_0NK_1, k_1) = -(-1)^i \operatorname{sn}(jv_0NK_1, k_1) = \pm j\frac{1}{\varepsilon_p} \end{aligned}$$

6. Design Example

To clarify the above design steps, we give the MATLAB code for calculating the zeros, poles, and transfer function of the elliptic example of Fig. 2.

```

fp = 4; fs = 4.5; Gp = 0.95; Gs = 0.05;           % filter specifications

Wp = 2*pi*fp; Ws = 2*pi*fs;

ep = sqrt(1/Gp^2 - 1); es = sqrt(1/Gs^2 - 1);    % ripples  $\varepsilon_p = 0.3287$ ,  $\varepsilon_s = 19.9750$ 

k = Wp/Ws;                                       %  $k = 0.8889$ 
k1 = ep/es;                                     %  $k_1 = 0.0165$ 

[K,Kp] = ellipk(k);                             % elliptic integrals  $K = 2.2353$ ,  $K' = 1.6646$ 
[K1,K1p] = ellipk(k1);                         % elliptic integrals  $K_1 = 1.5709$ ,  $K'_1 = 5.4937$ 

Nexact = (K1p/K1)/(Kp/K); N = ceil(Nexact);     %  $N_{\text{exact}} = 4.6961$ ,  $N = 5$ 

k = ellipdeg(N,k1);                             % recalculated  $k = 0.9143$ 

fs_new = fp/k;                                  % new stopband  $f_s = 4.3751$ 

L = floor(N/2); r = mod(N,2); i = (1:L)';      %  $L = 2$ ,  $r = 1$ ,  $i = [1; 2]$ 
u = (2*i-1)/N; zeta_i = cde(u,k);             %  $u_i = [0.2; 0.6]$ ,  $\zeta_i = [0.9808; 0.7471]$ 

za = Wp * j./(k*zeta_i);                       % filter zeros

v0 = -j*asne(j/ep, k1)/N;                     %  $v_0 = 0.2331$ 

pa = Wp * j*cde(u-j*v0, k);                   % filter poles
pa0 = Wp * j*sne(j*v0, k);

B = [ones(L,1), -2*real(1./za), abs(1./za).^2]; % numerator second-order sections
A = [ones(L,1), -2*real(1./pa), abs(1./pa).^2]; % denominator second-order sections

if r==0,                                       % prepend first-order sections
    B = [Gp, 0, 0; B];                       % DC gain is  $H_0 = G_p$ , if  $N$  is even
    A = [1, 0, 0; A];
else
    B = [1, 0, 0; B];                         % DC gain is  $H_0 = 1$ , if  $N$  is odd

```

```

    A = [1, -real(1/pa0), 0; A];
end

f = linspace(0,10,2001);

for n=1:length(f),
    s = j*2*pi*f(n);
    H(n) = prod((B(:,1) + B(:,2)*s + B(:,3)*s^2)./...
                (A(:,1) + A(:,2)*s + A(:,3)*s^2));
end
% calculate frequency response
% s = jΩ = 2πjf
% cascade filter sections
% alternatively, use H=fresp_a(B,A,f)

plot(f,abs(H),'r-');

xlim([0,10]); ylim([0,1.1]); grid off;
set(gca, 'xtick', 0:1:10); set(gca, 'ytick', 0:0.1:1);

title('Elliptic, N = 5');
xlabel('f'); ylabel('|H(f)|');

line([0,fp],[1,1]); line([fp,fp],[1,1.05]);
line([0,fp],[Gp,Gp]); line([fp,fp],[Gp,0]);
line([fs_new,10],[Gs,Gs]); line([fs,fs],[4*Gs,Gs]);
% draw brick-wall specs

```

The filter order was determined by calculating the exact value of N that satisfies the degree equation (34), that is, $N_{\text{exact}} = (K'_1/K_1)/(K'/K)$, and then, rounding it up to the next integer. With the slightly increased integer value of N , the degree equation is no longer satisfied with the given k, k_1 . To satisfy it exactly, we recalculate k from N, k_1 using Eq. (47). The resulting k is slightly larger than the original one, and hence, the effective stopband $f_s = f_p/k$ will be slightly smaller, making the transition width narrower. The calculated zeros and poles of the filter are, for $N = 5$ and $L = 2$:

$$\begin{aligned}
 z_1 &= 28.0265j & p_0 &= -15.1717 \\
 z_2 &= 36.7945j & p_1 &= -1.0115 + 25.4353j \\
 & & p_2 &= -6.2951 + 21.4113j
 \end{aligned}$$

The resulting first- and second-order numerator and denominator coefficients of the transfer function (57) are the rows of the matrices B and A , respectively:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0.00127 \\ 1 & 0 & 0.00074 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0.06591 & 0 \\ 1 & 0.00312 & 0.00154 \\ 1 & 0.02528 & 0.00201 \end{bmatrix} \quad (67)$$

Thus, the transfer function will be, with $s = 2\pi jf$:

$$H(s) = \frac{1}{1 + 0.06591s} \cdot \frac{1 + 0.00127s^2}{1 + 0.00312s + 0.00154s^2} \cdot \frac{1 + 0.00074s^2}{1 + 0.02528s + 0.00201s^2}$$

The following function `ellipap2.m` incorporates the above design steps and serves as a substitute for MATLAB's built-in function `ellipap`.

```

% ellipap2.m - analog lowpass elliptic filter design
%
% Usage: [z,p,H0,B,A] = ellipap2(N,Ap,As)
%
% N = filter order
% Ap = passband attenuation in dB
% As = stopband attenuation in dB
%
% z = vector of normalized filter zeros (in units of the passband frequency Ωp = 2π fp)
% p = vector of normalized filter poles

```

```

% H0 = DC gain factor
% B = matrix whose rows are the first- and second-order numerator coefficients
% A = matrix whose rows are the first- and second-order denominator coefficients
%
% notes: serves as a substitute for the built-in function ELLIPAP
%       the gain factor g returned by ELLIPAP is related to the dc gain by g = abs(H0*prod(p)/prod(z))
%
%       N = 2*L+r, r = mod(N,2), L = floor(N/2) = no. second-order sections
%
%       length(p) = N, length(z) = 2*L
%       transfer function:  $H(s) = H_0 \left[ \frac{1}{1-s/p_0} \right]^r \prod_{i=1}^L \left[ \frac{(1-s/z_i)(1-s/z_i^*)}{(1-s/p_i)(1-s/p_i^*)} \right]$ 
%
%       normalized s-plane variable,  $s = j\Omega/\Omega_p$ ,  $\Omega = 2\pi f$ ,  $\Omega_p = 2\pi f_p$ ,  $f_p$  = passband frequency

function [z,p,H0,B,A] = ellipap2(N,Ap,As)

if nargin==0, help ellipap2; return; end

Gp = 10^(-Ap/20); % passband gain
ep = sqrt(10^(Ap/10) - 1); % ripple factors
es = sqrt(10^(As/10) - 1);

k1 = ep/es;
k = ellipdeg(N,k1); % solve degree equation

L = floor(N/2); r = mod(N,2); % L is the number of second-order sections
i = (1:L)'; ui = (2*i-1)/N; zeta_i = cde(ui,k); % zeros of elliptic rational function

z = j./(k*zeta_i); % filter zeros = poles of elliptic rational function

v0 = -j*asne(j/ep, k1)/N; % solution of  $\text{sn}(jv_0NK_1, k_1) = j/\varepsilon_p$ 

p = j*cde(ui-j*v0, k); % filter poles
p0 = j*sne(j*v0, k); % first-order pole, needed when N is odd

B = [ones(L,1), -2*real(1./z), abs(1./z).^2]; % second-order numerator sections
A = [ones(L,1), -2*real(1./p), abs(1./p).^2]; % second-order denominator sections

if r==0, % prepend first-order sections
    B = [Gp, 0, 0; B]; A = [1, 0, 0; A];
else
    B = [1, 0, 0; B]; A = [1, -real(1/p0), 0; A];
end

z = cplxpair([z; conj(z)]); % append conjugate zeros
p = cplxpair([p; conj(p)]); % append conjugate poles
if r==1, p = [p; p0]; end % append first-order pole when N is odd

H0 = Gp^(1-r); % dc gain

```

7. Butterworth and Chebyshev Designs

For completeness, we discuss also the design of Butterworth, Chebyshev-1, and Chebyshev-2 analog filters. For given specifications $\{\Omega_p, \Omega_s, A_p, A_s\}$, one calculates the parameters k, k_1 from Eq. (3), and solves for the filter order from Eq. (9), rounding it up to the next integer. In all cases, the filter poles are obtained by solving the equation:

$$1 + \varepsilon_p^2 F_N^2(w) = 0 \quad \Rightarrow \quad F_N^2(w) = -\frac{1}{\varepsilon_p^2} \quad (68)$$

As in the elliptic case, we define the quantities:

$$r = \text{mod}(N, 2), \quad L = \frac{N-r}{2}, \quad u_i = \frac{2i-1}{N}, \quad i = 1, 2, \dots, L \quad (69)$$

For the Butterworth case, we obtain the conjugate poles $\{p_{ai}, p_{ai}^*\}$ of the second-order factors of Eq. (57) and the real-valued pole p_{a0} when N is odd (corresponding to the value $u_i = 1$):

$$\boxed{\begin{aligned} p_{ai} &= \Omega_p \varepsilon_p^{-1/N} j e^{j u_i \pi / 2}, \quad i = 1, 2, \dots, L \\ p_{a0} &= \Omega_p \varepsilon_p^{-1/N} j e^{j \pi / 2} = -\Omega_p \varepsilon_p^{-1/N} \end{aligned}} \quad (\text{Butterworth}) \quad (70)$$

We note that the quantity $\Omega_0 = \Omega_p \varepsilon_p^{-1/N}$ is the 3-dB frequency of the Butterworth filter. Indeed, since $\Omega_0^{2N} = \Omega_p^{2N} / \varepsilon_p^2$, the magnitude response can be written as

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_p^2 w^{2N}} = \frac{1}{1 + \varepsilon_p^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}} = \frac{1}{1 + \left(\frac{\Omega}{\Omega_0}\right)^{2N}}$$

In the Chebyshev-1 case, we have:

$$\boxed{\begin{aligned} p_{ai} &= \Omega_p j \cos((u_i - j\nu_0)\pi/2), \quad i = 1, 2, \dots, L \\ p_{a0} &= \Omega_p j \cos((1 - j\nu_0)\pi/2) = -\Omega_p \sinh(\nu_0\pi/2) \end{aligned}} \quad (\text{Chebyshev-1}) \quad (71)$$

where ν_0 is obtained from the solution of:

$$\sinh(N\nu_0\pi/2) = \frac{1}{\varepsilon_p} \Rightarrow \nu_0 = \frac{\text{asinh}(1/\varepsilon_p)}{N\pi/2} \quad (72)$$

In the Chebyshev-2 case, the transfer function (57) has both poles and zeros, the latter arising from the numerator of Eq. (6), that is,

$$|H(\Omega)|^2 = \frac{1}{1 + \varepsilon_p^2 k_1^{-2} / C_N^2(k^{-1}w^{-1})} = \frac{C_N^2(k^{-1}w^{-1})}{C_N^2(k^{-1}w^{-1}) + \varepsilon_p^2 k_1^{-2}}$$

The resulting conjugate s -plane zeros $\{z_{ai}, z_{ai}^*\}$ and poles $\{p_{ai}, p_{ai}^*\}$, and the extra real-valued pole p_{a0} when N is odd, are essentially the inverses of those of the Chebyshev-1 case because of the correspondence $w \rightarrow k^{-1}w^{-1}$:

$$\boxed{\begin{aligned} k^{-1}z_{ai}^{-1} &= \Omega_p^{-1} j \cos(u_i\pi/2), \quad i = 1, 2, \dots, L \\ k^{-1}p_{ai}^{-1} &= \Omega_p^{-1} j \cos((u_i - j\nu_0)\pi/2), \quad i = 1, 2, \dots, L \\ k^{-1}p_{a0}^{-1} &= \Omega_p^{-1} j \cos((1 - j\nu_0)\pi/2) = -\Omega_p^{-1} \sinh(\nu_0\pi/2) \end{aligned}} \quad (\text{Chebyshev-2}) \quad (73)$$

where ν_0 is the solution of the equation:

$$\sinh(N\nu_0\pi/2) = \varepsilon_p k_1^{-1} = \varepsilon_s \Rightarrow \nu_0 = \frac{\text{asinh}(\varepsilon_s)}{N\pi/2} \quad (74)$$

Because k is used in Eq. (73), it must be recalculated by solving the second of Eqs. (9) for k in terms of k_1 and the rounded-up value of N , that is, $k = 1/\cosh(\text{acosh}(k_1^{-1})/N)$.

