14

Spectrum Estimation and Array Processing

14.1 Spectrum Estimation by Autoregressive Modeling

When a block of signal samples is available, it may be too short to provide enough frequency resolution in the periodogram spectrum. Often, it may not even be correct to extend the length by collecting more samples, since this might come into conflict with the stationarity of the segment. In cases such as these, parametric representation of the spectra by means of autoregressive models can provide much better frequency resolution than the classical periodogram method [926]. This approach was discussed briefly in Sec. 1.13.

The spectrum estimation procedure is as follows: First, the given data segment $\{y_0, y_1, \ldots, y_{N-1}\}$ is subjected to one of the analysis methods discussed in Sec. 12.12 to extract estimates of the LPC model parameters $\{a_1, a_2, \ldots, a_M; E_M\}$. The choice of the order M is an important consideration. There are a number of criteria for model order selection [926], but there is no single one that works well under all circumstances. In fact, selecting the right order M is more often an art than science. As an example, we mention Akaike's final prediction error (FPE) criterion which selects the M that minimizes the quantity

$$E_M \cdot \frac{N+M+1}{N-M-1} = \min$$

where E_M is the estimate of the mean-square prediction error for the Mth order predictor, and N is the length of the sequence y_n. As M increases, the factor E_M decreases and the second factor increases, thus, there is a minimum value. Then, the spectrum estimate is given by

$$S_{AR}(\omega) = \frac{E_M}{|A_M(\omega)|^2} = \frac{E_M}{|1 + a_1 e^{-j\omega} + a_2 e^{-2j\omega} + \cdots + a_M e^{-Mj\omega}|^2} \tag{14.1.1}$$

Note that this would be the exact spectrum if y_n were autoregressive with the above set of model parameters. Generally, spectra that have a few dominant spectral peaks can be modeled quite successfully by such all-pole autoregressive models. One can also fit the given block of data to more general ARMA models. The decision to model a spectrum by ARMA, AR, or MA models should ultimately depend on some prior information

regarding the physics of the process y_n. The reader is referred to the exhaustive review article of Kay and Marple [926], and to [1076–1080]

Next, we compare by means of a simulation example the classical periodogram method, the Yule-Walker method, and Burg's method of computing spectrum estimates. Generally, the rule of thumb to follow is that Burg's method should work better than the other methods on short records of data, and that all three methods tend to improve as the data record becomes longer. For our simulation example, we chose a fourth order autoregressive model characterized by two very sharp peaks in its spectrum. The signal generator for the sequence y_n was

$$y_n + a_1 y_{n-1} + a_2 y_{n-2} + a_3 y_{n-3} + a_4 y_{n-4} = \epsilon_n$$

where ϵ_n was zero-mean, unit-variance, white noise. The prediction-error filter $A(z)$ was defined in terms of its four zeros:

$$\begin{aligned} A(z) &= 1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4} \\ &= (1 - z_1 z^{-1})(1 - z_1^* z^{-1})(1 - z_2 z^{-1})(1 - z_2^* z^{-1}) \end{aligned}$$

where the zeros were chosen as

$$z_1 = 0.99 \exp(0.2\pi j), \quad z_2 = 0.99 \exp(0.4\pi j)$$

This gives for the filter coefficients

$$a_1 = -2.2137, \quad a_2 = 2.9403, \quad a_3 = -2.1697, \quad a_4 = 0.9606$$

The exact spectrum is given by Eq. (14.1.1) with $E_4 = \sigma_\epsilon^2 = 1$. Since the two zeros z_1 and z_2, are near the unit circle, the spectrum will have two very sharp peaks at the normalized frequencies

$$\omega_1 = 0.2\pi, \quad \omega_2 = 0.4\pi \quad \text{[radians/sample]}$$

Using the above difference equation and a realization of ϵ_n, a sequence of length 20 of y_n samples was generated (the filter was run for a while until its transients died out and stationarity of y_n was reached). The same set of 20 samples was used to compute the ordinary periodogram spectrum and the autoregressive spectra using the Yule-Walker and Burg methods of extracting the model parameters. Then, the length of the data sequence y_n was increased to 100 and the periodogram, Yule-Walker, and Burg spectra were computed again. Fig. 14.1.1 shows the periodogram spectra for the two signal lengths of 20 and 100 samples. Fig. 14.1.2 show the Yule-Walker spectra, and Fig. 14.1.3, the Burg spectra.

The lack of sufficient resolution of both the periodogram and the Yule-Walker spectrum estimates for the shorter data record can be attributed to the windowing of the signal y_n. But as the length increases the effects of windowing become less pronounced and both methods improve. Burg's method is remarkable in that it works very well even on the basis of very short data records. The Burg spectral estimate is sometimes called the "maximum entropy" spectral estimate. The connection to entropy concepts is discussed in the above references.

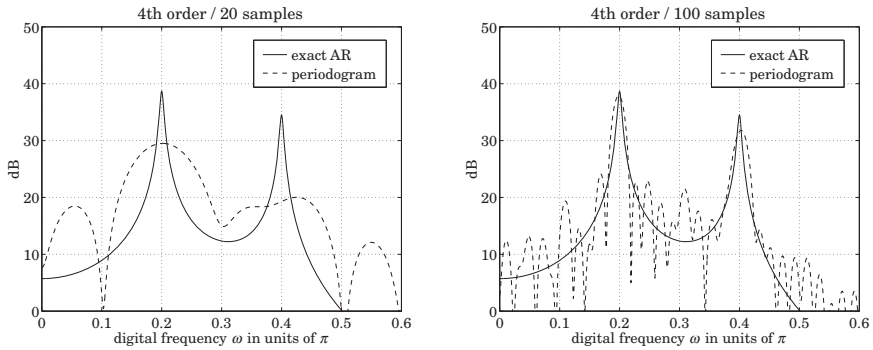


Fig. 14.1.1 Periodogram spectra based on 20 and 100 samples.

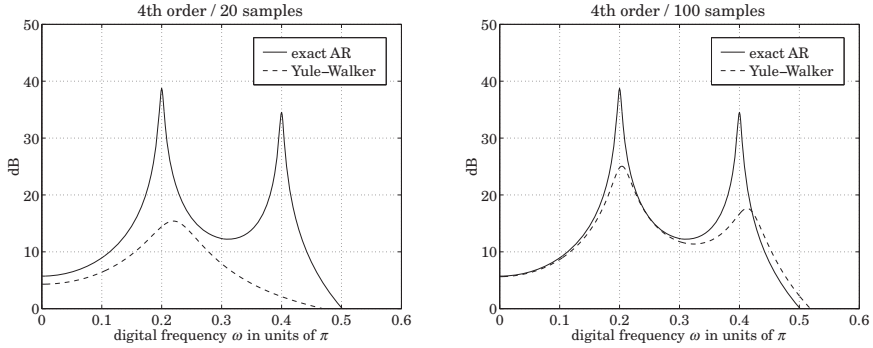


Fig. 14.1.2 Yule-Walker spectra based on 20 and 100 samples.

14.2 Spectral Analysis of Sinusoids in Noise

One of the most important signal processing problems is the estimation of the frequencies and amplitudes of sinusoidal signals buried in additive noise. In addition to its practical importance, this problem has served as the testing ground for all spectrum estimation techniques, new or old. In this section we discuss four approaches to this problem: (1) the classical method, based on the Fourier transform of the windowed autocorrelation; (2) the maximum entropy method, based on the autoregressive modeling of the spectrum; (3) the maximum likelihood, or minimum energy, method; and (4) Pisarenko's method of harmonic retrieval which offers the highest resolution.

Consider a signal consisting of L complex sinusoids with random phases in additive noise:

$$y_n = \nu_n + \sum_{i=1}^{L} A_i e^{j\omega_i n + j\phi_i} \tag{14.2.1}$$

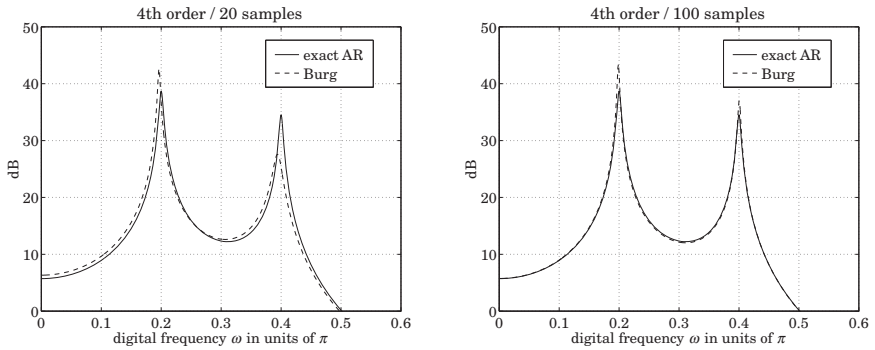


Fig. 14.1.3 Burg spectra based on 20 and 100 samples.

where the phases ϕ_i are uniformly distributed and independent of each other, and ν_n is zero-mean white noise of variance σ_ν^2, assumed to be independent of the phases ϕ_i:

$$E[\nu_n^* \nu_m] = \sigma_\nu^2 \delta_{nm}\,, \quad E[\phi_i \nu_n] = 0 \tag{14.2.2}$$

Under these assumptions, the autocorrelation of y_n is easily found to be

$$R(k) = E[y_{n+k} y_n^*] = \sigma_\nu^2 \delta(k) + \sum_{i=1}^{L} P_i e^{j\omega_i k} \tag{14.2.3}$$

where P_i denotes the power of the ith sinusoid; that is, $P_i = |A_i|^2$. The basic problem is to extract the set of frequencies $\{\omega_1, \omega_2, \ldots, \omega_L\}$ and powers $\{P_1, P_2, \ldots, P_L\}$ by appropriate processing a segment of signal samples y_n. The theoretical power spectrum is a line spectrum superimposed on a flat white-noise background:

$$S(\omega) = \sigma_\nu^2 + \sum_{i=1}^{L} P_i\, 2\pi\delta(\omega - \omega_i) \tag{14.2.4}$$

which is obtained by Fourier transforming Eq. (14.2.3):

$$S(\omega) = \sum_{k=-\infty}^{\infty} R(k)\, e^{-j\omega k} \tag{14.2.5}$$

Given a finite set of autocorrelation lags $\{R(0), R(1), \ldots, R(M)\}$, the classical spectrum analysis method consists of windowing these lags by an appropriate window and then computing the sum (14.2.5), truncated to $-M \le k \le M$. We will use the triangular or Bartlett window which corresponds to the mean value of the ordinary periodogram spectrum [12]. This window is defined by

$$w_B(k) = \begin{cases} \dfrac{M+1-|k|}{M+1}\,, & \text{if } -M \le k \le M \\ 0\,, & \text{otherwise} \end{cases}$$

Replacing $R(k)$ by $w_B(k)R(k)$ in Eq. (14.2.5), we obtain the classical Bartlett spectrum estimate:

$$\hat{S}_B(\omega) = \sum_{k=-M}^{M} w_B(k)R(k)e^{-j\omega k} \tag{14.2.6}$$

We chose the Bartlett window because this expression can be written in a compact matrix form by introducing the $(M+1)$-dimensional phase vector

$$\mathbf{s}_\omega = \begin{bmatrix} 1 \\ e^{j\omega} \\ e^{2j\omega} \\ \vdots \\ e^{Mj\omega} \end{bmatrix}$$

and the $(M+1)\times(M+1)$ autocorrelation matrix R, defined as

$$R_{km} = R(k-m) = \sigma_v^2\delta(k-m) + \sum_{i=1}^{L} P_i e^{j\omega_i(k-m)}, \quad 0 \le k, m \le M$$

Ignoring the $1/(M+1)$ scale factor arising from the definition of the Bartlett window, we may write Eq. (14.2.6) as

$$\hat{S}_B(\omega) = \mathbf{s}_\omega^\dagger R \mathbf{s}_\omega \qquad \text{(classical Bartlett spectrum)} \tag{14.2.7}$$

The autocorrelation matrix R of the sinusoids can also be written in terms of the phasing vectors as

$$R = \sigma_v^2 I + \sum_{i=1}^{L} P_i \mathbf{s}_{\omega_i}\mathbf{s}_{\omega_i}^\dagger \tag{14.2.8}$$

where I is the $(M+1)\times(M+1)$ identity matrix. It can be written even more compactly by introducing the $L\times L$ diagonal power matrix, and the $(M+1)\times L$ sinusoid matrix

$$P = \text{diag}\{P_1, P_2, \ldots, P_L\}, \quad S = [\mathbf{s}_{\omega_1}, \mathbf{s}_{\omega_2}, \ldots, \mathbf{s}_{\omega_L}]$$

Then, Eq. (14.2.8) becomes

$$R = \sigma_v^2 I + SPS^\dagger \tag{14.2.9}$$

Inserting Eq. (14.2.8) into (14.2.7) we find

$$\hat{S}_B(\omega) = \sigma_v^2 \mathbf{s}_\omega^\dagger \mathbf{s}_\omega + \sum_{i=1}^{L} P_i \mathbf{s}_\omega^\dagger \mathbf{s}_{\omega_i}\mathbf{s}_{\omega_i}^\dagger \mathbf{s}_\omega$$

Defining the function

$$W(\omega) = \sum_{m=0}^{M} e^{-j\omega m} = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}} = \frac{\sin\left(\frac{\omega(M+1)}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega M/2} \tag{14.2.10}$$

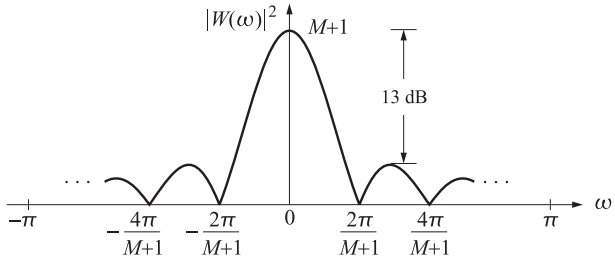
we note that

$$\mathbf{s}_\omega^\dagger \mathbf{s}_{\omega_i} = W(\omega-\omega_i) \quad \text{and} \quad \mathbf{s}_\omega^\dagger \mathbf{s}_\omega = W(0) = M+1$$

Then, in this notation, the Bartlett spectrum (14.2.7) becomes

$$\hat{S}_B(\omega) = \sigma_v^2(M+1) + \sum_{i=1}^{L} P_i \left| W(\omega-\omega_i) \right|^2 \tag{14.2.11}$$

The effect of $W(\omega-\omega_i)$ is to smear each spectral line $\delta(\omega-\omega_i)$ of the true spectrum. If the frequencies ω_i are too close to each other the smeared peaks will tend to overlap with a resulting loss of resolution. The function $W(\omega)$ is the Fourier transform of the rectangular window and is depicted below:



It has an effective resolution width of $\Delta\omega = 2\pi/(M+1)$. For fairly large Ms, the first side lobe is about 13 dB down from the main lobe. As M increases, the main lobe becomes higher and thinner, resembling more and more a delta function, which improves the frequency resolution capability of this estimate.

Next, we derive a closed form expression [1085] for the AR, or maximum entropy, spectral estimate. It is given by Eq. (14.1.1) and is obtained by fitting an order-M autoregressive model to the autocorrelation lags $\{R(0), R(1), \ldots, R(M)\}$. This can be done for any desired value of M. Autoregressive spectrum estimates generally work well in modeling "peaked" or resonant spectra; therefore, it is expected that they will work in this case, too. However, it should be kept in mind that AR models are not really appropriate for such sinusoidal signals. Indeed, AR models are characterized by all-pole stable filters that always result in autocorrelation functions $R(k)$ which decay exponentially with the lag k; whereas Eq. (14.2.3) is persistent in k and never decays.

As a rule, AR modeling of sinusoidal spectra works very well as long as the signal to noise ratios (SNRs) are fairly high. Pisarenko's method, to be discussed later, provides unbiased frequency estimates *regardless* of the SNRs. The LPC model parameters for the AR spectrum estimate (14.1.1) are obtained by minimizing the mean-square prediction error:

$$\mathcal{E} = E[e_n^* e_n] = \mathbf{a}^\dagger R \mathbf{a} = \min, \quad e_n = \sum_{m=0}^{M} a_m y_{n-m} \tag{14.2.12}$$

where $\mathbf{a} = [1, a_1, a_2, \ldots, a_M]^T$ is the prediction-error filter and R, the autocorrelation matrix (14.2.9). The minimization of $\mathcal{E}$ must be subject to the linear constraint that the first entry of $\mathbf{a}$ be unity. This constraint can be expressed in vector form

$$a_0 = \mathbf{u}_0^\dagger \mathbf{a} = 1 \tag{14.2.13}$$

where $\mathbf{u}_0 = [1,0,0,\ldots,0]^T$ is the unit vector consisting of 1 followed by M zeros. Incorporating this constraint with a Lagrange multiplier, we solve the minimization problem:

$$\mathcal{E} = \mathbf{a}^\dagger R\mathbf{a} + \mu(1 - \mathbf{u}_0^\dagger \mathbf{a}) = \min$$

Differentiating with respect to $\mathbf{a}$ we obtain the normal equations:

$$R\mathbf{a} = \mu\mathbf{u}_0$$

To fix the Lagrange multiplier, multiply from the left by $\mathbf{a}^\dagger$ and use Eq. (14.2.13) to get $\mathbf{a}^\dagger R\mathbf{a} = \mu\mathbf{a}^\dagger\mathbf{u}_0$, or, $\mathcal{E} = \mu$. Thus, μ is the minimized value of $\mathcal{E}$, which we denote by E. In summary, we have

$$R\mathbf{a} = E\mathbf{u}_0 \quad \Rightarrow \quad \mathbf{a} = ER^{-1}\mathbf{u}_0 \tag{14.2.14}$$

Multiplying from the left by $\mathbf{u}_0^\dagger$, we also find $1 = E(\mathbf{u}_0^\dagger R^{-1}\mathbf{u}_0)$, or

$$E^{-1} = \mathbf{u}_0^\dagger R^{-1}\mathbf{u}_0 = (R^{-1})_{00} \tag{14.2.15}$$

which is, of course, the same as Eq. (12.9.18). The special structure of R allows the computation of $\mathbf{a}$ and the AR spectrum in closed form. Applying the matrix inversion lemma to Eq. (14.2.9), we find the inverse of R:

$$R^{-1} = \frac{1}{\sigma_v^2}(I + SDS^\dagger) \tag{14.2.16}$$

where D is an $L\times L$ matrix given by

$$D = -\left[\sigma_v^2 P^{-1} + S^\dagger S\right]^{-1} \tag{14.2.17}$$

Equation (14.2.16) can also be derived directly by assuming such an expression for R^{-1} and then fixing D. The quantity $\sigma_v^2 P^{-1}$ in D is a matrix of noise to signal ratios. Inserting Eq. (14.2.16) into (14.2.14), we find for $\mathbf{a}$:

$$\mathbf{a} = ER^{-1}\mathbf{u}_0 = \frac{E}{\sigma_v^2}[\mathbf{u}_0 + SDS^\dagger\mathbf{u}_0] = \frac{E}{\sigma_v^2}[\mathbf{u}_0 + S\mathbf{d}]$$

where we used the fact that $\mathbf{s}_{\omega_i}^\dagger\mathbf{u}_0 = 1$, which implies that

$$S^\dagger\mathbf{u}_0 = \begin{bmatrix} \mathbf{s}_{\omega_1}^\dagger \\ \mathbf{s}_{\omega_2}^\dagger \\ \vdots \\ \mathbf{s}_{\omega_L}^\dagger \end{bmatrix}\mathbf{u}_0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \equiv \mathbf{v} \qquad \text{(i.e., a column of } L \text{ ones)}$$

and defined

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_L \end{bmatrix} = D\mathbf{v}, \quad \text{or,} \quad d_i = \sum_{j=1}^{L} D_{ij}$$

Using Eq. (14.2.15), we have also

$$E^{-1} = \mathbf{u}_0^\dagger R^{-1}\mathbf{u}_0 = \frac{1}{\sigma_v^2}\mathbf{u}_0^\dagger[I + SDS^\dagger] = \frac{1}{\sigma_v^2}[1 + \mathbf{v}^T D\mathbf{v}]$$

$$= \frac{1}{\sigma_v^2}[1 + \mathbf{v}^T\mathbf{d}] = \frac{1}{\sigma_v^2}\left[1 + \sum_{i=1}^{L} d_i\right]$$

and, therefore,

$$E = \sigma_v^2\left[1 + \sum_{i=1}^{L} d_i\right]^{-1} \tag{14.2.18}$$

We finally find for the prediction-error filter

$$\mathbf{a} = \frac{\mathbf{u}_0 + S\mathbf{d}}{1 + \mathbf{v}^T\mathbf{d}} = \frac{u_0 + \sum_{i=1}^{L} d_i\mathbf{s}_{\omega_i}}{1 + \sum_{i=1}^{L} d_i} \tag{14.2.19}$$

The frequency response $A(\omega)$ of the prediction-error filter is obtained by dotting the phasing vector $\mathbf{s}_\omega$ into $\mathbf{a}$:

$$A(\omega) = \sum_{m=0}^{M} a_m e^{-j\omega m} = \mathbf{s}_\omega^\dagger\mathbf{a} = \frac{1 + \sum_{i=1}^{L} d_i\mathbf{s}_\omega^\dagger\mathbf{s}_{\omega_i}}{1 + \sum_{i=1}^{L} d_i}$$

using the result that $\mathbf{s}_\omega^\dagger\mathbf{s}_{\omega_i} = W(\omega - \omega_i)$, we finally find:

$$A(\omega) = \frac{1 + \sum_{i=1}^{L} d_i W(\omega - \omega_i)}{1 + \sum_{i=1}^{L} d_i} \tag{14.2.20}$$

and for the AR, or maximum entropy, spectrum estimate:

$$\hat{S}_{AR}(\omega) = \frac{E}{|A(\omega)|^2} = \sigma_v^2 \frac{\left|1 + \sum_{i=1}^{L} d_i\right|}{\left|1 + \sum_{i=1}^{L} d_i W(\omega - \omega_i)\right|^2} \tag{14.2.21}$$

The frequency dependence is shown explicitly. Note, that the matrix $S^\dagger S$ appearing in the definition of D, can also be expressed in terms of $W(\omega)$. Indeed, the ijth element of $S^\dagger S$ is, for $0 \le i,j \le L$:

$$(S^\dagger S)_{ij} = \mathbf{s}_{\omega_i}^\dagger\mathbf{s}_{\omega_j} = W(\omega_i - \omega_j)$$

One interesting consequence of Eq. (14.2.21) is that in the limit of very weak noise $\sigma_v^2 \to 0$, it vanishes. In this limit the mean-square prediction error (14.2.18) vanishes. This is to be expected, since in this case the noise term v_n is absent from the sum (14.2.1), rendering y_n a deterministic signal; that is, one that can be predicted from a few past values with zero prediction error. To avoid such behavior when σ_v^2 is small, the factor E is sometimes dropped altogether from the spectral estimate resulting in the "pseudo-spectrum"

$$\hat{S}_{AR}(\omega) = \frac{1}{|A(\omega)|^2} \tag{14.2.22}$$

This expression will exhibit fairly sharp peaks at the sinusoid frequencies, but the magnitude of these peaks will no longer be representative of the power levels P_i. This expression can only be used to extract the frequencies ω_i. Up to a scale factor, Eq. (14.2.22) can also be written in the form

$$\hat{S}_{AR}(\omega) = \frac{1}{|\mathbf{s}_\omega^\dagger R^{-1} \mathbf{u}_0|^2}$$

Example 14.2.1: To see the effect of the SNR on the sharpness of the peaks in the AR spectrum, consider the case $M = L = 1$. Then,

$$S^\dagger S = \mathbf{s}_{\omega_1}^\dagger \mathbf{s}_{\omega_1} = [1,\ e^{-j\omega_1}] \begin{bmatrix} 1 \\ e^{j\omega_1} \end{bmatrix} = M + 1 = 2$$

$$D = -[\sigma_v^2 P_1^{-1} + 2]^{-1}$$

$$\mathbf{a} = \frac{\mathbf{u}_0 + d_1 \mathbf{s}_{\omega_1}}{1 + d_1} = \begin{bmatrix} 1 \\ \dfrac{d_1}{1+d_1} e^{j\omega_1} \end{bmatrix}$$

Using $d_1 = D$, we find

$$\mathbf{a} = \begin{bmatrix} 1 \\ -\dfrac{P_1}{P_1 + \sigma_v^2} e^{j\omega_1} \end{bmatrix}, \quad A(z) = 1 + a_1 z^{-1}$$

The prediction-error filter has a zero at

$$z_1 = -a_1 = \frac{P_1}{P_1 + \sigma_v^2} e^{j\omega_1}$$

The zero z_1 is inside the unit circle, as it should. The lower the $SNR = P_1/\sigma_v^2$, the more inside it lies, resulting in a more smeared peak about ω_1. As the SNR increases, the zero moves closer to the unit circle at the right frequency ω_1, resulting in a very sharp peak in the spectrum (14.2.22). □

Example 14.2.2: For the case of a single sinusoid and arbitrary order M, compute the 3-dB width of the spectral peak of AR spectrum and compare it with the width of the Bartlett spectrum. Using Eq. (14.2.20), we have

$$A(\omega) = \frac{1 + d_1 W(\omega - \omega_1)}{1 + d_1}, \quad d_1 = -[SNR^{-1} + M + 1]^{-1}$$

where we set $SNR = P_1/\sigma_v^2$. The value of $A(\omega)$ at the sinusoid frequency is

$$A(\omega_1) = \frac{1 + d_1 W(0)}{1 + d_1} = \frac{1}{1 + SNR \cdot M}$$

It is small in the limit of high SNR resulting in a high peak in the spectrum. The half-width at half-maximum of the AR spectrum is defined by the condition

$$\frac{S(\omega_1 + \Delta\omega)}{S(\omega_1)} = \frac{1}{2}, \quad \text{or, equivalently,} \quad \frac{|A(\omega_1 + \Delta\omega)|^2}{|A(\omega_1)|^2} = 2$$

To first order in $\Delta\omega$, we have

$$W(\Delta\omega) = \sum_{m=0}^{M} e^{-jm\Delta\omega} = \sum_{m=0}^{M} (1 - jm\Delta\omega) = (M+1) - \frac{1}{2} jM(M+1)\Delta\omega$$

where we used $\sum_{m=0}^{M} m = M(M+1)/2$. Then, we find

$$\frac{A(\omega_1 + \Delta\omega)}{A(\omega_1)} = \frac{1 + d_1 W(\Delta\omega)}{1 + d_1 W(0)} = 1 - \frac{1}{2} SNR \cdot jM(M+1)\Delta\omega$$

The condition for half-maximum requires that the above imaginary part be unity, which gives for the 3-dB width [1083]

$$(\Delta\omega)_{3\text{dB}} = 2\Delta\omega = \frac{4}{SNR \cdot M(M+1)}$$

Thus, the peak becomes narrower both with increasing SNR and with order M. Note that it depends on M like $O(1/M^2)$, which is a factor of M smaller than the Bartlett width that behaves like $O(1/M)$. □

More generally, in the case of multiple sinusoids, if the SNRs are high the spectrum (14.2.22) will exhibit sharp peaks at the desired sinusoid frequencies. The mechanism by which this happens can be seen qualitatively from Eq. (14.2.20) as follows: The matrix $S^\dagger S$ in D introduces cross-coupling among the various frequencies ω_i. However, if these frequencies are well separated from each other (by more than $2\pi/(M+1)$,) then the off-diagonal elements of $S^\dagger S$, namely $W(\omega_i - \omega_j)$ will be small, and for the purpose of this argument may be taken to be zero. This makes the matrix $S^\dagger S$ approximately diagonal. Since $W(0) = M + 1$ it follows that $S^\dagger S = (M+1)I$, and D will become diagonal with diagonal elements

$$d_i = D_{ii} = -[\sigma_v^2 P_i^{-1} + M + 1]^{-1} = -\frac{P_i}{\sigma_v^2 + (M+1)P_i}$$

Evaluating $A(\omega)$ at ω_i and keeping only the ith contribution in the sum we find, approximately,

$$A(\omega_i) \simeq \frac{1 + d_i W(0)}{1 + \sum_{j=0}^{L} d_j} = \frac{1}{1 + \sum_{j=0}^{L} d_j} \frac{1}{1 + (M+1)\left(\dfrac{P_i}{\sigma_v^2}\right)}$$

which shows that if the SNRs P_i/σ_v^2 are high, $A(\omega_i)$ will be very small, resulting in large spectral peaks in Eq. (14.2.22). The resolvability properties of the AR estimate improve both when the SNRs increase and when the order M increases. The mutual interaction of the various sinusoid components cannot be ignored altogether. One effect of this interaction is *biasing* in the estimates of the frequencies; that is, even if two nearby peaks are clearly separated, the peaks may not occur exactly at the desired sinusoid frequencies, but may be slightly shifted. The degree of bias depends on the relative separation of the peaks and on the SNRs. With the above qualifications in mind, we can state that the LPC approach to this problem is one of the most successful ones.