In all cases, the poles lie in the left-hand s -plane, that is, $\text{Re}(p_{ai}) < 0$. The overall transfer function is constructed by Eqs. (58)–(60), where in the Butterworth and Chebyshev-1 cases one may set $B_{i1} = B_{i2} = 0$ in the second-order numerator factors.

In all cases, the passband specification is matched exactly, while the stopband specification is exceeded because of the rounding of the exact N to the next integer.

The above design steps, as well as those for elliptic filters, have been incorporated into the MATLAB function `lpa.m`, listed below:

```
% lpa.m - lowpass analog filter design
%
% function [N,B,A] = lpa(Wp, Ws, Ap, As, type)
%
% Wp,Ws = passband and stopband frequencies in rad/sec
% Ap,As = passband and stopband attenuations in dB
% type = 0,1,2,3 for Butterworth, Chebyshev-1, Chebyshev-2, Elliptic
%
% N = filter order
% B,A = (L+1)x3 matrices of numerator and denominator coefficients, L=floor(N/2)

function [N,B,A] = lpa(Wp, Ws, Ap, As, type)

if nargin==0, help lpa; return; end

ep = sqrt(10^(Ap/10)-1); es = sqrt(10^(As/10)-1);

k = Wp/Ws; k1 = ep/es; % selectivity and discrimination parameters

switch type % determine order N
case 0
    Nexact = log(1/k1) / log(1/k);
    N = ceil(Nexact);
case 1
    Nexact = acosh(1/k1) / acosh(1/k);
    N = ceil(Nexact);
case 2
    Nexact = acosh(1/k1) / acosh(1/k);
    N = ceil(Nexact);
    k = 1/cosh(acosh(1/k1) / N); % recalculate k to satisfy degree equation
case 3
    [K,Kp] = ellipk(k);
    [K1,K1p] = ellipk(k1);
    Nexact = (K1p/K1)/(Kp/K);
    N = ceil(Nexact);
    k = ellipdeg(N,k1); % recalculate k to satisfy degree equation
end

r = mod(N,2); L = (N-r)/2; i = (1:L)'; u=(2*i-1)/N;

switch type % determine poles and zeros
case 0
    pa = Wp * j * exp(j*u*pi/2) / ep^(1/N);
    pa0 = -Wp / ep^(1/N);
case 1
    v0 = asinh(1/ep) / (N*pi/2);
    pa = Wp * j * cos((u-j*v0)*pi/2);
    pa0 = -Wp * sinh(v0*pi/2);
case 2
    v0 = asinh(es) / (N*pi/2);
    za = Wp ./ (j*k*cos(u*pi/2));
    pa = Wp ./ (j*k*cos((u-j*v0)*pi/2));
    pa0 = -Wp / (k*sinh(v0*pi/2));
case 3
    v0 = -j * asne(j/ep, k1) / N;
```

```

        za = Wp * j ./ (k*cde(u,k));
        pa = Wp * j * cde(u-j*v0, k);
        pa0 = Wp * j * sne(j*v0, k);
    end

    B = [ones(L+1,1), zeros(L+1,2)];
    A = [ones(L,1), -2*real(1./pa), abs(1./pa).^2]; % determine coefficient matrices

    A = [[1, -r*real(1/pa0), 0]; A];

    if type==2 | type==3,
        B(2:L+1,:) = [ones(L,1), -2*real(1./za), abs(1./za).^2];
    end

    Gp = 10^(-Ap/20);

    if type==1 | type==3, % adjust dc gain
        B(1,1) = Gp^(1-r);
    end
end

```

The elliptic portion of `lpa` is essentially equivalent to `ellipap2`. The function outputs the filter order N and the numerator and denominator coefficient matrices B, A , as given for example in Eq. (67), with the corresponding transfer function given by Eq. (59). The graphs of Fig. 2 can be generated by the following code fragment:

```

fp = 4; fs = 4.5;
Wp = 2*pi*fp; Ws = 2*pi*fs;
Gp = 0.95; Gs = 0.05;
Ap = -20*log10(Gp); As = -20*log10(Gs);

type = 3; % type = 0,1,2,3 for butter, cheby-1, cheby-2, elliptic

[N,B,A] = lpa(Wp,Ws,Ap,As,type);

f = linspace(0,10,1001);
s = j*2*pi*f; % s-domain

H = 1;
for i=1:size(B,1),
    H = H .* (B(i,1) + B(i,2)*s + B(i,3)*s.^2) ./ (A(i,1) + A(i,2)*s + A(i,3)*s.^2);
end

plot(f,abs(H),'r');

xtick(0:1:10); ylim([0,1.1]); ytick(0:0.1:1); grid off;

line([0,fp],[1,1],'LineStyle',':');
line([fp,fp],[1,1.05],'LineStyle',':');
line([0,fp],[Gp,Gp]); line([fp,fp],[Gp,0]);
line([fs,10],[Gs,Gs]); line([fs,fs],[4*Gs,Gs]);

```

8. Highpass, Bandpass, and Bandstop Analog Filters

The design of highpass, bandpass, or bandstop filters can be accomplished by applying an s -domain frequency transformation that maps a lowpass analog prototype to the desired filter. Such a transformation takes the following form, where s' is the equivalent lowpass variable:

$$s' = F(s) \quad (75)$$

The specifications of the desired highpass, bandpass, or bandstop filter are mapped by the same transformation into specifications, such as $\{\Omega'_p, \Omega'_s, A_p, A_s\}$, for the equivalent lowpass

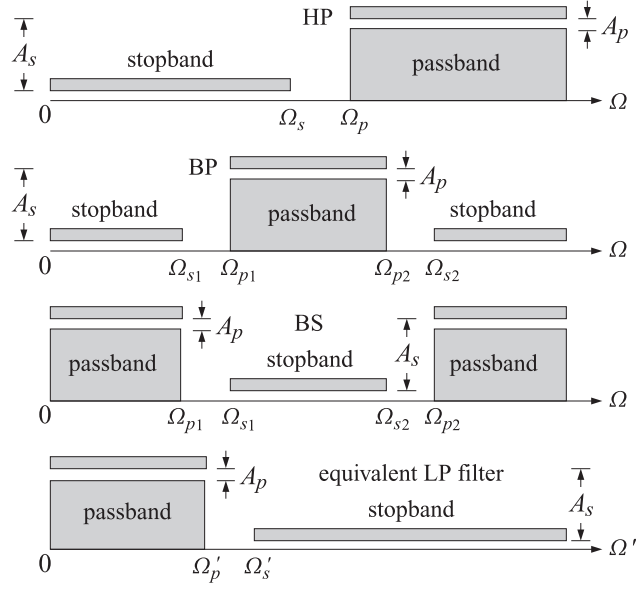


Fig. 11 Specifications of HP, BP, BS filters and of the equivalent LP filter.

filter. Based on the transformed specifications, the equivalent lowpass filter's transfer function can be designed in the following form (using for example the function 1pa):

$$H_{LP}(s') = H_0 \left[\frac{1}{1 + A_{01}s'} \right]^r \prod_{i=1}^L \left[\frac{1 + B_{i1}s' + B_{i2}s'^2}{1 + A_{i1}s' + A_{i2}s'^2} \right] \quad (76)$$

and then mapped onto the desired filter by:

$$H(s) = H_{LP}(s') = H_{LP}(F(s)) \quad (77)$$

The brick-wall specifications of the highpass, bandpass, and bandstop filters, and the corresponding specifications of the equivalent lowpass filter are shown in Fig. 11.

The mapping function $F(s)$ and the corresponding mapping specifications are given as follows in the three cases. For *highpass* designs, define:

$$s' = \frac{1}{s}, \quad \Omega'_p = \frac{1}{\Omega_p}, \quad \Omega'_s = \frac{1}{\Omega_s} \quad (78)$$

For *bandpass* designs, the bandwidth and center frequency of the passband are:

$$\Delta\Omega = \Omega_{p2} - \Omega_{p1}, \quad \Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} \quad (79)$$

Then, the corresponding LP parameters are:

$$s' = s + \frac{\Omega_0^2}{s}, \quad \Omega'_p = \Delta\Omega, \quad \Omega'_s = \min(|\Omega'_{s1}|, |\Omega'_{s2}|) \quad (80)$$

where

$$\Omega'_{s1} = \Omega_{s1} - \frac{\Omega_0^2}{\Omega_{s1}}, \quad \Omega'_{s2} = \Omega_{s2} - \frac{\Omega_0^2}{\Omega_{s2}} \quad (81)$$

These are justified as follows. Setting $s' = j\Omega'$ and $s = j\Omega$ into Eq. (80), we obtain the corresponding mapping of the bandpass frequency Ω to the lowpass frequency Ω' :

$$\Omega' = \Omega - \frac{\Omega_0^2}{\Omega}$$

and demand that the passband interval $[\Omega_{p1}, \Omega_{p2}]$ get mapped onto $[-\Omega'_p, \Omega'_p]$, that is,

$$-\Omega'_p = \Omega_{p1} - \frac{\Omega_0^2}{\Omega_{p1}}, \quad \Omega'_p = \Omega_{p2} - \frac{\Omega_0^2}{\Omega_{p2}}$$

These may be solved to give:

$$\Omega'_p = \Omega_{p2} - \Omega_{p1}, \quad \Omega_0^2 = \Omega_{p1}\Omega_{p2}$$

Once Ω_0 is fixed from the passband frequencies, it is no longer possible to map the stopband interval $[\Omega_{s1}, \Omega_{s2}]$ onto the symmetric lowpass interval $[-\Omega'_s, \Omega'_s]$. Therefore, Ω'_s is selected on the basis of the shorter of the two mapped stopband frequencies of Eq. (81).

For *bandstop* filters, the bandwidth $\Delta\Omega$ and center frequency Ω_0 are selected on the basis of the stopband interval:

$$\Delta\Omega = \Omega_{s2} - \Omega_{s1}, \quad \Omega_0 = \sqrt{\Omega_{s1}\Omega_{s2}} \quad (82)$$

Then, the corresponding LP parameters are:

$$s' = \frac{1}{s + \frac{\Omega_0^2}{s}}, \quad \Omega'_p = \max(|\Omega'_{p1}|, |\Omega'_{p2}|), \quad \Omega'_s = \frac{1}{\Delta\Omega} \quad (83)$$

where

$$\Omega'_{p1} = \frac{1}{\Omega_{p1} - \frac{\Omega_0^2}{\Omega_{p1}}}, \quad \Omega'_{p2} = \frac{1}{\Omega_{p2} - \frac{\Omega_0^2}{\Omega_{p2}}} \quad (84)$$

In all three cases, once the equivalent frequencies Ω'_p, Ω'_s have been determined, the selectivity parameter can be calculated by $k = \Omega'_p/\Omega'_s$. The discrimination parameter $k_1 = \varepsilon_p/\varepsilon_s$ remains the same. Based on the values of k, k_1 , the equivalent lowpass filter can be designed as a Butterworth, Chebyshev, or elliptic filter.

Fig. 12 shows some design examples. To clarify the design steps, the following code fragment implements the elliptic bandpass example:

```

fp1=3; fp2=6; fs1=2.5; fs2=6.5;
Wp1 = 2*pi*fp1; Wp2 = 2*pi*fp2; Ws1 = 2*pi*fs1; Ws2 = 2*pi*fs2;
Gp = 0.95; Gs = 0.05;
Ap = -20*log10(Gp); As = -20*log10(Gs);

Wp = Wp2-Wp1; W0 = sqrt(Wp1*Wp2);
W1 = Ws1 - W0^2/Ws1; W2 = Ws2 - W0^2/Ws2;
Ws = min(abs([W1,W2]));

type = 3;
[N,B,A] = lpa(Wp,Ws,Ap,As,type);

f = linspace(0,10,1001); s = j*2*pi*f;
s = s + W0^2./s; % division by zero warning can be ignored

H = 1;
for i=1:size(B,1),
    H = H .* (B(i,1) + B(i,2)*s + B(i,3)*s.^2) ./ (A(i,1) + A(i,2)*s + A(i,3)*s.^2);
end

```

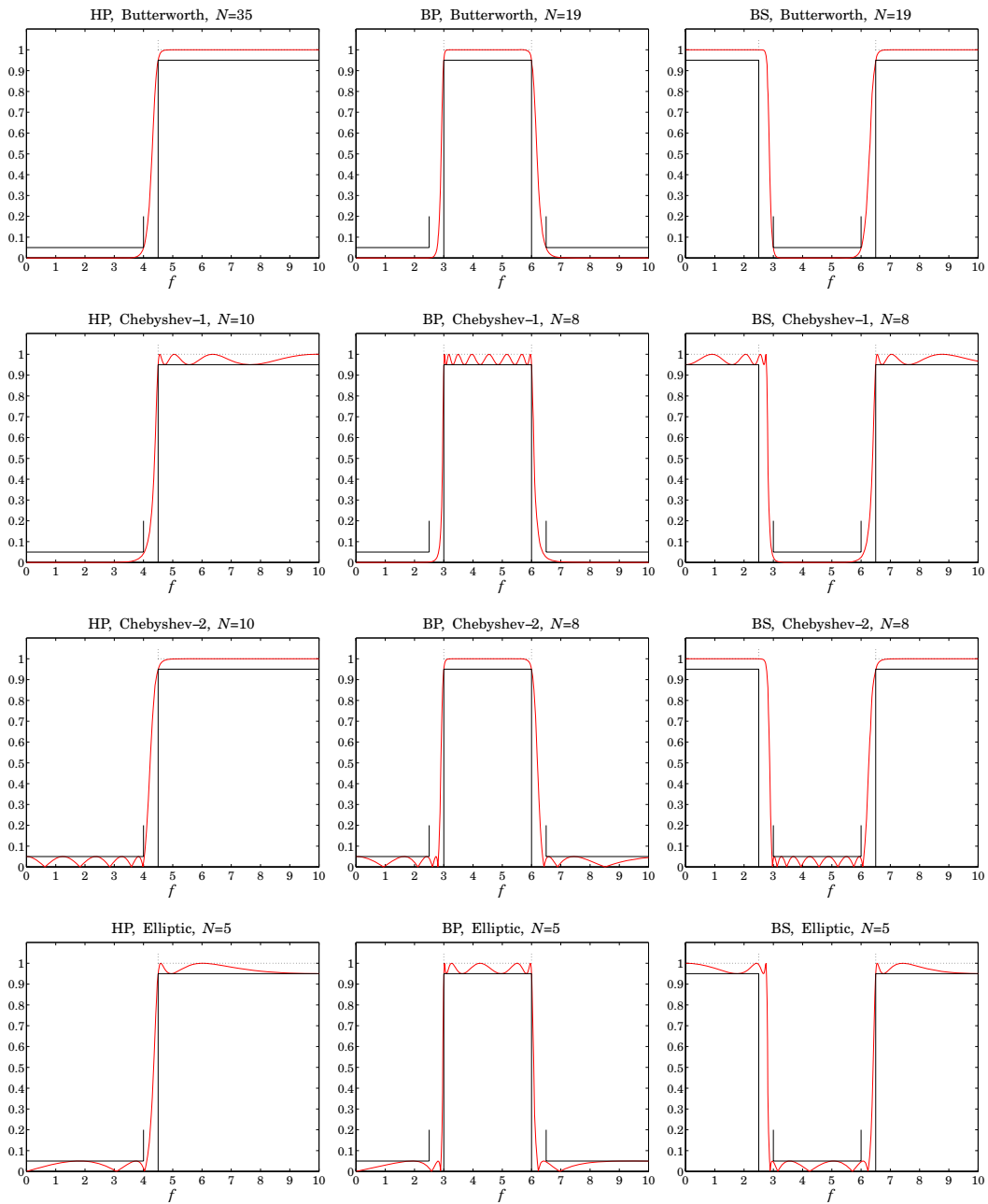



Fig. 12 Highpass, bandpass, and bandstop analog filters.

end

```
plot(f,abs(H),'r');
```

```
xtick(0:1:10); ylim([0,1.1]); ytick(0:0.1:1); grid off;
```

```
line([fp1,fp2],[1,1],'LineStyle',':');
```

```
line([fp1,fp1],[1,1.05],'LineStyle',':'); line([fp2,fp2],[1,1.05],'LineStyle',':');
```

```
line([fp1,fp2],[Gp,Gp]); line([fp1,fp1],[Gp,0]); line([fp2,fp2],[Gp,0]);
```

```

line([0,fs1],[Gs,Gs]); line([fs1,fs1],[4*Gs,Gs]);
line([fs2,10],[Gs,Gs]); line([fs2,fs2],[4*Gs,Gs]);

```

9. Digital Filter Design

Digital filters may be designed by the bilinear transformation method carried out by the following steps: (a) the specifications of the digital filter are transformed into equivalent specifications for a lowpass analog prototype filter, (b) the analog prototype is designed as a Butterworth, Chebyshev, or elliptic filter using the methods discussed above, and (c) the analog filter's transfer function $H_a(s)$ is transformed into the desired digital filter's transfer function $H(z)$ with an appropriate bilinear transformation, that is, a mapping between the s -plane and the z -plane of the form $s = f(z)$:

$$H(z) = H_a(s) \Big|_{s=f(z)} = H_a(f(z)) \quad (85)$$

The mappings used for lowpass, highpass, bandpass, and bandstop filters, and the corresponding frequency mappings obtained by setting $s = j\Omega$ and $z = e^{j\omega}$, where Ω is the equivalent analog frequency and $\omega = 2\pi f/f_s$, the digital frequency, are as follows:

$$\begin{aligned}
\text{(LP)} \quad s &= \frac{1 - z^{-1}}{1 + z^{-1}}, & \Omega &= \tan\left(\frac{\omega}{2}\right) \\
\text{(HP)} \quad s &= \frac{1 + z^{-1}}{1 - z^{-1}}, & \Omega &= -\cot\left(\frac{\omega}{2}\right) \\
\text{(BP)} \quad s &= \frac{1 - 2c_0z^{-1} + z^{-2}}{1 - z^{-2}}, & \Omega &= \frac{c_0 - \cos \omega}{\sin \omega} \\
\text{(BS)} \quad s &= \frac{1 - z^{-2}}{1 - 2c_0z^{-1} + z^{-2}}, & \Omega &= -\frac{\sin \omega}{c_0 - \cos \omega}
\end{aligned} \quad (86)$$

where $c_0 = \cos \omega_0$, with ω_0 corresponding to the center of the bandpass or bandstop filter.

10. Pole and Zero Transformations

We begin with lowpass digital filters constructed with the lowpass bilinear transformation:

$$s = \frac{1 - z^{-1}}{1 + z^{-1}} \quad (87)$$

Assuming that the equivalent lowpass analog filter $H_a(s)$ has already been constructed in terms of its zeros and poles, the lowpass digital filter's transfer function will be:

$$H_{\text{LP}}(z) = H_a(s) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = H_0 \left[\frac{1}{1 - s/p_{a0}} \right]^r \prod_{i=1}^L \left[\frac{(1 - s/z_{ai})(1 - s/z_{ai}^*)}{(1 - s/p_{ai})(1 - s/p_{ai}^*)} \right] \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \quad (88)$$

After mapping the analog s -plane poles and zeros $\{p_{ai}, z_{ai}\}$ to the digital z -plane poles and zeros $\{p_i, z_i\}$ by Eq. (87), the resulting digital transfer function will have the form:

$$H_{\text{LP}}(z) = H_0 \left[G_0 \frac{1 + z^{-1}}{1 - p_0 z^{-1}} \right]^r \prod_{i=1}^L \left[|G_i|^2 \frac{(1 - z_i z^{-1})(1 - z_i^* z^{-1})}{(1 - p_i z^{-1})(1 - p_i^* z^{-1})} \right] \quad (89)$$

Noting that the inverse of Eq. (87) is $z = \frac{1 + s}{1 - s}$, the digital poles and zeros are computed by:

$$p_0 = \frac{1 + p_{a0}}{1 - p_{a0}}, \quad p_i = \frac{1 + p_{ai}}{1 - p_{ai}}, \quad z_i = \frac{1 + z_{ai}}{1 - z_{ai}}, \quad i = 1, 2, \dots, L \quad (90)$$

and the gains factors, by:

$$G_0 = \frac{1 - p_0}{2}, \quad G_i = \frac{1 - p_i}{1 - z_i} \quad (91)$$

Eq. (89) follows from the transformation identity of each s -plane pole/zero factor:

$$\frac{1 - s/z_{ai}}{1 - s/p_{ai}} = G_i \frac{1 - z_i z^{-1}}{1 - p_i z^{-1}}, \quad G_i = \frac{1 - p_i}{1 - z_i} \quad (92)$$