Capon's maximum likelihood (ML), or minimum energy, spectral estimator is given by the expression [1081]

$$\hat{S}_{ML}(\omega) = \frac{1}{\mathbf{s}_\omega^\dagger R^{-1} \mathbf{s}_\omega} \tag{14.2.23}$$

It can be justified by envisioning a bank of narrowband filters, each designed to allow a sinewave through at the filter's center frequency and to attenuate all other frequency components. Thus, the narrowband filter with center frequency ω is required to let this frequency go through unchanged, that is,

$$A(\omega) = \mathbf{s}_\omega^\dagger \mathbf{a} = 1$$

while at the same time it is required to minimize the output power

$$\mathbf{a}^\dagger R \mathbf{a} = \min$$

The solution of this minimization problem subject to the above constraint is readily found to be

$$\mathbf{a} = \frac{R^{-1}\mathbf{s}_\omega}{\mathbf{s}_\omega^\dagger R^{-1} \mathbf{s}_\omega}$$

which gives for the minimized output power at this frequency

$$\mathbf{a}^\dagger R \mathbf{a} = \frac{1}{\mathbf{s}_\omega^\dagger R^{-1} \mathbf{s}_\omega}$$

Using Eq. (14.2.16), we find

$$\mathbf{s}_\omega^\dagger R^{-1} \mathbf{s}_\omega = \frac{1}{\sigma_v^2}\left[\mathbf{s}_\omega^\dagger \mathbf{s}_\omega + \sum_{i,j=1}^{L} D_{ij} \mathbf{s}_\omega^\dagger \mathbf{s}_{\omega_i} \mathbf{s}_{\omega_j}^\dagger \mathbf{s}_\omega \right]$$

$$= \frac{1}{\sigma_v^2}\left[(M+1) + \sum_{i,j=1}^{L} D_{ij} W(\omega - \omega_i) W^*(\omega - \omega_j) \right]$$

and the theoretical ML spectrum becomes in this case:

$$\hat{S}_{ML}(\omega) = \frac{\sigma_v^2}{\left[(M+1) + \sum_{i,j=1}^{L} D_{ij} W(\omega - \omega_i) W^*(\omega - \omega_j) \right]} \tag{14.2.24}$$

Example 14.2.3: Determine the matrix D and vector $\mathbf{d}$ for the case of $L = 2$ and arbitrary M.

The matrix $S^\dagger S$ is in this case

$$S^\dagger S = \begin{bmatrix} W(0) & W(\omega_1 - \omega_2) \\ W(\omega_2 - \omega_1) & W(0) \end{bmatrix} = \begin{bmatrix} M+1 & W_{12} \\ W_{12}^* & M+1 \end{bmatrix}$$

where $W_{12} = W(\omega_1 - \omega_2)$. Then, D becomes

$$D = -\begin{bmatrix} \sigma_v^2 P_1^{-1} + M + 1 & W_{12} \\ W_{12}^* & \sigma_v^2 P_2^{-1} + M + 1 \end{bmatrix}^{-1}, \quad \text{or,}$$

$$D = \frac{1}{|W_{12}|^2 - (\sigma_v^2 P_1^{-1} + M + 1)(\sigma_v^2 P_2^{-1} + M + 1)} \begin{bmatrix} \sigma_v^2 P_2^{-1} + M + 1 & -W_{12} \\ -W_{12}^* & \sigma_v^2 P_1^{-1} + M + 1 \end{bmatrix}$$

and, hence

$$\mathbf{d} = D \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{|W_{12}|^2 - (\sigma_v^2 P_1^{-1} + M + 1)(\sigma_v^2 P_2^{-1} + M + 1)} \begin{bmatrix} \sigma_v^2 P_2^{-1} + M + 1 - W_{12} \\ \sigma_v^2 P_1^{-1} + M + 1 - W_{12}^* \end{bmatrix}$$

Using the results of Example 14.2.3, we have carried out a computation illustrating the three spectral estimates. Fig. 14.2.1 shows the theoretical autoregressive, Bartlett, and maximum likelihood spectral estimates given by Eqs. (14.2.11), (14.2.22), and (14.2.24), respectively, for two sinusoids of frequencies

$$\omega_1 = 0.4\pi, \quad \omega_2 = 0.6\pi$$

and equal powers $SNR = 10\log_{10}(P_1/\sigma_v^2) = 6$ dB, and $M = 6$. To facilitate the comparison, all three spectra have been normalized to 0 dB at the frequency ω_1 of the first sinusoid. It is seen that the length $M = 6$ is too short for the Bartlett spectrum to resolve the two peaks. The AR spectrum is the best (however, close inspection of the graph will reveal a small bias in the frequency of the peaks, arising from the mutual interaction of the two sinewaves). The effect of increasing the SNR is shown on the right in Fig. 14.2.1, where the SNR has been changed to 12 dB. It is seen that the AR spectral peaks become narrower, thus increasing their resolvability.

To show the effect of increasing M, we kept $SNR = 6$ dB, and increased the order to $M = 12$ and $M = 18$. The resulting spectra are shown in Fig. 14.2.2. It is seen that all spectra tend to become better. The interplay between resolution, order, SNR, and bias has been studied in [1083,1085,1088].

The main motivation behind the definition (14.2.22) for the pseudospectrum was to obtain an expression that exhibits very sharp spectral peaks at the sinusoid frequencies ω_i. Infinite resolution can, in principle, be achieved if we can find a polynomial $A(z)$ that has zeros *on the unit circle* at the desired frequency angles; namely, at

$$z_i = e^{j\omega_i}, \quad i = 1, 2, \ldots, L \tag{14.2.25}$$

Pisarenko's method determines such a polynomial on the basis of the autocorrelation matrix R. The desired conditions on the polynomial are

$$A(z_i) = A(\omega_i) = 0, \quad i = 1, 2, \ldots, L \tag{14.2.26}$$

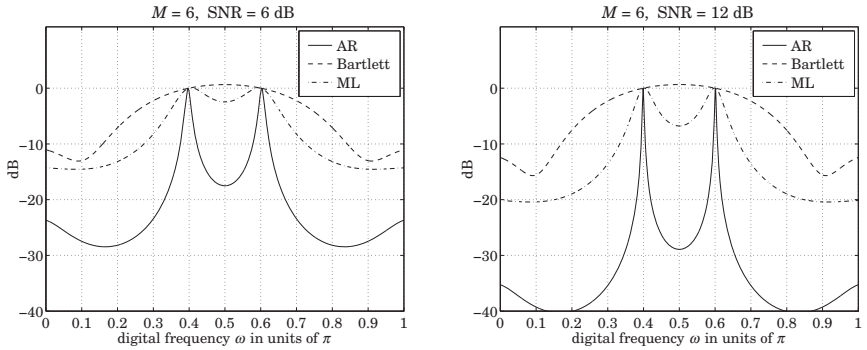


Fig. 14.2.1 AR, Bartlett, and ML spectrum estimates.

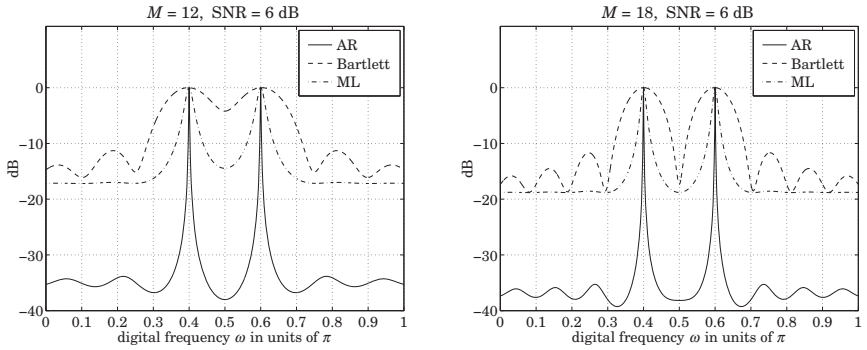


Fig. 14.2.2 AR, Bartlett, and ML spectrum estimates.

where we slightly abuse the notation and write $A(e^{j\omega}) = A(\omega)$. To satisfy these conditions, the degree M of the polynomial $A(z)$ must necessarily be $M \geq L$; then, the remaining $M-L$ zeros of $A(z)$ could be arbitrary. Let $\mathbf{a}$ be the vector of coefficients of $A(z)$, so that

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix}, \quad A(z) = a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}$$

Noting that $A(\omega) = \mathbf{s}_\omega^\dagger \mathbf{a}$, Eqs. (14.2.26) may be combined into one vectorial equation

$$S^\dagger \mathbf{a} = \begin{bmatrix} \mathbf{s}_{\omega_1}^\dagger \\ \mathbf{s}_{\omega_2}^\dagger \\ \vdots \\ \mathbf{s}_{\omega_L}^\dagger \end{bmatrix} \mathbf{a} = \begin{bmatrix} A(\omega_1) \\ A(\omega_2) \\ \vdots \\ A(\omega_L) \end{bmatrix} = 0 \tag{14.2.27}$$

But then, Eq. (14.2.9) implies that

$$R\mathbf{a} = \sigma_v^2 \mathbf{a} + SPS^\dagger \mathbf{a} = \sigma_v^2 \mathbf{a}$$

or, that σ_v^2 must be an *eigenvalue* of R with $\mathbf{a}$ the corresponding eigenvector:

$$R\mathbf{a} = \sigma_v^2 \mathbf{a} \tag{14.2.28}$$

The quantity σ_v^2 is actually the *smallest* eigenvalue of R. To see this, consider any other eigenvector $\mathbf{a}$ of R, and normalize it to unit norm

$$R\mathbf{a} = \lambda \mathbf{a}, \quad \text{with } \mathbf{a}^\dagger \mathbf{a} = 1 \tag{14.2.29}$$

Then, (14.2.9) implies that

$$\lambda = \lambda \mathbf{a}^\dagger \mathbf{a} = \mathbf{a}^\dagger R \mathbf{a} = \sigma_v^2 \mathbf{a}^\dagger \mathbf{a} + \mathbf{a} SPS^\dagger \mathbf{a}$$

$$= \sigma_v^2 + [A(\omega_1)^*, A(\omega_2)^*, \ldots, A(\omega_L)^*] \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_L \end{bmatrix} \begin{bmatrix} A(\omega_1) \\ A(\omega_2) \\ \vdots \\ A(\omega_L) \end{bmatrix}$$

$$= \sigma_v^2 + \sum_{i=1}^{L} P_i |A(\omega_i)|^2$$

which shows that λ is equal to σ_v^2 shifted by a non-negative amount. If the eigenvector satisfies the conditions (14.2.26), then the shift in λ vanishes. Thus, the desired polynomial $A(z)$ can be found by solving the eigenvalue problem (14.2.29) and selecting the eigenvector belonging to the minimum eigenvalue. This is Pisarenko's method [1084]. As a byproduct of the procedure, the noise power level σ_v^2 is also determined, which in turn allows the determination of the power matrix P, as follows. Writing Eq. (14.2.9) as

$$R - \sigma_v^2 I = SPS^\dagger$$

and acting by $S^\dagger$ and S from the left and right, we obtain

$$P = U^\dagger (R - \sigma_v^2 I) U, \quad \text{where} \quad U = S(S^\dagger S)^{-1} \tag{14.2.30}$$

Since there is freedom in selecting the *remaining* $M-L$ zeros of the polynomial $A(z)$, it follows that there are $(M-L)+1$ eigenvectors all belonging to the minimum eigenvalue σ_v^2. Thus, the $(M+1)$-dimensional eigenvalue problem (14.2.29) has two sets of eigenvalues: (a) $M+1-L$ *degenerate* eigenvalues equal to σ_v^2, and (b) L additional eigenvalues which are strictly greater than σ_v^2.

The $(M+1-L)$-dimensional subspace spanned by the degenerate eigenvectors belonging to σ_v^2 is called the *noise subspace*. The L-dimensional subspace spanned by the eigenvectors belonging to the remaining L eigenvalues is called the *signal subspace*. Since the signal subspace is orthogonal to the noise subspace. and the L linearly independent signal vectors $\mathbf{s}_{\omega_i}$, $i = 1, 2, \ldots, L$ are also orthogonal to the noise subspace, it follows that the signal subspace is *spanned* by the $\mathbf{s}_{\omega_i}$s.

In the special case when $L = M$ (corresponding to the Pisarenko's method), there is no degeneracy in the minimum eigenvalue, and there is a *unique* minimum eigenvector. In this case, all $M = L$ zeros of $A(z)$ lie on the unit circle at the desired angles ω_i.

Example 14.2.4: Consider the case $L = M = 2$. The matrix R is written explicitly as:

$$R = \sigma_v^2 I + P_1 \mathbf{s}_{\omega_1} \mathbf{s}_{\omega_1}^\dagger + P_2 \mathbf{s}_{\omega_2} \mathbf{s}_{\omega_2}^\dagger, \quad \text{or,}$$

$$R = \begin{bmatrix} \sigma_v^2 + P_1 + P_2 & P_1 e^{-j\omega_1} + P_2 e^{-j\omega_2} & P_1 e^{-2j\omega_1} + P_2 e^{-2j\omega_2} \\ P_1 e^{j\omega_1} + P_2 e^{j\omega_2} & \sigma_v^2 + P_1 + P_2 & P_1 e^{-j\omega_1} + P_2 e^{-j\omega_2} \\ P_1 e^{2j\omega_1} + P_2 e^{2j\omega_2} & P_1 e^{j\omega_1} + P_2 e^{j\omega_2} & \sigma_v^2 + P_1 + P_2 \end{bmatrix}$$

It is easily verified that the (unnormalized) vector

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -(e^{j\omega_1} + e^{j\omega_2}) \\ e^{j\omega_1} e^{j\omega_2} \end{bmatrix}$$

is an eigenvector of R belonging to $\lambda = \sigma_v^2$. In this case, the polynomial $A(z)$ is

$$\begin{aligned} A(z) &= a_0 + a_1 z^{-1} + a_2 z^{-2} = 1 - (e^{j\omega_1} + e^{j\omega_2}) z^{-1} + e^{j\omega_1} e^{j\omega_2} z^{-2} \\ &= (1 - e^{j\omega_1} z^{-1})(1 - e^{j\omega_2} z^{-1}) \end{aligned}$$

exhibiting the two desired zeros at the sinusoid frequencies. □

Example 14.2.5: Consider the case $M = 2$, $L = 1$. The matrix R is

$$R = \sigma_v^2 I + P_1 \mathbf{s}_{\omega_1} \mathbf{s}_{\omega_1}^\dagger = \begin{bmatrix} \sigma_v^2 + P_1 & P_1 e^{-j\omega_1} & P_1 e^{-2j\omega_1} \\ P_1 e^{j\omega_1} & \sigma_v^2 + P_1 & P_1 e^{-j\omega_1} \\ P_1 e^{2j\omega_1} & P_1 e^{j\omega_1} & \sigma_v^2 + P_1 \end{bmatrix}$$

It is easily verified that the three eigenvectors of R are

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ -e^{j\omega_1} \\ 0 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ -e^{j\omega_1} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 1 \\ e^{j\omega_1} \\ e^{2j\omega_1} \end{bmatrix}$$

belonging to the eigenvalues

$$\lambda = \sigma_v^2, \quad \lambda = \sigma_v^2, \quad \lambda = \sigma_v^2 + 3P_1$$

The first two eigenvectors span the noise subspace and the third, the signal subspace. Any linear combination of the noise eigenvectors also belongs to $\lambda = \sigma_v^2$. For example, if we take

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -e^{j\omega_1} \\ 0 \end{bmatrix} - \rho \begin{bmatrix} 0 \\ 1 \\ -e^{j\omega_1} \end{bmatrix} = \begin{bmatrix} 1 \\ -(\rho + e^{j\omega_1}) \\ \rho e^{j\omega_1} \end{bmatrix}$$

the corresponding polynomial is

$$A(z) = 1 - (\rho + e^{j\omega_1}) z^{-1} + \rho e^{j\omega_1} z^{-2} = (1 - e^{j\omega_1} z^{-1})(1 - \rho z^{-1})$$

showing one desired zero at $z_1 = e^{j\omega_1}$ and a spurious zero. □

The Pisarenko method can also be understood in terms of a *minimization* criterion of the type (14.2.12), as follows. For any set of coefficients $\mathbf{a}$, define the output signal

$$e_n = \sum_{m=0}^{M} a_m y_{n-m} = a_0 y_n + a_1 y_{n-1} + \cdots + a_M y_{n-M}$$

Then, the mean output power is expressed as

$$\mathcal{E} = E[e_n^* e_n] = \mathbf{a}^\dagger R \mathbf{a} = \sigma_v^2 \mathbf{a}^\dagger \mathbf{a} + \sum_{i=1}^{L} P_i |A(\omega_i)|^2$$

Imposing the quadratic constraint

$$\mathbf{a}^\dagger \mathbf{a} = 1 \tag{14.2.31}$$

we obtain

$$\mathcal{E} = E[e_n^* e_n] = \mathbf{a}^\dagger R \mathbf{a} = \sigma_v^2 + \sum_{i=1}^{L} P_i |A(\omega_i)|^2 \tag{14.2.32}$$

It is evident that the minimum of this expression is obtained when conditions (14.2.26) are satisfied. Thus, an equivalent formulation of the Pisarenko method is to minimize the performance index (14.2.32) subject to the quadratic constraint (14.2.31). The AR and the Pisarenko spectrum estimation techniques differ only in the type of constraint imposed on the filter weights $\mathbf{a}$.

We observed earlier that the AR spectral peaks become sharper as the SNR increases. One way to explain this is to note that in the high-SNR limit or, equivalently, in the noiseless limit $\sigma_v^2 \to 0$, the linear prediction filter tends to the Pisarenko filter, which has infinite resolution. This can be seen as follows. In the limit $\sigma_v^2 \to 0$, the matrix D defined in Eq. (14.2.17) tends to

$$D \to -(S^\dagger S)^{-1}$$

and therefore, R^{-1} given by Eq. (14.2.16) becomes singular, converging to

$$R^{-1} \to \frac{1}{\sigma_v^2} [I - S(S^\dagger S)^{-1} S^\dagger]$$

Thus, up to a scale factor the linear prediction solution, $R^{-1}\mathbf{u}_0$ will converge to

$$\mathbf{a} = [I - S(S^\dagger S)^{-1} S^\dagger] \mathbf{u}_0 \tag{14.2.33}$$

The matrix $[I - S(S^\dagger S)^{-1} S^\dagger]$ is the *projection* matrix onto the noise subspace, and therefore, $\mathbf{a}$ will lie in that subspace, that is, $S^\dagger \mathbf{a} = 0$. In the limit $\sigma_v^2 \to 0$, the noise subspace of R consists of all the eigenvectors with zero eigenvalue, $R\mathbf{a} = 0$. We note that the particular noise subspace eigenvector given in Eq. (14.2.33) corresponds to the so-called *minimum-norm* eigenvector, discussed in Sec. 14.6.

In his original method, Pisarenko considered the special case when the number of sinusoids was equal to the filter order, $L = M$. This implies that the noise subspace is one-dimensional, $M + 1 - L = 1$, consisting of a single eigenvector with zero eigenvalue, such that $R\mathbf{a} = 0$. In this case, the $(M + 1) \times (M + 1)$ singular matrix R has rank M and all its principal submatrices are nonsingular. As we mentioned in Sec. 12.5, such

singular Toeplitz matrices admit a general sinusoidal representation. It is obtained by setting $\sigma_v^2 = 0$ and $L = M$ in Eq. (14.2.8):

$$R = \sum_{i=1}^{L} P_i \mathbf{s}_{\omega_i} \mathbf{s}_{\omega_i}^\dagger, \quad \text{or,} \quad R(k) = \sum_{i=1}^{L} P_i e^{j\omega_i k}$$

In summary, we have discussed the theoretical aspects of four methods of estimating the frequencies of sinusoids in noise. In practice, an estimate of the correlation matrix R can be obtained in terms of the sample autocorrelations from a block of data values:

$$\hat{R}(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} y_{n+k} y_n^*, \quad k = 0, 1, \ldots, M$$

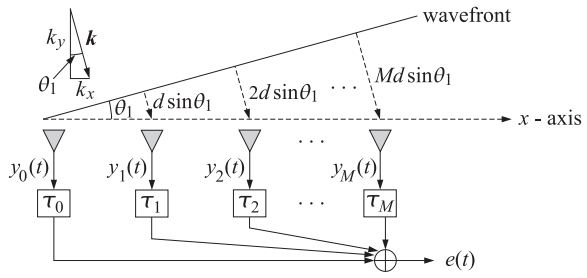
The quality of the resulting estimates of the eigenvectors will be discussed in Section 14.11. The AR and Pisarenko methods can also be implemented *adaptively*. The adaptive approach is based on the minimization criteria (14.2.12) and (14.2.32) and will be discussed in Chap. 16, where also some simulations will be presented.

14.3 Superresolution Array Processing

One of the main signal processing functions of sonar, radar, or seismic arrays of sensors is to detect the presence of one or more radiating point-sources. This is a problem of spectral analysis, and it is the *spatial frequency* analog of the problem of extracting sinusoids in noise discussed in the previous section. The same spectral analysis techniques can be applied to this problem. All methods aim at producing a high-resolution estimate of the spatial frequency power spectrum of the signal field incident on the array of sensors. The *directions* of point-source emitters can be extracted by identifying the sharpest peaks in this spectrum.

In this section, we discuss conventional (Bartlett) *beamforming*, as well as the *maximum likelihood*, *linear prediction*, and *eigenvector* based methods. We also discuss some aspects of *optimum beamforming* for interference nulling [1093–1095,1352,1167–1170].

Consider a linear array of $M + 1$ sensors equally spaced at distances d, and a plane wave incident on the array at an angle θ_1 with respect to the array normal, as shown below.



The conventional beamformer introduces appropriate delays at the outputs of each sensor to compensate for the propagation delays of the wavefront reaching the array. The output of the beamformer (the "beam") is the sum

$$e(t) = \sum_{m=0}^{M} y_m(t - \tau_m) \tag{14.3.1}$$

where $y_m(t)$, $m = 0, 1, \ldots, M$ is the signal at the mth sensor. To reach sensor 1, the wavefront must travel an extra distance $d\sin\theta_1$, to reach sensor 2 it must travel distance $2d\sin\theta_1$, and so on. Thus, it reaches these sensors with a propagation delay of $d\sin\theta_1/c$, $2d\sin\theta_1/c$, and so on. The last sensor is reached with a delay of $Md\sin\theta_1/c$ seconds. Thus, to time-align the first and the last sensor, the output of the first sensor must be delayed by $\tau_0 = Md\sin\theta_1/c$, and similarly, the mth sensor is time-aligned with the last one, with a delay of

$$\tau_m = \frac{1}{c}(M - m)d\sin\theta_1 \tag{14.3.2}$$

In this case, all terms in the sum (14.3.1) are equal to the value measured at the last sensor, that is, $y_m(t - \tau_m) = y_M(t)$, and the output of the beamformer is $e(t) = (M + 1)y_M(t)$, thus enhancing the received signal by a factor of $M + 1$ and hence its power by a factor $(M + 1)^2$. The concept of beamforming is the same as that of signal averaging. If there is additive noise present, it will contribute incoherently to the output power, that is, by a factor of $(M+1)$, whereas the signal power is enhanced by $(M+1)^2$. Thus, the gain in the signal to noise ratio at the output of the array (the array gain) is a factor of $M + 1$.

In the frequency domain, the above delay-and-sum operation becomes equivalent to linear weighting. Fourier transforming Eq. (14.3.1) we have

$$e(\omega) = \sum_{m=0}^{M} y_m(\omega) e^{-j\omega\tau_m}$$

which can be written compactly as:

$$e = \mathbf{a}^T \mathbf{y} \tag{14.3.3}$$

where $\mathbf{a}$ and $\mathbf{y}$ are the $(M + 1)$-vectors of weights and sensor outputs:

$$\mathbf{a} = \begin{bmatrix} e^{-j\omega\tau_0} \\ e^{-j\omega\tau_1} \\ \vdots \\ e^{-j\omega\tau_M} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_0(\omega) \\ y_1(\omega) \\ \vdots \\ y_M(\omega) \end{bmatrix}$$

From now on, we will concentrate on narrow-band arrays operating at a given frequency ω and the dependence on ω will not be shown explicitly. This assumes that the signals from all the sensors have been subjected to narrow-band prefiltering that leaves only the narrow operating frequency band. The beamformer now acts as a linear combiner, as shown in Fig. 14.3.1. A plane wave at the operating frequency ω, of amplitude

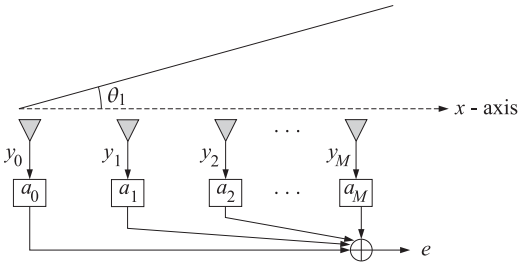


Fig. 14.3.1 Beamforming

A_1, and incident at the above angle θ_1, will have a value at the space-time point $(t,\mathbf{r})$ given by

$$A_1 e^{j\omega t - j\mathbf{k}\cdot\mathbf{r}}$$

Dropping the sinusoidal t-dependence and evaluating this expression on the x-axis, we have

$$A_1 e^{-jk_x x}$$

where k_x is the x-components of the wave vector $\mathbf{k}$

$$k_x = \frac{\omega}{c}\sin\theta_1$$

The value of this field at the mth sensor, $x_m = md$, is then

$$A_1 e^{-jmk_1}$$

where k_1 denotes the *normalized* wavenumber

$$k_1 = k_x d = \frac{\omega d}{c}\sin\theta_1 = \frac{2\pi d}{\lambda}\sin\theta_1\,, \quad \lambda = \text{wavelength} \tag{14.3.4}$$

This is the spatial analog of the digital frequency. To avoid aliasing effects arising from the spatial sampling process, the spatial sampling frequency $1/d$ must be greater than or equal to twice the spatial frequency $1/\lambda$ of the wave. Thus, we must have $d^{-1} \geq 2\lambda^{-1}$, or $d \leq \lambda/2$. Since $\sin\theta_1$ has magnitude less than one, the sampling condition forces k_1 to lie within the Nyquist interval

$$-\pi \leq k_1 \leq \pi$$

In this case the correspondence between k_1 and θ_1, is unique. For any angle θ and corresponding normalized wavenumber k, we introduce the *phasing*, or *steering vector*

$$\mathbf{s}_k = \begin{bmatrix} 1 \\ e^{jk} \\ e^{2jk} \\ \vdots \\ e^{Mjk} \end{bmatrix}, \quad k = \frac{2\pi d}{\lambda}\sin\theta \tag{14.3.5}$$

In this notation, the plane wave measured at the sensors is represented by the vector

$$\mathbf{y} = A_1\mathbf{s}_{k_1}^* = A_1\begin{bmatrix} 1 \\ e^{-jk_1} \\ e^{-2jk_1} \\ \vdots \\ e^{-Mjk_1} \end{bmatrix}$$

The steering vector of array weights $\mathbf{a}$, steered towards an arbitrary direction θ, is also expressed in terms of the phasing vector $\mathbf{s}_k$; we have

$$a_m = e^{-j\omega\tau_m} = e^{-j\omega(M-m)(d\sin\theta/c)} = e^{-jMk}e^{jmk}$$

or, ignoring the overall common phase e^{-jMk}, we have

$$\mathbf{a} = \mathbf{s}_k \qquad \text{(steering vector towards } k = \frac{2\pi d}{\lambda}\sin\theta\text{)} \tag{14.3.6}$$

The output of the beamformer, steered towards θ, is

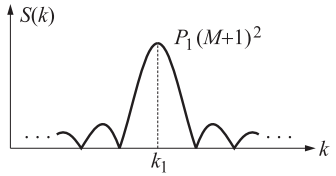
$$e = \mathbf{a}^T\mathbf{y} = \mathbf{s}_k^T\mathbf{y} = A_1\,\mathbf{s}_k^T\mathbf{s}_{k_1}^* = A_1\,\mathbf{s}_{k_1}^\dagger\mathbf{s}_k = A_1 W(k-k_1)^*$$

where $W(\cdot)$ was defined in Sec. 14.2. The mean output power of the beamformer steered towards k is

$$S(k) = E[e^*e] = \mathbf{a}^\dagger E[\mathbf{y}^*\mathbf{y}^T]\mathbf{a} = \mathbf{a}^\dagger R\mathbf{a} = \mathbf{s}_k^\dagger R\mathbf{s}_k$$

Using $\mathbf{y} = A_1\mathbf{s}_{k_1}^*$, we find $R = E[\mathbf{y}^*\mathbf{y}^T] = P_1\mathbf{s}_{k_1}\mathbf{s}_{k_1}^\dagger$, where $P_1 = E[|A_1|^2]$, and

$$\begin{aligned} S(k) &= \mathbf{s}_k^\dagger R\mathbf{s}_k = P_1\mathbf{s}_k^\dagger\mathbf{s}_{k_1}\mathbf{s}_{k_1}^\dagger\mathbf{s}_k \\ &= P_1|W(k-k_1)|^2 \end{aligned}$$



If the beam is steered on target, that is, if $\theta = \theta_1$, or, $k = k_1$, then $S(k_1) = P_1(M+1)^2$ and the output power is enhanced. The response pattern of the array has the same shape as the function $W(k)$, and therefore its resolution capability is limited to the width $\Delta k = 2\pi/(M+1)$ of the main lobe of $W(k)$. Setting $\Delta k = (2\pi d/\lambda)\Delta\theta$, we find the basic angular resolution to be $\Delta\theta = \lambda/\big((M+1)d\big)$, or, $\Delta\theta = \lambda/D$, where $D = (M+1)d$ is the effective aperture of the array. This is the classical *Rayleigh limit* on the resolving power of an optical system with aperture D [1092].