The zeros of the Butterworth and Chebyshev-1 cases are simply $z_i = -1$ as follows by setting $z_{ai} = \infty$ in Eq. (90). Highpass digital filters can be designed by mapping the lowpass analog prototype by the highpass version of the bilinear transformation:

$$s = \frac{1 + z^{-1}}{1 - z^{-1}} \quad (93)$$

which amounts to making the replacement $z^{-1} \rightarrow -z^{-1}$ in the lowpass case. Replacing the lowpass poles and zeros $\{p_i, z_i\}$ by their negatives, one obtains the highpass transfer function:

$$\begin{aligned} H_{\text{HP}}(z) &= H_0 \left[\frac{1}{1 - s/p_{a0}} \right]^r \prod_{i=1}^L \left[\frac{(1 - s/z_{ai})(1 - s/z_{ai}^*)}{(1 - s/p_{ai})(1 - s/p_{ai}^*)} \right] \Bigg|_{s=\frac{1+z^{-1}}{1-z^{-1}}} \\ &= H_0 \left[G_0 \frac{1 - z^{-1}}{1 - p_0 z^{-1}} \right]^r \prod_{i=1}^L \left[|G_i|^2 \frac{(1 - z_i z^{-1})(1 - z_i^* z^{-1})}{(1 - p_i z^{-1})(1 - p_i^* z^{-1})} \right] \end{aligned} \quad (94)$$

where the highpass digital poles and zeros are now given by:

$$p_0 = -\frac{1 + p_{a0}}{1 - p_{a0}}, \quad p_i = -\frac{1 + p_{ai}}{1 - p_{ai}}, \quad z_i = -\frac{1 + z_{ai}}{1 - z_{ai}}, \quad i = 1, 2, \dots, L \quad (95)$$

and the gains factors, by:

$$G_0 = \frac{1 + p_0}{2}, \quad G_i = \frac{1 + p_i}{1 + z_i} \quad (96)$$

More generally, the lowpass analog filter can be mapped first into a lowpass digital filter using Eq. (87), which can subsequently be mapped by a z -domain frequency transformation [21,22] to a highpass, bandpass, or bandstop digital filter:

$$\boxed{\begin{array}{c} \text{LP analog} \\ s\text{-plane} \\ H_a(s) \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{LP digital} \\ \hat{z}\text{-plane} \\ \hat{H}(\hat{z}) \end{array}} \longrightarrow \boxed{\begin{array}{c} \text{HP/BP/BS digital} \\ z\text{-plane} \\ H(z) \end{array}} \quad (97)$$

For bandpass designs, the required transformations take the two-step form:

$$s = \frac{1 - \hat{z}^{-1}}{1 + \hat{z}^{-1}} = \frac{1 - 2c_0 z^{-1} + z^{-2}}{1 - z^{-2}}, \quad \hat{z}^{-1} = \frac{z^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}} \quad (98)$$

where $c_0 = \cos \omega_0$ and $\omega_0 = 2\pi f_0/f_s$ is the center frequency of the bandpass filter. Setting $\omega_0 = \pi$ or $c_0 = -1$ yields the highpass case, which has $\hat{z}^{-1} = -z^{-1}$. The lowpass case corresponds to $\omega_0 = 0$ or $c_0 = 1$, which gives $\hat{z} = z$.

The mapping $s \rightarrow \hat{z}$ transforms a lowpass analog filter $H_a(s)$ into lowpass digital filter $\hat{H}(\hat{z})$, which is further transformed into a bandpass digital filter $H_{BP}(z)$ by the mapping $\hat{z} \rightarrow z$, that is, the transfer functions are related by:

$$H_{BP}(z) = \hat{H}(\hat{z}) \Big|_{\hat{z}^{-1} = \frac{z^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}}} = H_a(s) \Big|_{s = \frac{1 - \hat{z}^{-1}}{1 + \hat{z}^{-1}}} \quad (99)$$

For bandstop designs, we may use:

$$s = \frac{1 - \hat{z}^{-1}}{1 + \hat{z}^{-1}} = \frac{1 - z^{-2}}{1 - 2c_0 z^{-1} + z^{-2}}, \quad \hat{z}^{-1} = -\frac{z^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}} \quad (100)$$

with transfer functions related by:

$$H_{BS}(z) = \hat{H}(\hat{z}) \Big|_{\hat{z}^{-1} = -\frac{z^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}}} = H_a(s) \Big|_{s = \frac{1 - \hat{z}^{-1}}{1 + \hat{z}^{-1}}} \quad (101)$$

The bandpass and bandstop cases can be combined into one by defining:

$$s = \frac{1 - \hat{z}^{-1}}{1 + \hat{z}^{-1}}, \quad \hat{z}^{-1} = q \frac{z^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}}, \quad q = \begin{cases} +1, & \text{BP case} \\ -1, & \text{BS case} \end{cases} \quad (102)$$

The transfer functions will be related by:

$$H(z) = \hat{H}(\hat{z}) \Big|_{\hat{z}^{-1} = \frac{qz^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}}} = H_a(s) \Big|_{s = \frac{1 - \hat{z}^{-1}}{1 + \hat{z}^{-1}}} \quad (103)$$

Setting $s = j\Omega$, $\hat{z} = e^{j\hat{\omega}}$, and $z = e^{j\omega}$, we find the corresponding frequency relationships:

$$\Omega = \tan\left(\frac{\hat{\omega}}{2}\right) = \begin{cases} \frac{c_0 - \cos \omega}{\sin \omega}, & \text{if } q = 1 \\ -\frac{\sin \omega}{c_0 - \cos \omega}, & \text{if } q = -1 \end{cases} \quad (104)$$

Because \hat{z} depends quadratically on z , each lowpass analog s -plane pole p_{ai} will first get mapped into a lowpass digital \hat{z} -plane pole \hat{p}_i , which will then be mapped into two z -plane poles, say, p_i^+ , p_i^- . The pole \hat{p}_i is constructed from the analog pole p_{ai} via Eq. (102):

$$p_{ai} = \frac{1 - \hat{p}_i^{-1}}{1 + \hat{p}_i^{-1}} \Rightarrow \boxed{\hat{p}_i = \frac{1 + p_{ai}}{1 - p_{ai}}} \quad (105)$$

then, the pole pairs p_i^\pm are obtained as the two solutions for p_i of the equation:

$$\hat{p}_i = q \frac{p_i(c_0 - p_i)}{1 - c_0 p_i} \Rightarrow p_i^\pm = \frac{1}{2} \left[c_0(1 + q\hat{p}_i) \pm \sqrt{c_0^2(1 + q\hat{p}_i)^2 - 4q\hat{p}_i} \right] \quad (106)$$

Because \hat{z} remains invariant under the substitution $z \rightarrow z' = (c_0 - z)/(1 - c_0 z)$, that is,

$$\hat{z} = q \frac{z(c_0 - z)}{1 - c_0 z} = q \frac{z'(c_0 - z')}{1 - c_0 z'}, \quad \text{with } z' = \frac{c_0 - z}{1 - c_0 z}, \quad z = \frac{c_0 - z'}{1 - c_0 z'}, \quad (107)$$

it follows that the pair p_i^+ , p_i^- can be constructed, alternatively, by:

$$\boxed{p_i^+ = \frac{1}{2} \left[c_0(1 + q\hat{p}_i) + \sqrt{c_0^2(1 + q\hat{p}_i)^2 - 4q\hat{p}_i} \right]}, \quad p_i^- = \frac{c_0 - p_i^+}{1 - c_0 p_i^+} \quad (108)$$

Thus, Eqs. (105) and (108) allow the mapping of the analog poles to the final digital filter poles. The analog zeros z_{ai} are mapped in a similar way to the zeros \hat{z}_i and then to z_i^+, z_i^- . Using Eqs. (102), (105), and (108), the following identity can be verified easily:

$$\frac{1 - s/z_{ai}}{1 - s/p_{ai}} = G_i \frac{1 - \hat{z}_i \hat{z}^{-1}}{1 - \hat{p}_i \hat{z}^{-1}} = G_i \frac{(1 - z_i^+ z^{-1})(1 - z_i^- z^{-1})}{(1 - p_i^+ z^{-1})(1 - p_i^- z^{-1})}, \quad G_i = \frac{1 - \hat{p}_i}{1 - \hat{z}_i} \quad (109)$$

It follows that the transfer function can be expressed in the equivalent factored forms:

$$\begin{aligned} H(z) &= H_0 \left[\frac{1}{1 - s/p_{a0}} \right]^r \prod_{i=1}^L \left[\frac{(1 - s/z_{ai})(1 - s/z_{ai}^*)}{(1 - s/p_{ai})(1 - s/p_{ai}^*)} \right] \Bigg|_{s=\frac{1-\hat{z}^{-1}}{1+\hat{z}^{-1}}} \\ &= H_0 \left[G_0 \frac{1 + \hat{z}^{-1}}{1 - \hat{p}_0 \hat{z}^{-1}} \right]^r \prod_{i=1}^L \left[|G_i|^2 \frac{(1 - \hat{z}_i \hat{z}^{-1})(1 - \hat{z}_i^* \hat{z}^{-1})}{(1 - \hat{p}_i \hat{z}^{-1})(1 - \hat{p}_i^* \hat{z}^{-1})} \right] \Bigg|_{\hat{z}^{-1}=\frac{qz^{-1}(c_0-z^{-1})}{1-c_0z^{-1}}} \\ &= H_0 \left[G_0 \frac{(1 - z_0^+ z^{-1})(1 - z_0^- z^{-1})}{(1 - p_0^+ z^{-1})(1 - p_0^- z^{-1})} \right]^r \\ &\quad \prod_{i=1}^L \left[|G_i| \frac{(1 - z_i^+ z^{-1})(1 - z_i^{+*} z^{-1})}{(1 - p_i^+ z^{-1})(1 - p_i^{+*} z^{-1})} \right] \cdot \prod_{i=1}^L \left[|G_i| \frac{(1 - z_i^- z^{-1})(1 - z_i^{-*} z^{-1})}{(1 - p_i^- z^{-1})(1 - p_i^{-*} z^{-1})} \right] \end{aligned} \quad (110)$$

The poles p_0^\pm arise from the mapping of the analog pole p_{a0} :

$$\hat{p}_0 = \frac{1 + p_{a0}}{1 - p_{a0}} \quad \Rightarrow \quad p_0^\pm = \frac{1}{2} \left[c_0(1 + q\hat{p}_0) \pm \sqrt{c_0^2(1 + q\hat{p}_0)^2 - 4q\hat{p}_0} \right] \quad (111)$$

Because p_{a0} is real, p_0^\pm are either both real-valued or conjugate pairs. The zeros z_0^\pm arise from mapping the analog zero $z_{a0} = \infty$ to $\hat{z}_0 = -1$, and are given in the two cases $q = \pm 1$ by:

$$\hat{z}_0 = -1 \quad \Rightarrow \quad z_0^\pm = \frac{1}{2} \left[c_0(1 + q\hat{z}_0) \pm \sqrt{c_0^2(1 + q\hat{z}_0)^2 - 4q\hat{z}_0} \right] = \begin{cases} \pm 1, & \text{if } q = 1 \\ e^{\pm j\omega_0}, & \text{if } q = -1 \end{cases} \quad (112)$$

Thus, the corresponding numerator second-order factor becomes:

$$(1 - z_0^+ z^{-1})(1 - z_0^- z^{-1}) = \begin{cases} 1 - z^{-2}, & \text{if } q = 1 \\ 1 - 2c_0 z^{-1} + z^{-2}, & \text{if } q = -1 \end{cases} \quad (113)$$

Such are also the other numerator factors in the Butterworth and Chebyshev-1 cases.

In summary, Eq. (110) expresses $H(z)$ as a product of second-order sections, which is usually the preferred form. By combining the last two groups of L second-order factors, we may express $H(z)$ as a cascade of L fourth-order sections:

$$\begin{aligned} H(z) &= H_0 \left[G_0 \frac{(1 - z_0^+ z^{-1})(1 - z_0^- z^{-1})}{(1 - p_0^+ z^{-1})(1 - p_0^- z^{-1})} \right]^r \\ &\quad \prod_{i=1}^L \left[|G_i|^2 \frac{(1 - z_i^+ z^{-1})(1 - z_i^{+*} z^{-1})(1 - z_i^- z^{-1})(1 - z_i^{-*} z^{-1})}{(1 - p_i^+ z^{-1})(1 - p_i^{+*} z^{-1})(1 - p_i^- z^{-1})(1 - p_i^{-*} z^{-1})} \right] \end{aligned} \quad (114)$$

Eq. (110) includes also the LP and HP cases, which have $q = 1$ and $c_0 = \pm 1$ resulting in $\hat{z} = \pm z$. The second group of L sections reduces to unity because $p_i^- = (c_0 - p_i^+) / (1 - c_0 p_i^+) = \pm 1$ when $c_0 = \pm 1$, and similarly $z_i^- = \pm 1$, as well as, $p_0^- = z_0^- = \pm 1$.