Next, we consider the problem of resolving the directions of arrival of multiple plane waves incident on an array in the presence of background noise. We assume L planes waves incident on an array of $M+1$ sensors from angles θ_i, $i = 1, 2, \ldots, L$. The incident field is sampled at the sensors giving rise to a series of "snapshots." At the nth snapshot time instant, the field received at the mth sensor has the form [1394]

$$y_m(n) = v_m(n) + \sum_{i=1}^{L} A_i(n)\,e^{-jmk_i}\,, \quad m = 0, 1, \ldots, M \tag{14.3.7}$$

where $A_i(n)$ is the amplitude of the ith wave (it would be a constant independent of time if we had exact sinusoidal dependence at the operating frequency), and k_i are the normalized wavenumbers related to the angles of arrival by

$$k_i = \frac{2\pi d}{\lambda}\sin\theta_i\,, \quad i = 1, 2, \ldots, L \tag{14.3.8}$$

and $v_m(n)$ is the background noise, which is assumed to be *spatially incoherent*, and also uncorrelated with the signal amplitudes $A_i(n)$; that is,

$$E[v_m(n)^* v_k(n)] = \sigma_v^2 \delta_{mk}\,, \quad E[v_m(n)^* A_i(n)] = 0 \tag{14.3.9}$$

Eq. (14.3.7) can be written in vector form as follows

$$\mathbf{y}(n) = \mathbf{v}(n) + \sum_{i=1}^{L} A_i(n)\mathbf{s}_{k_i}^* \tag{14.3.10}$$

The autocorrelation matrix of the signal field sensed by the array is

$$R = E[\mathbf{y}(n)^*\mathbf{y}(n)^T] = \sigma_v^2 I + \sum_{i,j=1}^{L} \mathbf{s}_{k_i} P_{ij} \mathbf{s}_{k_j}^\dagger \tag{14.3.11}$$

where I is the $(M+1)\times(M+1)$ unit matrix, and P_{ij} is the amplitude correlation matrix

$$P_{ij} = E[A_i(n)^* A_j(n)]\,, \quad 1 \le i, j \le L \tag{14.3.12}$$

If the sources are uncorrelated with respect to each other, the power matrix P_{ij} is diagonal. Introducing the $(M+1)\times L$ signal matrix

$$S = [\mathbf{s}_{k_1},\ \mathbf{s}_{k_2},\ \ldots,\ \mathbf{s}_{k_L}]$$

we may write Eq. (14.3.11) as

$$R = \sigma_v^2 I + SPS^\dagger \tag{14.3.13}$$

which is the same as Eq. (14.2.9) of the previous section. Therefore, the analytical expressions of the various spectral estimators can be transferred to this problem as well. We summarize the various spectrum estimators below:

$$\hat{S}_B(k) = \mathbf{s}_k^\dagger R \mathbf{s}_k \qquad \text{(conventional Bartlett beamformer)}$$

$$\hat{S}_{LP}(k) = \frac{1}{|\mathbf{s}_k^\dagger R^{-1}\mathbf{u}_0|^2} \qquad \text{(LP spectrum estimate)}$$

$$\hat{S}_{ML}(k) = \frac{1}{\mathbf{s}_k^\dagger R^{-1}\mathbf{s}_k} \qquad \text{(ML spectrum estimate)}$$

For example, for uncorrelated sources $P_{ij} = P_i\delta_{ij}$, the Bartlett spatial spectrum will be

$$\hat{S}_B(k) = \mathbf{s}_k^\dagger R \mathbf{s}_k = \sigma_v^2(M+1) + \sum_{i=1}^{L} P_i |W(k-k_i)|^2$$

which gives rise to peaks at the desired wavenumbers k_i from which the angles θ_i can be extracted. When the beam is steered towards the ith plane wave, the measured power at the output of the beamformer will be

$$\hat{S}_B(k_i) = \sigma_v^2(M+1) + P_i(M+1)^2 + \sum_{j\neq i} P_j |W(k_i - k_j)|^2$$

Ignoring the third term for the moment, we observe the basic improvement in the SNR offered by beamforming:

$$\frac{P_i(M+1)^2}{\sigma_v^2(M+1)} = \frac{P_i}{\sigma_v^2}(M+1)$$

If the sources are too close to each other [closer than the beamwidth of $W(k)$], the resolution ability of the beamformer worsens. In such cases, the alternative spectral estimates offer better resolution, with the LP estimate typically having a better performance. The resolution capability of both the ML and the LP estimates improves with higher SNR, whereas that of the conventional beamformer does not.

The Pisarenko method can also be applied here. As discussed in the previous section, the $(M+1)$-dimensional eigenvalue problem $R\mathbf{a} = \lambda\mathbf{a}$ has an L-dimensional *signal subspace* with eigenvalues greater than σ_v^2, and an $(M+1-L)$-dimensional *noise subspace* spanned by the degenerate eigenvectors belonging to the minimum eigenvalue of σ_v^2. Any vector $\mathbf{a}$ in the noise subspace will have at least L zeros at the desired wavenumber frequencies k_i, that is, the polynomial

$$A(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}$$

will have L zeros at

$$z_i = e^{jk_i}\,, \quad i = 1, 2, \ldots, L$$

and $(M-L)$ other spurious zeros. This can be seen as follows: If $R\mathbf{a} = \sigma_v^2\mathbf{a}$, then Eq. (14.3.13) implies that

$$(\sigma_v^2 I + SPS^\dagger)\mathbf{a} = \sigma_v^2\mathbf{a} \quad \Rightarrow \quad SPS^\dagger\mathbf{a} = 0$$

Dotting with $\mathbf{a}^\dagger$, we find that $\mathbf{a}^\dagger SPS^\dagger\mathbf{a} = 0$, and since P is assumed to be strictly positive definite, it follows that $S^\dagger\mathbf{a} = 0$, or

$$S^\dagger\mathbf{a} = \begin{bmatrix} A(k_1) \\ A(k_2) \\ \vdots \\ A(k_L) \end{bmatrix} = 0$$

The L largest eigenvalues of R correspond to the signal subspace eigenvectors and can be determined by reducing the original $(M+1)\times(M+1)$ eigenvalue problem for R into a smaller $L\times L$ eigenvalue problem.

Let $\mathbf{e}$ be any eigenvector in the signal subspace, that is, $R\mathbf{e} = \lambda\mathbf{e}$, with $\lambda > \sigma_v^2$. It follows that $SPS^\dagger\mathbf{e} = (\lambda - \sigma_v^2)\mathbf{e}$. Multiplying both sides by $S^\dagger$ we obtain $(S^\dagger SP)(S^\dagger\mathbf{e}) = (\lambda - \sigma_v^2)(S^\dagger\mathbf{e})$, which states that the L-dimensional vector $S^\dagger\mathbf{e}$ is an eigenvector of the

$L\times L$ matrix $S^\dagger SP$. We can turn this into a hermitian eigenvalue problem by factoring the power matrix P into its square root factors, $P = GG^\dagger$, and multiplying both sides of the reduced eigenvalue problem by $G^\dagger$. This gives $(G^\dagger S^\dagger SG)\,(G^\dagger S^\dagger \mathbf{e}) = (\lambda - \sigma_v^2)\,(G^\dagger S^\dagger \mathbf{e})$. Thus, we obtain the $L\times L$ hermitian eigenvalue problem

$$F\,\mathbf{f} = (\lambda - \sigma_v^2)\mathbf{f}\,, \quad \text{where} \quad F = G^\dagger S^\dagger SG\,, \quad \mathbf{f} = G^\dagger S^\dagger \mathbf{e} \tag{14.3.14}$$

The L signal subspace eigenvalues are obtained from the solution of this reduced eigenproblem. From each L-dimensional eigenvector $\mathbf{f}$, one can also construct the corresponding $(M+1)$-dimensional eigenvector $\mathbf{e}$. Because $\mathbf{e}$ lies in the signal subspace, it can be expressed as a linear combination of the plane waves

$$\mathbf{e} = \sum_{i=1}^{L} c_i \mathbf{s}_{k_i} = [\mathbf{s}_{k_1},\ \mathbf{s}_{k_2},\ \ldots,\ \mathbf{s}_{k_L}] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_L \end{bmatrix} = S\mathbf{c}$$

It, then, follows from Eq. (14.3.14) that

$$\mathbf{f} = G^\dagger S^\dagger \mathbf{e} = G^\dagger S^\dagger S\mathbf{c} \quad \Rightarrow \quad \mathbf{c} = (S^\dagger S)^{-1} G^{-\dagger}\mathbf{f}$$

and therefore,

$$\mathbf{e} = S\mathbf{c} = S(S^\dagger S)^{-1} G^{-\dagger}\mathbf{f} \tag{14.3.15}$$

Example 14.3.1: Using the above reduction method, determine the signal subspace eigenvectors and eigenvalues for the case of two equal-power uncorrelated plane waves and arbitrary M. The 2×2 matrix P becomes proportional to the identity matrix $P = P_1 I$. The reduced matrix F is then

$$F = P_1 S^\dagger S = P_1 \begin{bmatrix} \mathbf{s}_1^\dagger \mathbf{s}_1 & \mathbf{s}_1^\dagger \mathbf{s}_2 \\ \mathbf{s}_2^\dagger \mathbf{s}_1 & \mathbf{s}_2^\dagger \mathbf{s}_2 \end{bmatrix} = P_1 \begin{bmatrix} M+1 & W_{12} \\ W_{12}^* & M+1 \end{bmatrix}$$

where $\mathbf{s}_1 = \mathbf{s}_{k_1}$, $\mathbf{s}_2 = \mathbf{s}_{k_2}$, and $W_{12} = W(k_1 - k_2)$. In the equal-power case, F is always proportional to $S^\dagger S$, and therefore, $\mathbf{f}$ is an eigenvector of that. It follows that $(S^\dagger S)^{-1}\mathbf{f}$ will be a scalar multiple of $\mathbf{f}$ and that Eq. (14.3.15) can be simplified (up to a scalar factor) to $\mathbf{e} = S\mathbf{f}$. The two eigenvalues and eigenvectors of F are easily found to be

$$\lambda - \sigma_v^2 = P_1(M + 1 \pm |W_{12}|)\,, \quad \mathbf{f} = \begin{bmatrix} 1 \\ \pm e^{-j\theta_{12}} \end{bmatrix}$$

where θ_{12} is the phase of W_{12}. Using $\mathbf{e} = S\mathbf{f}$, it follows that the two signal subspace eigenvectors will be

$$\mathbf{e} = \mathbf{s}_1 \pm e^{-j\theta_{12}}\mathbf{s}_2$$

The eigenvalue spread of R is in this case

$$\frac{\lambda_{\max}}{\lambda_{\min}} = \frac{\sigma_v^2 + (M + 1 + |W_{12}|)P_1}{\sigma_v^2} = 1 + SNR\cdot(M + 1 + |W_{12}|)$$

where $SNR = P_1/\sigma_v^2$. It can be written in the form

$$\frac{\lambda_{\max}}{\lambda_{\min}} = 1 + SNR_{\text{eff}}\cdot(1 + |\cos\phi_{12}|)$$

where $SNR_{\text{eff}} = SNR\cdot(M+1)$ is the effective SNR of the array, or the array gain, and ϕ_{12} is the angle between the two signal vectors, that is, $\cos\phi_{12} = \mathbf{s}_1^\dagger \mathbf{s}_2/(\|\mathbf{s}_1\|\cdot\|\mathbf{s}_2\|)$. □

In practice, estimates of the covariance matrix R are used. For example, if the sensor outputs are recorded over N snapshots, that is, $\mathbf{y}(n)$, $n = 0, 1, \ldots, N-1$, then, the covariance matrix R may be estimated by replacing the ensemble average of Eq. (14.3.11) with the time-average:

$$\hat{R} = \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{y}(n)^*\mathbf{y}(n)^T \qquad \text{(empirical } R\text{)}$$

Since the empirical R will not be of the exact theoretical form of Eq. (14.3.11) the degeneracy of the noise subspace will be lifted somewhat. The degree to which this happens depends on how much the empirical R differs from the theoretical R. One can still use the minimum eigenvector $\mathbf{a}$ to define the polynomial $A(z)$ and from it an approximate Pisarenko spectral estimator

$$\hat{S}_P(k) = \frac{1}{|A(z)|^2}\,, \quad \text{where } z = e^{jk}$$

which will have sharp and possibly biased peaks at the desired wavenumber frequencies.