We note finally that the mappings defined in Eq. (102) preserve the causality and stability of the filters, in the sense that they map left-hand s -plane poles to poles inside the \hat{z} -plane unit circle, to poles inside the z -plane unit circle. These follows from the relationships:

$$\operatorname{Re}(s) = \frac{1 - |\hat{z}|^2}{|\hat{z} + 1|^2}, \quad 1 - |\hat{z}|^2 = (1 - |z|^2) \frac{|c_0 - z|^2 + s_0^2}{|1 - c_0 z|^2}, \quad s_0 = \sin \omega_0 \quad (115)$$

which show that $\operatorname{Re}(s) \leq 0$ is equivalent to $|\hat{z}| \leq 1$ and to $|z| \leq 1$.

11. Transformation of the Frequency Specifications

The design specifications for lowpass, highpass, bandpass, and bandstop digital filters, and the specifications of the equivalent lowpass analog filter are shown in Fig. 13, where f is in Hz and f_s is the sampling rate. The frequency transformation equations (86) give rise to the following procedures for mapping the given specifications into the analog ones.

For *lowpass* filters having passband and stopband frequencies $f_{\text{pass}} < f_{\text{stop}} < f_s/2$, we calculate $\omega_{\text{pass}} = 2\pi f_{\text{pass}}/f_s$, $\omega_{\text{stop}} = 2\pi f_{\text{stop}}/f_s$, and

$$\Omega_p = \tan\left(\frac{\omega_{\text{pass}}}{2}\right), \quad \Omega_s = \tan\left(\frac{\omega_{\text{stop}}}{2}\right) \quad (116)$$

For *highpass* designs with passband and stopband frequencies $f_{\text{stop}} < f_{\text{pass}} < f_s/2$, we calculate $\omega_{\text{pass}} = 2\pi f_{\text{pass}}/f_s$, $\omega_{\text{stop}} = 2\pi f_{\text{stop}}/f_s$, and:

$$\Omega_p = \cot\left(\frac{\omega_{\text{pass}}}{2}\right), \quad \Omega_s = \cot\left(\frac{\omega_{\text{stop}}}{2}\right) \quad (117)$$

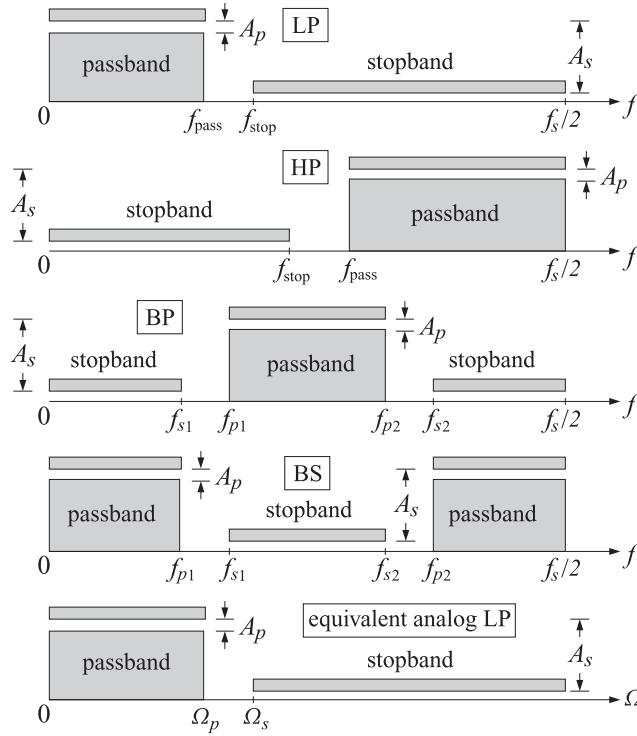


Fig. 13 Specifications of digital and equivalent analog filters (f_s is the sampling frequency).

For *bandpass* designs with a passband interval $[f_{p1}, f_{p2}]$ given as a subset of a stopband interval $[f_{s1}, f_{s2}]$, we calculate the analog filter's parameters as follows, where $\omega_{p1} = 2\pi f_{p1}/f_s$, $\omega_{p2} = 2\pi f_{p2}/f_s$, $\omega_{s1} = 2\pi f_{s1}/f_s$, $\omega_{s2} = 2\pi f_{s2}/f_s$:

$$\begin{aligned} \cos \omega_0 &= \frac{\sin(\omega_{p1} + \omega_{p2})}{\sin \omega_{p1} + \sin \omega_{p2}}, \quad \Omega_{s1} = \frac{\cos \omega_0 - \cos \omega_{s1}}{\sin \omega_{s1}}, \quad \Omega_{s2} = \frac{\cos \omega_0 - \cos \omega_{s2}}{\sin \omega_{s2}} \\ \Omega_p &= \tan\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right), \quad \Omega_s = \min(|\Omega_{s1}|, |\Omega_{s2}|) \end{aligned} \quad (118)$$

These choices match the passband specifications. An alternative choice that matches the stopband specifications is as follows:

$$\begin{aligned} \cos \omega_0 &= \frac{\sin(\omega_{s1} + \omega_{s2})}{\sin \omega_{s1} + \sin \omega_{s2}}, \quad \Omega_{p1} = \frac{\cos \omega_0 - \cos \omega_{p1}}{\sin \omega_{p1}}, \quad \Omega_{p2} = \frac{\cos \omega_0 - \cos \omega_{p2}}{\sin \omega_{p2}} \\ \Omega_p &= \max(|\Omega_{p1}|, |\Omega_{p2}|), \quad \Omega_s = \tan\left(\frac{\omega_{s2} - \omega_{s1}}{2}\right) \end{aligned} \quad (119)$$

For *bandstop* designs with a stopband interval $[f_{s1}, f_{s2}]$ given as a subset of a passband interval $[f_{p1}, f_{p2}]$, we calculate the analog filter's parameters as follows, where $\omega_{p1} = 2\pi f_{p1}/f_s$, $\omega_{p2} = 2\pi f_{p2}/f_s$, $\omega_{s1} = 2\pi f_{s1}/f_s$, $\omega_{s2} = 2\pi f_{s2}/f_s$, choosing to match the passband as in [12]:

$$\begin{aligned} \cos \omega_0 &= \frac{\sin(\omega_{p1} + \omega_{p2})}{\sin \omega_{p1} + \sin \omega_{p2}}, \quad \Omega_{s1} = \frac{\sin \omega_{s1}}{\cos \omega_0 - \cos \omega_{s1}}, \quad \Omega_{s2} = \frac{\sin \omega_{s2}}{\cos \omega_0 - \cos \omega_{s2}} \\ \Omega_p &= \cot\left(\frac{\omega_{p2} - \omega_{p1}}{2}\right), \quad \Omega_s = \min(|\Omega_{s1}|, |\Omega_{s2}|) \end{aligned} \quad (120)$$

Alternatively, we may match the stopband specifications:

$$\begin{aligned} \cos \omega_0 &= \frac{\sin(\omega_{s1} + \omega_{s2})}{\sin \omega_{s1} + \sin \omega_{s2}}, \quad \Omega_{p1} = \frac{\sin \omega_{p1}}{\cos \omega_0 - \cos \omega_{p1}}, \quad \Omega_{p2} = \frac{\sin \omega_{p2}}{\cos \omega_0 - \cos \omega_{p2}} \\ \Omega_p &= \max(|\Omega_{p1}|, |\Omega_{p2}|), \quad \Omega_s = \cot\left(\frac{\omega_{s2} - \omega_{s1}}{2}\right) \end{aligned} \quad (121)$$

12. MATLAB Implementation and Examples

Once the parameters Ω_p, Ω_s have been determined, the equivalent lowpass analog prototype filter can be designed, based on the specifications $\{\Omega_p, \Omega_s, A_p, A_s\}$ using for example the function `lpa`, and the analog zeros and poles may be mapped into those of the digital filter.

In order to emulate MATLAB's built-in filter design functions, the above design steps have been divided into two stages. In the first stage, the function `dford.m` uses the given specifications to determine the analog filter order N and the frequency and attenuation level that must be matched exactly:

```
% dford.m - digital filter order determination
%
% Usage: [N,Ad,wd] = dford(wp,ws,Ap,As,type,match);
%
% wp,ws = passband and stopband frequencies in rads/sample, (1d for LP/HP, 2d for BP/BS)
% Ap,As = passband and stopband attenuations in dB
% type = 0,1,2,3 for Butterworth, Chebyshev-1, Chebyshev-2, Elliptic
% match = 'p', 's', if omitted it defaults to 'p' for type=0,1,3, 's' for type=2
%
% N = filter order of LP analog prototype
```

```

% Ad = attenuation in dB to be matched exactly at design frequency wd
% wd = design frequency (1d or 2d) that must be matched exactly
%
% the outputs N,Ad,wd may be passed directly to DFDES to design the filter
%
% it determines the type LP/HP/BP/BS from wp,ws using the following conventions:
%
%   match='p' matches passband specs exactly
%   match='s' matches stopband specs exactly
%
%   LP: wp,ws are scalars with wp < ws
%   HP: wp,ws are scalars with wp > ws
%   BP: wp = [wp1,wp2], ws = [ws1,ws2], with ws1 < wp1 < wp2 < ws2
%   BS: wp = [wp1,wp2], ws = [ws1,ws2], with wp1 < ws1 < ws2 < wp2

```

This function serves as substitute for the built-in functions `buttord`, `cheb1ord`, `cheb2ord`, and `ellipord`. In the second stage, the function `dfdes.m` uses the calculated filter order and the matched frequency band and attenuation to calculate the analog filter poles and zeros, remap them to the digital ones, and construct the second-order or fourth-order section coefficients using Eq. (110) or (114):

```

% dfdes.m - digital filter design with the bilinear transformation
%
% Usage: [B,A,w0] = dfdes(N,Ad,wd,type,shape,coeffs)
%
%   [B,A,w0] = dfdes(N,Ad,wd,type,shape)           % equivalent to coeffs='4os'
%   [B,A,w0] = dfdes(N,Ad,wd,type,shape,'4os')    % fourth-order sections
%   [B,A,w0] = dfdes(N,Ad,wd,type,shape,'sos')    % second-order sections
%   [B,A,w0] = dfdes(N,Ad,wd,type,shape,'hsos')   % hat-second-order sections
%   [B,A,w0] = dfdes(N,Ad,wd,type,shape,'dir')    % direct-form coefficients
%
% N      = filter order of LP analog prototype, digital filter order is 2*N
% Ad     = attenuation in dB at the design frequency wd, (Ad is 1d, 2d, or 3d row)
% wd     = design frequency in radians/sample, (wd is 1d for LP/HP, or, 2d for BP/BS)
% type   = 0,1,2,3 for Butterworth, Chebyshev-1, Chebyshev-2, Elliptic
% shape  = 'LP', 'HP', 'BP', 'BS', for lowpass, highpass, bandpass, bandstop designs
% coeffs = '4os', 'sos', 'hsos', 'dir'
%
% B,A = (L+1)x5 for '4os', (L+1)x3 for 'hsos', (2L+1)x3 for 'sos', (2N+1)x1 for 'dir'
% w0 = center freq (rads/sample) of the wd band for BP/BS, w0=0 or pi for LP or HP
%
% note that N = 2L+r, r=0,1, L = floor(N/2) = number of SOSs of LP analog prototype
%
% Ad,wd are entered using the following conventions:
%
% [B,A,w0] = dfdes(N,Ap,wp,0,shape,coeffs)        match Ap at wp
% [B,A,w0] = dfdes(N,Ap,wp,1,shape,coeffs)        match Ap at wp
% [B,A,w0] = dfdes(N,As,ws,2,shape,coeffs)        match As at ws
% [B,A,w0] = dfdes(N,[Ap,As],wp,3,shape,coeffs)   match Ap at wp, equiripple at Ap,As
%
% [B,A,w0] = dfdes(N,[Ap,Ab],wb,1,shape,coeffs)   match Ab at wb, equiripple at Ap
% [B,A,w0] = dfdes(N,[Ab,As],wb,2,shape,coeffs)   match Ab at wb, equiripple at As
% [B,A,w0] = dfdes(N,[Ap,Ab,As],wb,3,shape,coeffs) match Ab at wb, equiripple at Ap,As
% [B,A,w0] = dfdes(N,[Ap,As,As],ws,3,shape,coeffs) match As at ws, equiripple at Ap,As

```

This function serves as substitute for the built-in functions `butter`, `cheby1`, `cheby2`, and `ellip`. Fig. 14 depicts Butterworth, Chebyshev-1, Chebyshev-2, and elliptic digital filter designs of lowpass, highpass, bandpass, and bandstop filters designed with the same passband and stopband attenuations as in Eq. (10) and with the following frequency specifications (in kHz):

LP case: $f_s = 20$, $f_{\text{pass}} = 4.0$, $f_{\text{stop}} = 4.5$

HP case: $f_s = 20$, $f_{\text{pass}} = 4.5$, $f_{\text{stop}} = 4.0$

BP case: $f_s = 20$, $[f_{p1}, f_{p2}] = [3.0, 6.0]$, $[f_{s1}, f_{s2}] = [2.5, 6.5]$

BS case: $f_s = 20$, $[f_{p1}, f_{p2}] = [2.5, 6.5]$, $[f_{s1}, f_{s2}] = [3.0, 6.0]$

For example, the elliptic designs may be generated by the code fragment:

```
Gp = 0.95; Ap = -20*log10(Gp);
Gs = 0.05; As = -20*log10(Gs);

fsamp=20;
f = linspace(0,10,1001);
w = 2*pi*f/fsamp;

% ----- LP -----
fp = 4; fs = 4.5; wp = 2*pi*fp/fsamp; ws = 2*pi*fs/fsamp;
[N,Ad,wd] = dford(wp,ws,Ap,As,3,'s');
[B,A] = dfdes(N,Ad,wd,3,'LP','sos');
H = fresp(B,A,w); figure; plot(f,abs(H),'r');
% ----- HP -----
fp = 4.5; fs = 4; wp = 2*pi*fp/fsamp; ws = 2*pi*fs/fsamp;
[N,Ad,wd] = dford(wp,ws,Ap,As,3,'s');
[B,A] = dfdes(N,Ad,wd,3,'HP','sos');
H = fresp(B,A,w); figure; plot(f,abs(H),'r');
% ----- BP -----
fp1=3; fp2=6; fs1=2.5; fs2=6.5; wp = 2*pi*[fp1,fp2]/fsamp; ws = 2*pi*[fs1,fs2]/fsamp;
[N,Ad,wd] = dford(wp,ws,Ap,As,3,'s');
[B,A] = dfdes(N,Ad,wd,3,'BP','sos');
H = fresp(B,A,w); figure; plot(f,abs(H),'r');
% ----- BS -----
fp1=2.5; fp2=6.5; fs1=3; fs2=6; wp = 2*pi*[fp1,fp2]/fsamp; ws = 2*pi*[fs1,fs2]/fsamp;
[N,Ad,wd] = dford(wp,ws,Ap,As,3,'s');
[B,A] = dfdes(N,Ad,wd,3,'BS','sos');
H = fresp(B,A,w); figure; plot(f,abs(H),'r');
```

where we have chosen to match the stopbands exactly and output the filter coefficients as second-order sections. The frequency response evaluation of the cascaded sections was implemented with the help of the MATLAB function `fresp.m`, borrowed from [30]. The designed transfer functions and coefficient matrices are as follows:

LP case:

$$B = \begin{bmatrix} 0.3204 & 0.3204 & 0 \\ 0.8591 & -0.2363 & 0.8591 \\ 0.4534 & 0.1206 & 0.4534 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -0.3593 & 0 \\ 1 & -0.4436 & 0.9255 \\ 1 & -0.5547 & 0.5821 \end{bmatrix}, \quad N = 5$$

$$H_{\text{LP}}(z) = \frac{0.3204(1+z^{-1})}{1-0.3593z^{-1}} \cdot \frac{0.8591-0.2363z^{-1}+0.8591z^{-2}}{1-0.4436z^{-1}+0.9255z^{-2}} \cdot \frac{0.4534+0.1206z^{-1}+0.4534z^{-2}}{1-0.5547z^{-1}+0.5821z^{-2}}$$

HP case:

$$B = \begin{bmatrix} 0.4317 & -0.4317 & 0 \\ 0.8986 & -0.5866 & 0.8986 \\ 0.5615 & -0.6118 & 0.5615 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0.1366 & 0 \\ 1 & -0.4582 & 0.9257 \\ 1 & -0.1727 & 0.5621 \end{bmatrix}, \quad N = 5$$

BP case:

$$B = \begin{bmatrix} 0.9500 & 0 & 0 \\ 0.8161 & -1.1771 & 0.8161 \\ 0.4017 & -0.7171 & 0.4017 \\ 0.8161 & 0.7778 & 0.8161 \\ 0.4017 & 0.6260 & 0.4017 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1.2501 & 0.9253 \\ 1 & -0.8124 & 0.6129 \\ 1 & 0.6965 & 0.9093 \\ 1 & 0.2530 & 0.5697 \end{bmatrix}, \quad N = 4$$

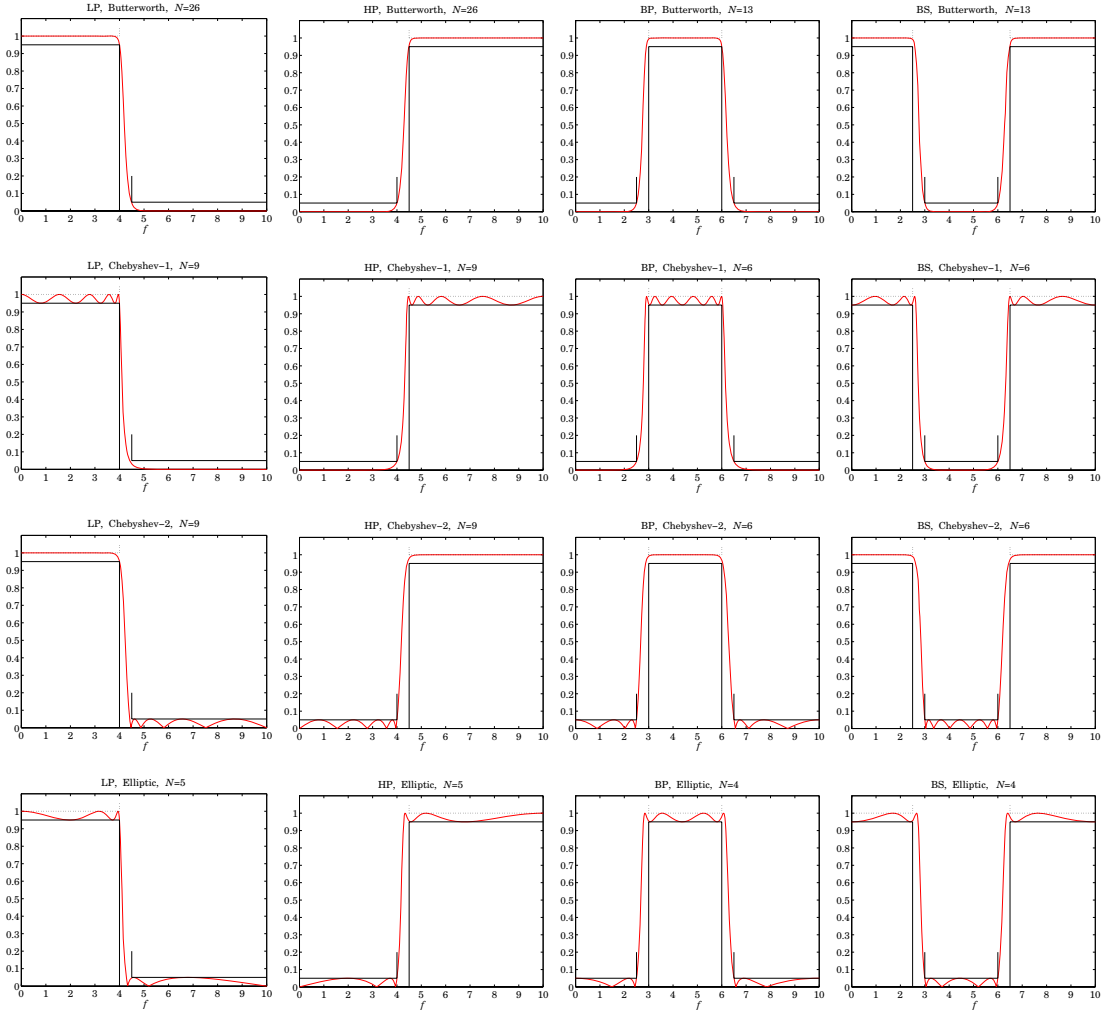


Fig. 14 Digital LP, HP, BP, and BS filters.

BS case:

$$B = \begin{bmatrix} 0.9500 & 0 & 0 \\ 0.9081 & -1.0417 & 0.9081 \\ 0.6221 & -0.4912 & 0.6221 \\ 0.9081 & 0.5257 & 0.9081 \\ 0.6221 & 0.0778 & 0.6221 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1.2399 & 0.9239 \\ 1 & -1.0384 & 0.5163 \\ 1 & 0.7432 & 0.9090 \\ 1 & 0.6453 & 0.4377 \end{bmatrix}, \quad N = 4$$

The functions `lpa`, `dford`, and `dfdes` are available from [23].

13. Frequency-Shifted Realizations

Besides the conventional realizations based on second- or fourth-order sections, it is possible to realize the digital filter as a cascade of second-order sections in the \hat{z}^{-1} variable. These realizations are also generated by the function `dfdes` and have transfer function:

$$\begin{aligned}
H(z) &= H_0 \left[G_0 \frac{1 + \hat{z}^{-1}}{1 - \hat{p}_0 \hat{z}^{-1}} \right]^r \prod_{i=1}^L \left[|G_i|^2 \frac{(1 - \hat{z}_i \hat{z}^{-1})(1 - \hat{z}_i^* \hat{z}^{-1})}{(1 - \hat{p}_i \hat{z}^{-1})(1 - \hat{p}_i^* \hat{z}^{-1})} \right] \\
&\equiv H_0 \left[\frac{\hat{b}_{00} + \hat{b}_{01} \hat{z}^{-1}}{1 + \hat{a}_{01} \hat{z}^{-1}} \right]^r \prod_{i=1}^L \left[\frac{\hat{b}_{i0} + \hat{b}_{i1} \hat{z}^{-1} + \hat{b}_{i2} \hat{z}^{-2}}{1 + \hat{a}_{i1} \hat{z}^{-1} + \hat{a}_{i2} \hat{z}^{-2}} \right]
\end{aligned} \tag{122}$$

where \hat{z}^{-1} must be replaced by

$$\hat{z}^{-1} = q \frac{z^{-1}(c_0 - z^{-1})}{1 - c_0 z^{-1}} \tag{123}$$

This transformation may be represented by the block diagram shown in Fig. 15, where $c_0 = \cos \omega_0$ and $s_0 = \sin \omega_0$. This diagram is the so-called normalized lattice realization of Eq. (123). Other realizations of (123) are, of course, possible [24-27].