Example 14.3.2: Consider the case $L = M = 1$, defined by the theoretical autocorrelation matrix

$$R = \sigma_v^2 I + P_1 \mathbf{s}_{k_1}\mathbf{s}_{k_1}^\dagger = \begin{bmatrix} \sigma_v^2 + P_1 & P_1 e^{-jk_1} \\ P_1 e^{jk_1} & \sigma_v^2 + P_1 \end{bmatrix}$$

Its eigenvectors are:

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ -e^{jk_1} \end{bmatrix}, \quad \mathbf{e}_1 = \mathbf{s}_{k_1} = \begin{bmatrix} 1 \\ e^{jk_1} \end{bmatrix}$$

belonging to the eigenvalues $\lambda_0 = \sigma_v^2$ and $\lambda_1 = \sigma_v^2 + 2P_1$, respectively. Selecting as the array vector

$$\mathbf{a} = \mathbf{e}_0 = \begin{bmatrix} 1 \\ -e^{jk_1} \end{bmatrix}$$

we obtain a polynomial with a zero at the desired location:

$$A(z) = a_0 + a_1 z^{-1} = 1 - e^{jk_1} z^{-1}$$

Now, suppose that the analysis is based on an empirical autocorrelation matrix R which differs from the theoretical one by a small amount:

$$\hat{R} = R + \Delta R$$

Using standard first-order perturbation theory, we find the correction to the minimum eigenvalue λ_0 and eigenvector $\mathbf{e}_0$

$$\hat{\lambda}_0 = \lambda_0 + \Delta\lambda_0\,, \quad \hat{\mathbf{e}}_0 = \mathbf{e}_0 + \Delta c\,\mathbf{e}_1$$

where the first-order correction terms are

$$\Delta\lambda_0 = \frac{\mathbf{e}_0^\dagger(\Delta R)\mathbf{e}_0}{\mathbf{e}_0^\dagger\mathbf{e}_0}\,, \quad \Delta c = \frac{\mathbf{e}_1^\dagger(\Delta R)\mathbf{e}_0}{(\lambda_0 - \lambda_1)\mathbf{e}_1^\dagger\mathbf{e}_1}$$

The change induced in the zero of the eigenpolynomial is found as follows

$$\hat{\mathbf{a}} = \hat{\mathbf{e}}_0 = \begin{bmatrix} 1 \\ -e^{jk_1} \end{bmatrix} + \Delta c \begin{bmatrix} 1 \\ e^{jk_1} \end{bmatrix} = \begin{bmatrix} 1+\Delta c \\ -(1-\Delta c)e^{jk_1} \end{bmatrix}$$

so that

$$\hat{A}(z) = (1+\Delta c) - (1-\Delta c)e^{jk_1}z^{-1}$$

and the zero is now at

$$z_1 = \frac{1-\Delta c}{1+\Delta c}\, e^{jk_1} \simeq (1-2\Delta c)e^{jk_1}$$

to first-order in Δc. Since Δc is generally complex, the factor $(1-2\Delta c)$ will cause both a change (bias) in the phase of the zero e^{jk_1}, and will move it off the unit circle reducing the resolution. Another way to see this is to compute the value of the polynomial steered on target; that is,

$$\hat{A}(k_1) = \mathbf{s}_{k_1}^\dagger \mathbf{a} = \mathbf{s}_{k_1}^\dagger (\mathbf{e}_0 + \Delta c\, \mathbf{e}_1) = \Delta c\, \mathbf{s}_{k_1}^\dagger \mathbf{e}_1 = 2\Delta c$$

which is small but not zero. □

The high resolution properties of the Pisarenko and other eigenvector methods depend directly on the assumption that the background noise field is spatially incoherent, resulting in the special structure of the autocorrelation matrix R. When the noise is spatially coherent, a different eigenanalysis must be carried out. Suppose that the covariance matrix of the noise field $\mathbf{v}$ is

$$E[\mathbf{v}^*\mathbf{v}^T] = \sigma_v^2 Q$$

where Q reflects the spatial coherence of $\mathbf{v}$. Then the covariance matrix of Eq. (14.3.13) is replaced by

$$R = \sigma_v^2 Q + SPS^\dagger \tag{14.3.16}$$

The relevant eigenvalue problem is now the *generalized* eigenvalue problem

$$R\mathbf{a} = \lambda Q\mathbf{a} \tag{14.3.17}$$

Consider any such generalized eigenvector $\mathbf{a}$, and assume it is normalized such that

$$\mathbf{a}^\dagger Q\mathbf{a} = 1 \tag{14.3.18}$$

Then, the corresponding eigenvalue is expressed as

$$\lambda = \lambda \mathbf{a}^\dagger Q\mathbf{a} = \mathbf{a}^\dagger R\mathbf{a} = \sigma_v^2 \mathbf{a}^\dagger Q\mathbf{a} + \mathbf{a}^\dagger SPS^\dagger \mathbf{a} = \sigma_v^2 + \mathbf{a}^\dagger SPS^\dagger \mathbf{a}$$

which shows that the minimum eigenvalue is σ_v^2 and is attained whenever $\mathbf{a}^\dagger SPS^\dagger \mathbf{a} = 0$, or equivalently (assuming that P has full rank), $S^\dagger \mathbf{a} = 0$, or, $A(k_i) = 0$, $i = 1, 2, \ldots, L$. Therefore, the eigenpolynomial $A(z)$ can be used to determine the wavenumbers k_i.

Thus, the procedure is to solve the generalized eigenvalue problem and select the minimum eigenvector. This eigenvalue problem is also equivalent to the minimization problem

$$\mathcal{E} = \mathbf{a}^\dagger R\mathbf{a} = \min, \quad \text{subject to } \mathbf{a}^\dagger Q\mathbf{a} = 1 \tag{14.3.19}$$

This criterion, and its solution as the minimum eigenvector, is equivalent to the unconstrained minimization of the *Rayleigh quotient*, that is,

$$\frac{\mathbf{a}^\dagger R\mathbf{a}}{\mathbf{a}^\dagger Q\mathbf{a}} = \min \quad \Leftrightarrow \quad R\mathbf{a} = \lambda_{\min} Q\mathbf{a} \tag{14.3.20}$$

The practical implementation of the method requires knowledge of the noise covariance matrix Q, which is not always possible to obtain. Covariance difference methods [1135–1138] can be used in the case of unknown Q. Such methods work with measurements from two different arrays, translated or rotated with respect to each other. Assuming that the background noise is invariant under translation or rotation, the covariance matrices of the two arrays will be $R_1 = S_1 P_1 S_1^\dagger + \sigma_v^2 Q$ and $R_2 = S_2 P_2 S_2^\dagger + \sigma_v^2 Q$. The eigenstructure of the covariance difference $R_1 - R_2 = S_1 P_1 S_1^\dagger - S_2 P_2 S_2^\dagger$ can be used to extract the signal information.

The two spectral analysis problems discussed in this and the previous section—direction finding and harmonic retrieval—are *dual* to each other; one dealing with spatial frequencies and the other with time frequencies. The optimum processing part is the same in both cases. The optimum processor does not care how its inputs are supplied, it only "sees" the *correlations* among the inputs and its function is to "break down" these correlations thereby extracting the sinusoidal components. The two cases differ only in the way the inputs to the optimum processor are supplied. This conceptual separation between the input part and the optimum processing part is shown in Fig. 14.3.2. In the time series case, the correlations among the inputs are *sequential correlations* in time, whereas in the array case they are *spatial correlations*, such as those that exist along a coherent wavefront.

A problem related, but not identical, to direction finding is that of *optimum beamforming* for interference nulling [1093–1095,1352,1167–1170]. In this case, one of the plane waves, say, $\mathbf{s}_{k_1}$, is assumed to be a desired plane wave with *known* direction of arrival θ_1, or wavenumber k_1. The other plane waves are considered as interferers or jammers to be nulled. Assuming for simplicity uncorrelated sources, the covariance matrix (14.3.11) may be decomposed into a part due to the desired signal and a part due to the noise plus interference:

$$R = \sigma_v^2 I + \sum_{i=1}^{L} P_i \mathbf{s}_i \mathbf{s}_i^\dagger = P_1 \mathbf{s}_1 \mathbf{s}_1^\dagger + \left[\sigma_v^2 I + \sum_{i=2}^{L} P_i \mathbf{s}_i \mathbf{s}_i^\dagger\right] = P_1 \mathbf{s}_1 \mathbf{s}_1^\dagger + R_n$$

where we denoted $\mathbf{s}_i = \mathbf{s}_{k_i}$. The output power of the array with weights $\mathbf{a}$ will be

$$\mathcal{E} = \mathbf{a}^\dagger R\mathbf{a} = P_1 |\mathbf{s}_1^\dagger \mathbf{a}|^2 + \mathbf{a}^\dagger R_n \mathbf{a} \tag{14.3.21}$$

The first term is the output power due to the desired signal; the second term is due to the presence of noise plus interference. This expression suggests two possible optimization criteria for $\mathbf{a}$. First, choose $\mathbf{a}$ to maximize the relative *signal to noise plus interference ratio* (SNIR):

$$SNIR = \frac{P_1 |\mathbf{s}_1^\dagger \mathbf{a}|^2}{\mathbf{a}^\dagger R_n \mathbf{a}} = \max \tag{14.3.22}$$

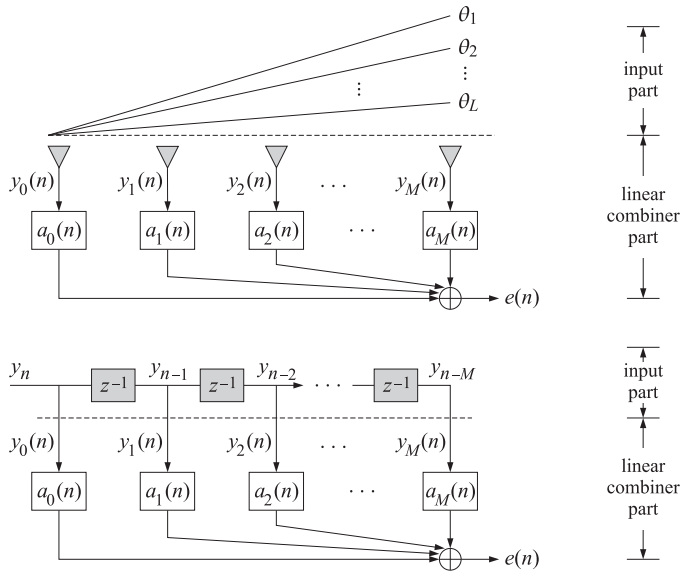


Fig. 14.3.2 Duality between time series and array problems.

The second criterion is to keep the output of the array toward the look direction $\mathbf{s}_1$ fixed, while minimizing the output power:

$$\mathbf{s}_1^\dagger \mathbf{a} = 1 \quad \text{and} \quad \mathcal{E} = \mathbf{a}^\dagger R \mathbf{a} = P_1 + \mathbf{a}^\dagger R_n \mathbf{a} = \min \tag{14.3.23}$$

This is equivalent to minimizing the noise plus interference term $\mathbf{a}^\dagger R_n \mathbf{a}$. These two criteria are essentially equivalent. This is seen as follows. Equation (14.3.22) is equivalent to *minimizing* the inverse function $SNIR^{-1}$. Adding one to it, we obtain the equivalent criterion

$$1 + SNIR^{-1} = 1 + \frac{\mathbf{a}^\dagger R_n \mathbf{a}}{P_1 |\mathbf{s}_1^\dagger \mathbf{a}|^2} = \frac{\mathbf{a}^\dagger R \mathbf{a}}{P_1 |\mathbf{s}_1^\dagger \mathbf{a}|^2} = \min$$

This is identical to the Rayleigh quotient (14.3.20) with the choice $Q = P_1 \mathbf{s}_1 \mathbf{s}_1^\dagger$. It is equivalent to the minimum eigenvector solution of

$$R\mathbf{a} = \lambda Q \mathbf{a} = \lambda P_1 \mathbf{s}_1 \mathbf{s}_1^\dagger \mathbf{a} = \mu \mathbf{s}_1 \quad \Rightarrow \quad \mathbf{a} = \mu R^{-1} \mathbf{s}_1$$

where we put all the scalar factors into μ. Similarly, the constraint $\mathbf{s}_1^\dagger \mathbf{a} = 1$ implies that $\mathbf{a}^\dagger Q_1 \mathbf{a} = 1$ with $Q_1 = \mathbf{s}_1 \mathbf{s}_1^\dagger$. It follows from Eq. (14.3.19), applied with Q_1, that the solution of Eq. (14.3.23) is again the generalized eigenvector

$$R\mathbf{a} = \lambda_1 Q_1 \mathbf{a} = \lambda_1 \mathbf{s}_1 \mathbf{s}_1^\dagger \mathbf{a} = \mu_1 \mathbf{s}_1 \quad \Rightarrow \quad \mathbf{a} = \mu_1 R^{-1} \mathbf{s}_1$$

Thus, up to a scale factor, the optimum solution for both criteria is

$$\mathbf{a} = R^{-1} \mathbf{s}_1 \tag{14.3.24}$$

This solution admits, yet, a third interpretation as the *Wiener solution* of an ordinary mean-square estimation problem. The term $\mathbf{y}_1(n) = A_1(n)\mathbf{s}_1^*$ of Eq. (14.3.10) is the desired signal. A reference signal $x(n)$ could be chosen to correlate highly with this term and not at all with the other terms in Eq. (14.3.10). For example, $x(n) = f(n)A_1(n)$. The array weights can be designed by demanding that the scalar output of the array, $\mathbf{a}^T \mathbf{y}(n)$, be the best mean-square estimate of $x(n)$. This gives the criterion

$$E\big[|x(n) - \mathbf{a}^T \mathbf{y}(n)|^2\big] = E\big[|x(n)|^2\big] - \mathbf{a}^\dagger \mathbf{r} - \mathbf{r}^\dagger \mathbf{a} + \mathbf{a}^\dagger R \mathbf{a}$$

where we set $\mathbf{r} = E[x(n)\mathbf{y}(n)^*]$. Minimizing with respect to $\mathbf{a}$ (and $\mathbf{a}^*$) gives the Wiener solution $\mathbf{a} = R^{-1}\mathbf{r}$. Now, because $x(n)$ is correlated only with $\mathbf{y}_1(n)$, it follows that $\mathbf{r}$ will be proportional to $\mathbf{s}_1$:

$$\mathbf{r} = E[x(n)\mathbf{y}(n)^*] = E[x(n)\mathbf{y}_1(n)^*] = E[x(n)A_1(n)^*]\,\mathbf{s}_1$$

Thus, again up to a scale, we obtain the solution (14.3.24). Using the matrix inversion lemma (see Problem 14.6), we can write the inverse of $R = P_1 \mathbf{s}_1 \mathbf{s}_1^\dagger + R_n$, in the form

$$R^{-1} = R_n^{-1} - c R_n^{-1} \mathbf{s}_1 \mathbf{s}_1^\dagger R_n^{-1}\,, \quad c = (P_1^{-1} + \mathbf{s}_1^\dagger R_n^{-1} \mathbf{s}_1)^{-1}$$

Acting by both sides on $\mathbf{s}_1$, we find

$$R^{-1}\mathbf{s}_1 = c_1 R_n^{-1} \mathbf{s}_1\,, \quad c_1 = c P_1^{-1}$$

Therefore, the optimal solution can also be written (up to another scale factor) in terms of the noise plus interference covariance matrix R_n:

$$\mathbf{a} = R_n^{-1} \mathbf{s}_1 \tag{14.3.25}$$

These solutions, known as *steered solutions*, are sometimes modified to include arbitrary tapering weights for the array—replacing the steering vector $\mathbf{s}_1$ with a generalized steering vector

$$\mathbf{s} = \begin{bmatrix} b_0 \\ b_1 e^{jk_1} \\ \vdots \\ b_M e^{jk_1 M} \end{bmatrix} = B\,\mathbf{s}_1\,, \quad B = \operatorname{diag}\{b_0, b_1, \ldots, b_M\} \tag{14.3.26}$$