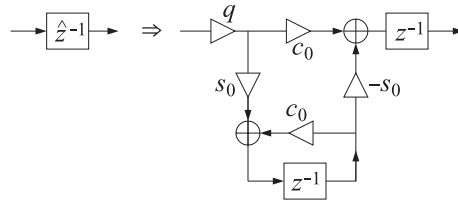


Fig. 15 LP to BP/BS frequency transformation, $q = 1$ for BP, $q = -1$ for BS.

If one has a realization of Eq. (122), then each unit-delay \hat{z}^{-1} may be replaced by the block diagram of Fig. 15, resulting in a realization of the final digital filter. For example, Fig. 16 shows the transposed realization of one of the second-order sections and its shifted version.

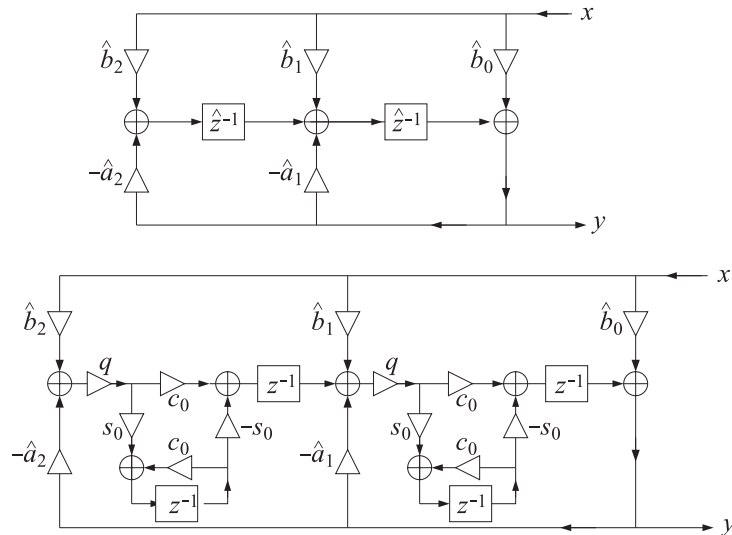


Fig. 16 Frequency-shifted transposed realization.

The advantage of such realizations is that they decouple the dependence on the center frequency ω_0 . The hat-coefficients depend only on the desired bandwidth and attenuations, and not on ω_0 . Thus, one can design a lowpass filter and shift it to any center frequency ω_0 , transforming it into a bandpass (or bandstop) filter.

The MATLAB function `dfdes` calculates (with argument `coeffs` set to ‘`hsos`’) the hat-coefficients from the order N of the analog prototype and a prescribed frequency band $[\omega_{d1}, \omega_{d2}]$ that is matched exactly at a desired attenuation level A_d (chosen to be either the passband or the stopband.)

The bandwidth $\Delta\omega_d = \omega_{d2} - \omega_{d1}$ is used internally by `dfdes` to calculate the analog design parameter $\Omega_d = \tan(\Delta\omega_d/2)$. Then, Ω_d is mapped to the passband parameter Ω_p , which is used to design of the analog prototype filter.

To design a bandpass filter with a given bandwidth of $\Delta\omega_d$ and center frequency ω_0 , one may start by designing a lowpass digital filter with cutoff frequency $\hat{\omega}_d = \Delta\omega_d$ matched at level A_d , and then shift it to ω_0 . Since ω_0 and $\Delta\omega_d$ are given, the bandedge frequencies $[\omega_{d1}, \omega_{d2}]$ cannot be independently specified, but may be calculated by the following formulas [30]:

$$\cos \omega_{d1} = \frac{c_0 + \Omega_d \sqrt{\Omega_d^2 + s_0^2}}{1 + \Omega_d^2}, \quad \cos \omega_{d2} = \frac{c_0 - \Omega_d \sqrt{\Omega_d^2 + s_0^2}}{1 + \Omega_d^2} \quad (124)$$

where $\Omega_d = \tan(\Delta\omega_d/2)$, and $c_0 = \cos \omega_0$, $s_0 = \sin \omega_0$. These are derived by demanding that the interval $[\omega_{d1}, \omega_{d2}]$ be mapped onto the analog lowpass interval $[-\Omega_d, \Omega_d]$ through the bilinear transformation of Eq. (86), that is,

$$-\Omega_d = \frac{c_0 - \cos \omega_{d1}}{\sin \omega_{d1}}, \quad \Omega_d = \frac{c_0 - \cos \omega_{d2}}{\sin \omega_{d2}}$$

If the bandedge frequencies $[\omega_{d1}, \omega_{d2}]$ are specified, then, $\Delta\omega_d = \omega_{d2} - \omega_{d1}$, and the center frequency ω_0 is calculated by

$$\cos \omega_0 = \frac{\sin(\omega_{d1} + \omega_{d2})}{\sin \omega_{d1} + \sin \omega_{d2}} \quad (125)$$

Eqs. (124) and (125) are valid also for shifted bandstop filters, but now the lowpass filter’s design frequency must be measured from Nyquist, that is, $\hat{\omega}_d = \pi - \Delta\omega_d$, because the LP stopband will be transformed to the BS stopband.

Next, we look at some design examples. Fig. 17 shows a Chebyshev-2 bandpass digital filter with sampling rate $f_s = 20$ kHz and bandwidth of $\Delta f_p = 3$ kHz measured at the passband level of $A_p = -20 \log_{10}(0.95)$ dB, and shifted to the passband center frequency of $f_0 = 4$ kHz.

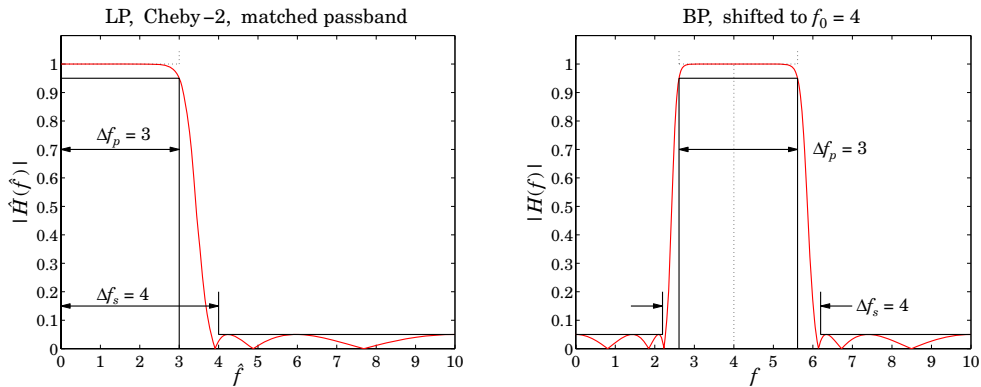


Fig. 17 LP Chebyshev-2 shifted to BP with passband centered at $f_0 = 4$.

The bandpass filter was obtained by shifting a lowpass digital filter that was designed with passband frequency $\hat{f}_{\text{pass}} = \Delta f_p = 3$ kHz at level A_p and stopband frequency $\hat{f}_{\text{stop}} = 4$ kHz at

the stopband level of $A_s = -10 \log_{10}(0.05)$ dB. The passband and stopband edge frequencies of the shifted filter were calculated from Eq. (124) and are shown as brick-walls on Fig. 17:

$$[f_{p1}, f_{p2}] = [2.6121, 5.6121] \text{ kHz}, \quad [f_{s1}, f_{s2}] = [2.1957, 6.1957] \text{ kHz}$$

The passband frequencies satisfy $\Delta f_p = f_{p2} - f_{p1} = 3$ kHz. The stopband edge frequencies were calculated by using the center frequency $f_0 = 4$ and the width of the lowpass filter's stopband, that is, $\Delta f_s = \hat{f}_{\text{stop}} = 4$ kHz. The resulting filter order was $N = 6$. The hat-second-order section coefficients were:

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6796 & -0.4558 & 0.6796 \\ 0.4768 & -0.0352 & 0.4768 \\ 0.2919 & 0.4366 & 0.2919 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -0.8721 & 0.7755 \\ 1 & -0.4583 & 0.3767 \\ 1 & -0.0335 & 0.0539 \end{bmatrix}$$

The c_0 parameter was equal to 0.3090. The MATLAB code used to generate Fig. 17 was:

```
Gp = 0.95; Gs = 0.05; Ap = -20*log10(Gp); As = -20*log10(Gs);

fsamp = 20; fpass = 3; fstop = 4; f0 = 4;
wp = 2*pi*fpass/fsamp; ws = 2*pi*fstop/fsamp; w0 = 2*pi*f0/fsamp;

[N,Ad,wd] = dford(wp,ws,Ap,As,2,'p'); % match passband
[Bh,Ah] = dfdes(N,Ad,wd,2,'LP','hsos'); % hat-sos sections

f = linspace(0,10,1001); w = 2*pi*f/fsamp;

HLP = fresp(Bh,Ah,w); % frequency response of LP digital filter
HBP = fresp(Bh,Ah,w,w0); % frequency response of shifted LP filter

figure; plot(f,abs(HLP),'r');
figure; plot(f,abs(HBP),'r');

c0 = cos(w0); s0=sin(w0); % calculate bandedge frequencies
Wp = tan(wp/2);
wp1 = acos((c0 + Wp*sqrt(Wp^2+s0^2))/(1+Wp^2)); fp1 = wp1*fsamp/2/pi;
wp2 = acos((c0 - Wp*sqrt(Wp^2+s0^2))/(1+Wp^2)); fp2 = wp2*fsamp/2/pi;

Ws = tan(ws/2);
ws1 = acos((c0 + Ws*sqrt(Ws^2+s0^2))/(1+Ws^2)); fs1 = ws1*fsamp/2/pi;
ws2 = acos((c0 - Ws*sqrt(Ws^2+s0^2))/(1+Ws^2)); fs2 = ws2*fsamp/2/pi;
```

Fig. 18 shows another example in which the same lowpass digital filter was transformed to a bandpass filter with a passband given by $[f_{p1}, f_{p2}] = [2, 5]$ kHz, which has the same bandwidth $\Delta f_p = 5 - 2 = 3$ kHz as the previous example. In this case, because the bandedge frequencies were given, the center frequency ω_0 was calculated by Eq. (125), and the stopband edge frequencies by Eq. (124):

$$f_0 = 3.2982 \text{ kHz}, \quad [f_{s1}, f_{s2}] = [1.6476, 5.6476] \text{ kHz}$$

By construction, the stopband width was as before, that is, $\Delta f_s = f_{s2} - f_{s1} = 4$ kHz. The hat-coefficients were the same as in the previous example, but the new value of c_0 was 0.5095.