The weights b_m can be chosen to attain a prescribed shape for the quiescent response of the array in absence of interference. Typical choices are (with $k_1 = 0$)

$$\mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

To appreciate the properties of the optimum solution, we consider the case of one jammer, so that

$$R = P_1 \mathbf{s}_1 \mathbf{s}_1^\dagger + R_n\,, \quad R_n = \sigma_v^2 I + P_2 \mathbf{s}_2 \mathbf{s}_2^\dagger$$

Using the matrix inversion lemma on R_n^{-1}, we obtain

$$R_n^{-1} = \frac{1}{\sigma_v^2}\left[I - \frac{1}{\sigma_v^2 P_2^{-1} + \mathbf{s}_2^\dagger \mathbf{s}_2}\,\mathbf{s}_2\mathbf{s}_2^\dagger\right]$$

Therefore, the optimum solution given by Eq. (14.3.25) becomes

$$\mathbf{a} = R_n^{-1}\mathbf{s}_1 = \frac{1}{\sigma_v^2}\left[\mathbf{s}_1 - \frac{P_2 W(k_2-k_1)}{\sigma_v^2 + P_2(M+1)}\,\mathbf{s}_2\right]$$

where we used $\mathbf{s}_2^\dagger\mathbf{s}_2 = M+1$ and $\mathbf{s}_2^\dagger\mathbf{s}_1 = W(k_2-k_1)$. Dropping the overall factor of $1/\sigma_v^2$, we find for the array pattern as a function of wavenumber k or angle θ

$$A(k) = \mathbf{s}_k^\dagger\mathbf{a} = W(k-k_1) - \frac{P_2 W(k_2-k_1)}{\sigma_v^2 + P_2(M+1)}\,W(k-k_2) \tag{14.3.27}$$

In the absence of the jammer, $P_2 = 0$, we obtain the usual quiescent Bartlett response, $W(k-k_1)$. The presence of the second term, called a *retrodirective beam*, will partially distort the quiescent pattern but it will suppress the jammer. Indeed, the array response steered toward the jammer at $k = k_2$, becomes

$$A(k_2) = W(k_2-k_1) - \frac{P_2 W(k_2-k_1)}{\sigma_v^2 + P_2(M+1)}\,W(0) = \frac{W(k_2-k_1)}{\sigma_v^2 + P_2(M+1)}$$

The ratio $A(k_2)/W(k_2-k_1)$ is the array response, in the direction of the jammer, relative to the quiescent response. Thus, if the signal to noise ratio $SNR_2 = P_2/\sigma_v^2$ is large, the jammer will be suppressed. Only in the limit of infinite SNR is the jammer completely nulled.

The reason for the incomplete nulling can be traced, as in the case of linear prediction, to the linear constraint on the weights (14.3.23). To get exact nulling of the jammers, we must force the zeros of the polynomial $\mathbf{a}$ to lie on the unit circle at the jammer positions. As suggested in Problem 14.13, this can be accomplished by imposing a quadratic constraint $\mathbf{a}^\dagger Q\mathbf{a} = \text{const.}$, where Q must be chosen as $Q = \sigma_v^2 I + P_1\mathbf{s}_1\mathbf{s}_1^\dagger$ instead of $Q = P_1\mathbf{s}_1\mathbf{s}_1^\dagger$. The optimum weight is the minimum eigenvector solution of the generalized eigenproblem $R\mathbf{a} = \lambda Q\mathbf{a}$ and will have exact zeros at the jammer positions. As in the linear prediction case, the linearly constrained optimum beamformer solution tends to this eigenvector solution in the limit $\sigma_v^2 \to 0$.

14.4 Eigenvector Methods

The single most important property of eigenvector methods is that, at least in principle, they produce unbiased frequency estimates with infinite resolution, regardless of the signal to noise ratios. This property is not shared by the older methods. For example, the resolution of the Bartlett method is limited by the array aperture, and the resolution of the linear prediction and maximum likelihood methods degenerates with decreasing SNRs. Because of this property, eigenvector methods have received considerable attention in signal processing and have been applied to several problems, such as harmonic retrieval, direction finding, echo resolution, and pole identification [1084,1109–1163].

In the remainder of this chapter, we discuss the theoretical aspects of eigenvector methods in further detail, and present several versions of such methods, such as MUSIC, Minimum-Norm, and ESPRIT.

We have seen that the eigenspace of the covariance matrix R consists of two mutually orthogonal parts: the $(M+1-L)$-dimensional noise subspace spanned by the eigenvectors belonging to the minimum eigenvalue σ_v^2, and the L-dimensional signal subspace spanned by the remaining L eigenvectors having eigenvalues strictly greater than σ_v^2. Let $\mathbf{e}_i$, $i = 0, 1, \ldots, M$, denote the orthonormal eigenvectors of R in order of increasing eigenvalue, and let $K = M+1-L$ denote the dimension of the noise subspace. Then, the first K eigenvectors, $\mathbf{e}_i$, $i = 0, 1, \ldots, K-1$, form an orthonormal basis for the noise subspace, and the last L eigenvectors, $\mathbf{e}_i$, $i = K, K+1, \ldots, M$, form a basis for the signal subspace. We arrange these basis vectors into the eigenvector matrices:

$$E_N = [\mathbf{e}_0,\ \mathbf{e}_1,\ \ldots,\ \mathbf{e}_{K-1}], \quad E_S = [\mathbf{e}_K,\ \mathbf{e}_{K+1},\ \ldots,\ \mathbf{e}_M] \tag{14.4.1}$$

Their dimensions are $(M+1)\times K$ and $(M+1)\times L$. The full eigenvector matrix of R is:

$$E = [E_N, E_S] = [\mathbf{e}_0,\ \mathbf{e}_1,\ \ldots,\ \mathbf{e}_{K-1},\ \mathbf{e}_K,\ \mathbf{e}_{K+1},\ \ldots,\ \mathbf{e}_M] \tag{14.4.2}$$

The *orthonormality* of the eigenvectors is expressed by the unitarity property $E^\dagger E = I$, where I is the $(M+1)$-dimensional unit matrix. The unitarity can be written in terms of the submatrices (14.4.1):

$$E_N^\dagger E_N = I_K\,, \quad E_N^\dagger E_S = 0\,, \quad E_S^\dagger E_S = I_L \tag{14.4.3}$$

where I_K and I_L are the $K\times K$ and $L\times L$ unit matrices. The *completeness* of the eigenvectors is expressed also by the unitarity of E, i.e., $EE^\dagger = I$. In terms of the submatrices, it reads:

$$E_N E_N^\dagger + E_S E_S^\dagger = I \tag{14.4.4}$$

These two terms are the *projection* matrices onto the noise and signal subspaces. We have seen that the L signal direction vectors $\mathbf{s}_{k_i}$ belong to the signal subspace, and therefore, are expressible as linear combinations of E_S. It follows that the signal matrix $S = [s_{k_1}, \ldots, s_{k_L}]$ is a non-orthogonal basis of the signal subspace and must be related to E_S by $S = E_S C$, where C is an $L\times L$ *invertible* matrix. Using the orthonormality of E_S, it follows that $S^\dagger S = C^\dagger E_S^\dagger E_S C = C^\dagger C$. Thus, the projector onto the signal subspace may be written as

$$P_S = E_S E_S^\dagger = (SC^{-1})(C^{-\dagger}S^\dagger) = S(C^\dagger C)^{-1}S^\dagger = S(S^\dagger S)^{-1}S^\dagger \tag{14.4.5}$$

We may also obtain a non-orthogonal, but useful, basis for the noise subspace. We have seen that an $(M+1)$-dimensional vector $\mathbf{e}$ lies in the noise subspace—equivalently, it is an eigenvector belonging to the minimum eigenvalue σ_v^2—if and only if the corresponding order-M eigenfilter $E(z)$ has L zeros on the unit circle at the desired signal zeros, $z_i = e^{jk_i}$, $i = 1, 2, \ldots, L$, and has $M - L = K - 1$ other spurious zeros. Such a polynomial will factor into the product:

$$E(z) = A(z)F(z) = A(z)\left[f_0 + f_1 z^{-1} + \cdots + f_{K-1}z^{-(K-1)}\right] \tag{14.4.6}$$

where the zeros of $F(z)$ are the spurious zeros, and $A(z)$ is the *reduced-order* polynomial of order L whose zeros are the desired zeros; that is,

$$A(z) = \prod_{i=1}^{L} (1 - e^{jk_i} z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_L z^{-L} \tag{14.4.7}$$

Introducing the K delayed polynomials:

$$B_i(z) = z^{-i} A(z)\,, \quad i = 0, 1, \ldots, K-1 \tag{14.4.8}$$

we may write Eq. (14.4.6) in the form

$$E(z) = f_0 B_0(z) + f_1 B_1(z) + \cdots + f_{K-1} B_{K-1}(z) = \sum_{i=0}^{K-1} f_i B_i(z) \tag{14.4.9}$$

and in coefficient form

$$\mathbf{e} = \sum_{i=0}^{K-1} f_i \mathbf{b}_i = [\mathbf{b}_0,\ \mathbf{b}_1,\ \ldots,\ \mathbf{b}_{K-1}] \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{K-1} \end{bmatrix} \equiv B\,\mathbf{f} \tag{14.4.10}$$

Because each of the polynomials $B_i(z)$ has L desired zeros, it follows that the corresponding vectors $\mathbf{b}_i$ will lie in the noise subspace. Thus, the matrix B defined in Eq. (14.4.10) will be a non-orthogonal basis of the noise subspace. It is a useful basis because the expansion coefficients $\mathbf{f}$ of any noise subspace vector $\mathbf{e}$ are the coefficients of the spurious polynomial $F(z)$ in the factorization (14.4.6). Put differently, Eq. (14.4.10) parametrizes explicitly the spurious degrees of freedom arising from the K-fold degeneracy of the minimum eigenvalue. The basis vectors $\mathbf{b}_i$, considered as $(M+1)$-dimensional vectors, are simply the delayed versions of the vector of coefficients, $\mathbf{a} = [1, a_1, \ldots, a_L]^T$, of the polynomial $A(z)$, that is,

$$\mathbf{b}_i = [\underbrace{0,\ \ldots,\ 0}_{i \text{ zeros}},\ 1,\ a_1,\ \ldots,\ a_L,\ \underbrace{0,\ \ldots,\ 0}_{K-1-i \text{ zeros}}]^T \tag{14.4.11}$$

For example, in the case $L = 2$ and $M = 5$, we have $K = M + 1 - L = 4$ and B is:

$$B = [\mathbf{b}_0,\ \mathbf{b}_1,\ \mathbf{b}_2,\ \mathbf{b}_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1 & 1 & 0 & 0 \\ a_2 & a_1 & 1 & 0 \\ 0 & a_2 & a_1 & 1 \\ 0 & 0 & a_2 & a_1 \\ 0 & 0 & 0 & a_2 \end{bmatrix}$$

It follows that the basis B must be linearly related to the orthonormal basis E_N by $B = E_N C$, where C is a $K \times K$ invertible matrix. Then, $B^\dagger B = C^\dagger C$ and the projector onto the noise subspace becomes:

$$P_N = E_N E_N^\dagger = (BC^{-1})\,(C^{-\dagger} B^\dagger) = B(C^\dagger C)^{-1} B^\dagger = B(B^\dagger B)^{-1} B^\dagger \tag{14.4.12}$$

Combining Eqs. (14.4.12) and (14.4.5), we may write the completeness relation (14.4.4) in terms of the non-orthogonal bases B and S:

$$B(B^\dagger B)^{-1} B^\dagger + S(S^\dagger S)^{-1} S^\dagger = I \tag{14.4.13}$$

The objective of all eigenvector methods is to estimate the signal zeros $z_i = e^{jk_i}$, $i = 1, 2, \ldots, L$. All methods begin with an eigenanalysis of R, such that E_N and E_S are available. In practice, the eigenanalysis is based on the sample covariance matrix $\hat{R}$ defined on the basis of a finite number of snapshots, say N:

$$\hat{R} = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{y}(n)^* \mathbf{y}(n)^T \tag{14.4.14}$$

Sometimes, a symmetrized version is preferred, obtained from $\hat{R}$ by

$$\hat{R}_s = \frac{1}{2}(\hat{R} + J\hat{R}^* J) \tag{14.4.15}$$

where J the $(M+1)$-dimensional reversing matrix. The matrix $\hat{R}_s$ is invariant under reversal, that is, $J\hat{R}_s J = \hat{R}_s^*$. This version is appropriate when the theoretical R is Toeplitz. This case arises if and only if the $L \times L$ power matrix P is diagonal; that is, when the L sources are mutually uncorrelated. As the number of snapshots increases, the eigenstructure of $\hat{R}$ or $\hat{R}_s$ becomes a better and better approximation of the eigenstructure of R. Such asymptotic statistical properties will be discussed in Sec. 14.11. Next, we discuss several practical approaches.

14.5 MUSIC method

Let $E_i(z)$ denote the eigenfilters of the noise subspace eigenvectors $\mathbf{e}_i$, $i = 0, 1, \ldots, K-1$. According to Eq. (14.4.5), we can write $E_i(z) = A(z) F_i(z)$, which shows that $E_i(z)$ have a common set of L zeros at the desired signal locations, but each may have a different set of $K-1$ spurious zeros. It is possible for these spurious zeros to lie very close to or on the unit circle. Therefore, if only one eigenfilter is used, there may be an ambiguity in distinguishing the desired zeros from the spurious ones. The *multiple signal classification* (MUSIC) method [1110] attempts to average out the effect of the spurious zeros by forming the sum of the magnitude responses of the K noise subspace eigenfilters, that is, setting $z = e^{jk}$,

$$\frac{1}{K} \sum_{i=0}^{K-1} |E_i(k)|^2 = |A(k)|^2 \frac{1}{K} \sum_{i=0}^{K-1} |F_i(k)|^2$$

Because the polynomials $F_i(z)$ are all different, the averaging operation will tend to smear out any spurious zero of any individual term in the sum. Thus, the above expression will effectively vanish only at the L desired zeros of the common factor $|A(k)|^2$. The MUSIC pseudospectrum is defined as the inverse

$$S_{MUS}(k) = \frac{1}{\dfrac{1}{K} \displaystyle\sum_{i=0}^{K-1} |E_i(k)|^2} \tag{14.5.1}$$

It will exhibit peaks at the L desired wavenumbers k_i, $i = 0, 1, \ldots, L$. The sum may also be replaced by a weighted sum [1118]. The sum may be written compactly in terms of the projection matrices onto the noise or signal subspaces. Noting that $|E_i(k)|^2 = \mathbf{s}_k^\dagger(\mathbf{e}_i\mathbf{e}_i^\dagger)\mathbf{s}_k$, we find

$$\sum_{i=0}^{K-1} |E_i(k)|^2 = \mathbf{s}_k^\dagger \left[\sum_{i=0}^{K-1} \mathbf{e}_i\mathbf{e}_i^\dagger\right]\mathbf{s}_k = \mathbf{s}_k^\dagger E_N E_N^\dagger \mathbf{s}_k = \mathbf{s}_k^\dagger (I - E_S E_S^\dagger)\mathbf{s}_k$$

where we used Eq. (14.4.4). The practical version of the MUSIC method is summarized below:

1. Based on a finite number of snapshots, compute the sample covariance matrix $\hat{R}$, solve its eigenproblem, and obtain the estimated eigenvector matrix E with eigenvalues arranged in increasing order.
2. Estimate the dimension K of the noise subspace as the number of the smallest, approximately equal, eigenvalues. This can be done systematically using the AIC or MDL criteria discussed later. The estimated number of plane waves will be $L = M + 1 - K$. Divide E into its noise and signal subspace parts, E_N and E_S.
3. Compute the spectrum (14.5.1) and extract the desired wavenumbers k_i from the L peaks in this spectrum.