In the example shown in Fig. 19, the digital lowpass filter was designed with the same specifications as the previous two examples, but the stopband was matched exactly. The resulting hat-coefficients were:

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0.6843 & -0.3796 & 0.6843 \\ 0.4830 & 0.0262 & 0.4830 \\ 0.3065 & 0.4749 & 0.3065 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -0.7805 & 0.7695 \\ 1 & -0.3760 & 0.3683 \\ 1 & 0.0340 & 0.0539 \end{bmatrix}$$

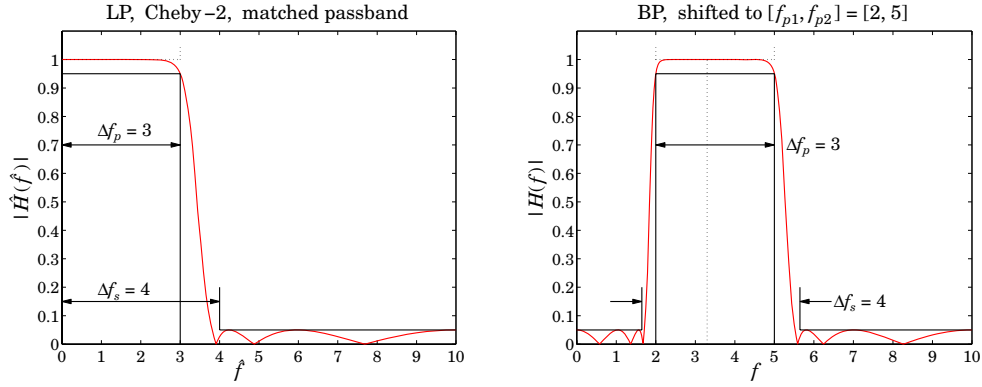


Fig. 18 LP Chebyshev-2 shifted to BP with passband at $[f_{p1}, f_{p2}] = [2, 5]$.

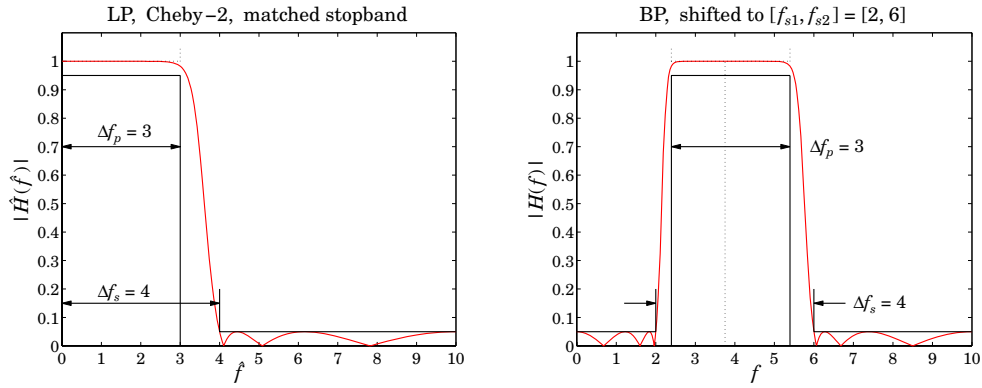


Fig. 19 LP Chebyshev-2 shifted to BP with stopband at $[f_{s1}, f_{s2}] = [2, 6]$.

The 4-kHz stopband frequency of the lowpass filter was shifted to the stopband interval $[f_{s1}, f_{s2}] = [2, 6]$ kHz, so that $\Delta f_s = 6 - 2 = 4$ kHz. The center frequency f_0 was calculated on the basis of the stopband frequencies using Eq. (125), and then, the passband frequencies were determined using (124) and satisfying $\Delta f_p = f_{p2} - f_{p1} = 3$ kHz:

$$f_0 = 3.7525 \text{ kHz}, \quad c_0 = 0.3820, \quad [f_{p1}, f_{p2}] = [2.3946, 5.3946] \text{ kHz}$$

Fig. 20 shows a bandstop design with passband and stopband bandwidths of $\Delta f_p = 4$ and $\Delta f_s = 3$ kHz. The corresponding passband and stopband frequencies of the equivalent lowpass digital filter must be measured from Nyquist, that is,

$$\hat{f}_{\text{pass}} = \frac{f_s}{2} - \Delta f_p = 10 - 4 = 6 \text{ kHz}, \quad \hat{f}_{\text{stop}} = \frac{f_s}{2} - \Delta f_s = 10 - 3 = 7 \text{ kHz}$$

The lowpass digital filter was designed with the same attenuation A_p, A_s as the previous three examples with matching the passband exactly. The lowpass filter was shifted to the center frequency $f_0 = 4$ kHz, from which the passband and stopband edge frequencies of the bandstop filter were found to be:

$$[f_{p1}, f_{p2}] = [2.1957, 6.1957] \text{ kHz}, \quad [f_{s1}, f_{s2}] = [2.6121, 5.6121] \text{ kHz}$$

The c_0 parameter was 0.3090, and the hat-coefficients:

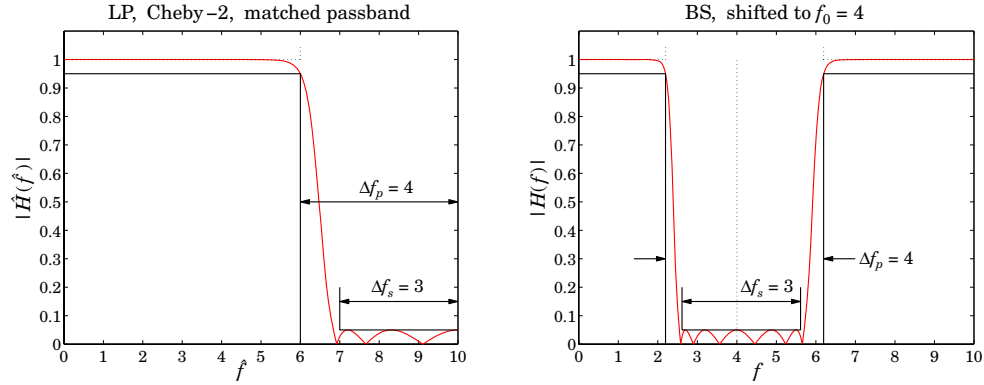


Fig. 20 LP Chebyshev-2 shifted to BS with passband centered at $f_0 = 4$.

$$\hat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0.8043 & 0.9141 & 0.8043 \\ 0.6460 & 0.9598 & 0.6460 \\ 0.5565 & 1.0698 & 0.5565 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.7548 & 0.7680 \\ 1 & 0.8176 & 0.4342 \\ 1 & 0.9320 & 0.2508 \end{bmatrix}$$

The MATLAB code used to generate Fig. 20 was:

```
Gp = 0.95; Gs = 0.05; Ap = -20*log10(Gp); As = -20*log10(Gs);

fsamp = 20; Dfp = 4; Dfs = 3; f0 = 4;
Dwp = 2*pi*Dfp/fsamp; Dws = 2*pi*Dfs/fsamp; w0 = 2*pi*f0/fsamp;

fpass = fsamp/2 - Dfp; fstop = fsamp/2 - Dfs;
wp = 2*pi*fpass/fsamp; ws = 2*pi*fstop/fsamp;

[N,Ad,wd] = dford(wp,ws,Ap,As,2,'p'); % match passband
[Bh,Ah] = dfdes(N,Ad,wd,2,'LP','hsos'); % output hat-sos sections

f = linspace(0,10,1001); w = 2*pi*f/fsamp;

HLP = fresp(Bh,Ah,w); % frequency response of LP digital filter
q = -1; % shift LP to BS
HBS = fresp(Bh,Ah,w,w0,q); % frequency response of shifted LP filter

figure; plot(f,abs(HLP),'r');
figure; plot(f,abs(HBS),'r');

c0 = cos(w0), s0 = sin(w0); % calculate bandedge frequencies
Wp = tan(Dwp/2);
wp1 = acos((c0 + Wp*sqrt(Wp^2+s0^2))/(1+Wp^2)); fp1 = wp1*fsamp/2/pi;
wp2 = acos((c0 - Wp*sqrt(Wp^2+s0^2))/(1+Wp^2)); fp2 = wp2*fsamp/2/pi;

Ws = tan(Dws/2);
ws1 = acos((c0 + Ws*sqrt(Ws^2+s0^2))/(1+Ws^2)); fs1 = ws1*fsamp/2/pi;
ws2 = acos((c0 - Ws*sqrt(Ws^2+s0^2))/(1+Ws^2)); fs2 = ws2*fsamp/2/pi;
```

The shifted frequency response was computed with the help of the function `fresp` which was modified from that of [30] to handle the bandstop case with $q = -1$ as an additional input.

Fig. 21 redesigns the lowpass filter of Fig. 20 to match the stopband exactly, and then shifts the stopband interval $[f_{s1}, f_{s2}] = [2, 5]$ kHz, from which the center frequency f_0 , shift parameter

c_0 , and passband edge frequencies can be calculated:

$$f_0 = 3.2982 \text{ kHz}, \quad c_0 = 0.5095, \quad [f_{p1}, f_{p2}] = [1.6476, 5.6476] \text{ kHz}$$

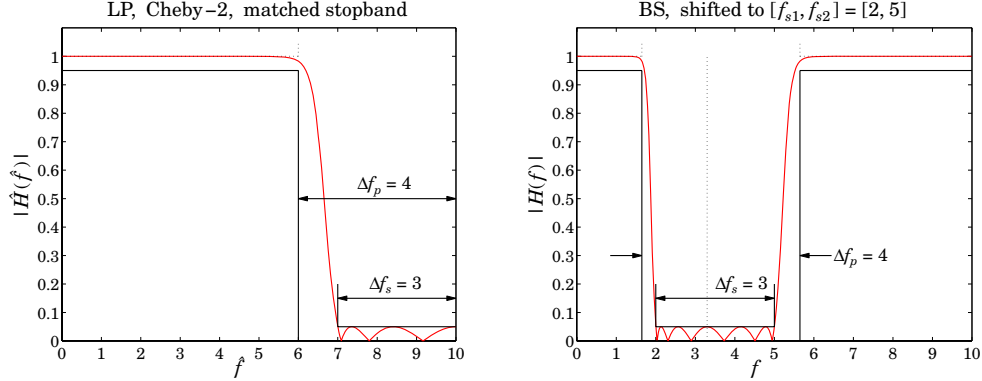


Fig. 21 LP Chebyshev-2 shifted to BS with stopband at $[f_{s1}, f_{s2}] = [2, 5]$.

14. High-Order Parametric Equalizer Design

A recent generalization of the above methods to the design of high-order digital parametric audio equalizers is given in [30], with some earlier approaches in [28,29].

The method starts with a lowpass analog shelving equalizer filter having a magnitude response of the form:

$$|H(\Omega)|^2 = \frac{G^2 + G_0^2 \varepsilon^2 F_N^2(w)}{1 + \varepsilon^2 F_N^2(w)}, \quad w = \frac{\Omega}{\Omega_B}$$

and determines the corresponding analog transfer function, and then transforms that into a digital bandpass equalizer. The quantity Ω_B is an effective bandwidth frequency that corresponds to a desired bandwidth level G_B (such as the 3-dB level). The parameter ε is calculated from the condition that $F_N(w) = 1$ at $\Omega = \Omega_B$:

$$\frac{G^2 + G_0^2 \varepsilon^2}{1 + \varepsilon^2} = G_B^2 \quad \Rightarrow \quad \varepsilon = \sqrt{\frac{G^2 - G_B^2}{G_B^2 - G_0^2}}$$

Here, G is the peak or cut gain, and G_0 a reference gain, typically chosen to be $G_0 = 1$ for cascaded equalizers. All the above designs of lowpass, highpass, and bandpass analog filters correspond to the special case of $G = 1, G_0 = 0$. The web page [31] includes a MATLAB toolbox of functions for the design and implementation of such parametric equalizers. The toolbox may also be used to design ordinary lowpass, highpass, bandpass, and bandstop filters, as an alternative to the methods discussed in these notes.

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