The Akaike (AIC) and minimum description length (MDL) information-theoretic criteria have been suggested to determine the number of plane waves that are present, or equivalently, the dimension of the noise subspace [1125]. They are defined by

$$\begin{aligned} AIC(k) &= -2Nk\mathcal{L}(k) + 2(M+1-k)(M+1+k) \\ MDL(k) &= -Nk\mathcal{L}(k) + \frac{1}{2}(M+1-k)(M+1+k)\log(N) \end{aligned} \tag{14.5.2}$$

for $k = 1, 2, \ldots, M+1$, where N is the number of snapshots and $\mathcal{L}(k)$ is a likelihood function defined as the log of the ratio of the harmonic and arithmetic means of the first k estimated eigenvalues $\{\hat{\lambda}_0, \hat{\lambda}_1, \ldots, \hat{\lambda}_{k-1}\}$ of $\hat{R}$; namely,

$$\mathcal{L}(k) = \ln\left[\frac{(\hat{\lambda}_0\hat{\lambda}_1\cdots\hat{\lambda}_{k-1})^{1/k}}{\frac{1}{k}(\hat{\lambda}_0 + \hat{\lambda}_1 + \cdots + \hat{\lambda}_{k-1})}\right]$$

The dimension K of the noise subspace is chosen to be that k that *minimizes* the functions $AIC(k)$ or $MDL(k)$. The above definition is equivalent to that of [1125], but produces the value of K instead of L. The function **aicmdl** takes as inputs the $M+1$ estimated eigenvalues in increasing order and the number N, and computes the values of the AIC and MDL functions. Once K is known, an estimate of the minimum eigenvalue can be obtained by

$$\hat{\sigma}_v^2 = \hat{\lambda}_{\min} = \frac{1}{K}(\hat{\lambda}_0 + \hat{\lambda}_1 + \cdots + \hat{\lambda}_{K-1}) \tag{14.5.3}$$

Next, we present some simulation examples. First, we compare the MUSIC method against the linear prediction method. We considered two uncorrelated equal-power

plane waves incident on an array of 8 sensors ($M = 7$). The SNR of the waves, defined by $SNR_i = 10\log_{10}(P_i/\sigma_v^2)$, was -5 dB and their wavenumbers $k_1 = 0.2\pi$ and $k_2 = 0.4\pi$. For half-wavelength array spacing ($d = \lambda/2$), these correspond, through (14.3.8), to the angles of arrival $\theta_1 = 11.54^o$ and $\theta_2 = 23.58^o$.

The number of snapshots was $N = 500$. The snapshots were simulated using Eq. (14.3.10). Each $\mathbf{v}(n)$ was generated as a complex vector of $M+1$ zero-mean independent gaussian components of variance $\sigma_v^2 = 1$.

Note that to generate a zero-mean complex random variable v of variance σ_v^2, one must generate two zero-mean independent real random variables v_1 and v_2, each with variance $\sigma_v^2/2$ and set $v = v_1 + jv_2$; then, $E[v^*v] = E[v_1^2] + E[v_2^2] = 2(\sigma_v^2/2) = \sigma_v^2$. The amplitudes $A_i(n)$ were assumed to have only random phases; that is, $A_i(n) = (P_i)^{1/2}e^{j\phi_{in}}$, where ϕ_{in}, were independent angles uniformly distributed in $[0, 2\pi]$. The function **snap** takes as input an integer seed, generates a snapshot vector $\mathbf{y}$, and updates the seed. Successive calls to **snap**, in conjunction with the (complex version) of the function **sampcov**, can be used to generate the sample covariance matrix $\hat{R}$. In this particular example, we used the symmetrized version $\hat{R}_s$, because the two sources were uncorrelated.

Fig. 14.5.1 shows the MUSIC spectrum computed using Eq. (14.5.1) together with the LP spectrum $S_{LP}(k) = 1/|\mathbf{s}_k^\dagger\mathbf{a}|^2$, where $\mathbf{a} = \hat{R}_s^{-1}\mathbf{u}_0$. Because each term in the sum (14.5.1) arises from a unit-norm eigenvector, we have normalized the LP vector $\mathbf{a}$ also to unit norm for the purpose of plotting the two spectra on the same graph. Increasing the number of snapshots will improve the MUSIC spectrum because the covariance matrix $\hat{R}_s$ will become a better estimate of R, but it will not improve the LP spectrum because the theoretical LP spectrum does not perform well at low SNRs.

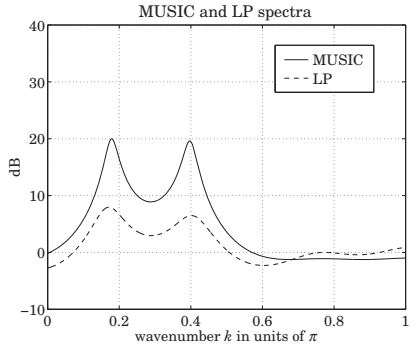


Fig. 14.5.1 MUSIC and LP spectra.

To facilitate the computation and manipulation of spectra, we have included the following small functions. The built-in function **norm** converts a vector a to a unit-norm vector and the function **freqz** computes the magnitude response squared, $|A(k)|^2 = |\mathbf{s}_k^\dagger\mathbf{a}|^2$, of an Mth order filter $\mathbf{a}$ at a specified number of equally-spaced frequency points within the right-half of the Nyquist interval, $0 \le k \le \pi$. It can be modified easily to include the entire Nyquist interval or any subinterval. The function **invresp** inverts a

given spectrum, $S(k) \to 1/S(k)$. The functions **abs2db** and **db2abs** convert a spectrum from absolute units to decibels and back, $S(k) = 10\log_{10} S(k)$. The function **select** picks out any eigenvector from the $M+1$ ones of the eigenvector matrix E. The function **music** computes Eq. (14.5.1) over a specified number of frequency points. It is built out of the functions **select**, **freqz**, and **invresp**.

In the second simulation example, we increased the SNR of the two plane waves to 10 dB and reduced the number of snapshots to $N = 100$. The theoretical and empirical eigenvalues of R and $\hat{R}_s$, were found to be

i	0	1	2	3	4	5	6	7
λ_i	1	1	1	1	1	1	61.98	100.02
$\hat{\lambda}_i$	0.70	0.76	0.83	0.87	1.05	1.28	64.08	101.89

The values of the AIC and MDL functions were

k	1	2	3	4	5	6	7	8
$AIC(k)$	126.0	120.3	111.4	98.7	87.2	81.1	2544.2	3278.2
$MDL(k)$	145.1	138.3	127.4	111.9	94.4	77.0	1291.6	1639.1

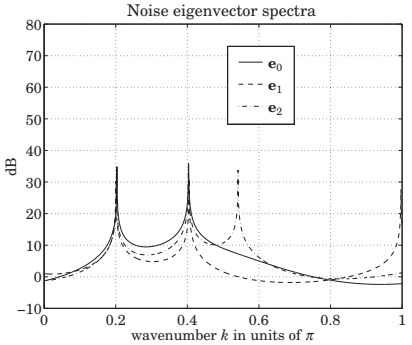


Fig. 14.5.2 Spectra of the first three noise subspace eigenvectors.

Both functions achieve their minimum value at $K = 6$ and therefore, $L = M + 1 - K = 2$. The estimated value of σ_v^2, computed by Eq. (14.5.3), was $\hat{\sigma}_v^2 = 0.915$. Fig. 14.5.2 shows the spectra of the first three noise subspace eigenvectors; namely, $S_i(k) = 1/|E_i(k)|^2 = 1/|\mathbf{s}_k^\dagger \mathbf{e}_i|^2$, for $i = 0, 1, 2$. We note the presence of a common set of peaks at the two desired wavenumbers and several spurious peaks. The spurious peaks are different, however, in each spectrum and therefore, the averaging operation will tend to eliminate them. The averaged MUSIC spectrum, based on all $K = 6$ noise subspace eigenvectors, is plotted in Fig. 14.6.1 using the same scale as in Fig. 14.5.2.

The averaging operation has had two effects. First, the removal of all spurious peaks and second, the broadening and reduction in sharpness of the two desired peaks. This broadening is the result of statistical sampling; that is, using $\hat{R}$ instead of R, causes small biases in the peaks of individual eigenvectors about their true locations. These biases

are not inherent in the theoretical method, as they are in the linear prediction case; they are statistical in nature and disappear in the limit of large number of snapshots. Fig. 14.6.1 also shows the performance of the minimum-norm method, which we discuss next. It appears to produce somewhat sharper peaks than MUSIC, but it can sometimes exhibit higher levels of spurious peaks.

14.6 Minimum-Norm Method

The minimum-norm method [1117] attempts to eliminate the effect of spurious zeros by pushing them inside the unit circle, leaving the L desired zeros on the circle. This is accomplished by finding a noise subspace vector $\mathbf{d} = [d_0, d_1, \ldots, d_M]^T$ such that the corresponding eigenfilter $D(z)$ will have all its spurious zeros within the unit circle. This means that in the factorization (14.4.6), $D(z) = A(z)F(z)$, the spurious polynomial $F(z)$ must be chosen to have all its zeros strictly inside the unit circle, equivalently, $F(z)$ must be a minimum-phase polynomial. If $F(z)$ were the prediction-error filter of a linear prediction problem, then it would necessarily be a minimum-phase filter. Thus, the design strategy for $\mathbf{d}$ is to make $F(z)$ a linear prediction filter. This can be done by requiring that $\mathbf{d}$ have minimum norm subject to the constraint that its first coefficient be unity; that is,

$$\mathbf{d}^\dagger \mathbf{d} = \min, \quad \text{subject to} \quad \mathbf{u}_0^\dagger \mathbf{d} = d_0 = 1 \tag{14.6.1}$$

The minimization is carried over the noise subspace vectors. In the B basis (14.4.10), the vector $\mathbf{d}$ is expressed by $\mathbf{d} = B\mathbf{f}$, where $\mathbf{f}$ are the coefficients of $F(z)$, and the constraint equation becomes $\mathbf{u}_0^\dagger B\mathbf{f} = 1$. With the exception of $\mathbf{b}_0$, all basis vectors $\mathbf{b}_i$ start with zero; therefore, $\mathbf{u}_0^\dagger B = [\mathbf{u}_0^\dagger \mathbf{b}_0, \mathbf{u}_0^\dagger \mathbf{b}_1, \ldots, \mathbf{u}_0^\dagger \mathbf{b}_{K-1}] = [1, 0, \ldots, 0] \equiv \mathbf{u}^\dagger$, that is, a K-dimensional unit vector. Therefore, in the B basis Eq. (14.6.1) becomes

$$\mathbf{d}^\dagger \mathbf{d} = \mathbf{f}^\dagger R_{aa} \mathbf{f} = \min, \quad \text{subject to} \quad \mathbf{u}^\dagger \mathbf{f} = 1 \tag{14.6.2}$$

where we set $R_{aa} = B^\dagger B$. This is recognized as the Toeplitz matrix of autocorrelations of the filter $\mathbf{a}$, as defined in Eq. (1.19.5) of Sec. 1.19. For the 6×4 example above, we verify,

$$R_{aa} = B^\dagger B = \begin{bmatrix} R_{aa}(0) & R_{aa}(1)^* & R_{aa}(2)^* & 0 \\ R_{aa}(1) & R_{aa}(0) & R_{aa}(1)^* & R_{aa}(2)^* \\ R_{aa}(2) & R_{aa}(1) & R_{aa}(0) & R_{aa}(1)^* \\ 0 & R_{aa}(2) & R_{aa}(1) & R_{aa}(0) \end{bmatrix}$$

where $R_{aa}(0) = |a_0|^2 + |a_1|^2 + |a_2|^2$, $R_{aa}(1) = a_1 a_0^* + a_2 a_1^*$, $R_{aa}(2) = a_2 a_0^*$, and $R_{aa}(3) = 0$. Note that the autocorrelation function of an order-M filter $\mathbf{a}$ vanishes for lags greater than $M+1$. It follows that Eq. (14.6.2) represents an ordinary linear prediction problem and its solution $\mathbf{f}$ will be a minimum-phase filter with all its zeros inside the unit circle. Up to a scale factor, we may write this solution as $\mathbf{f} = R_{aa}^{-1}\mathbf{u} = (B^\dagger B)^{-1}\mathbf{u}$. Writing $\mathbf{u} = B^\dagger \mathbf{u}_0$, we have $\mathbf{f} = (B^\dagger B)^{-1} B^\dagger \mathbf{u}_0$, and the solution for $\mathbf{d}$ becomes

$$\mathbf{d} = B\mathbf{f} = B(B^\dagger B)^{-1} B^\dagger \mathbf{u}_0 = E_N E_N^\dagger \mathbf{u}_0 \tag{14.6.3}$$

This is the solution of criterion (14.6.1) up to a scale. Interestingly, the locations of the spurious zeros do not depend on the signal to noise ratios, but depend only on the desired zeros on the unit circle. This follows from the fact that the solution for $\mathbf{f}$ depends only on B. Using Eq. (14.4.13), we may also write $\mathbf{d}$ in terms of the signal subspace basis

$$\mathbf{d} = [I - E_S E_S^\dagger]\mathbf{u}_0 = [I - S(S^\dagger S)^{-1}S^\dagger]\mathbf{u}_0$$

Recall from Sec. 14.2 that this is the large-SNR limit of the LP solution. Noting that $E_N^\dagger \mathbf{u}_0$, is the complex conjugate of the top row of the eigenvector matrix E_N, we write Eq. (14.6.3) explicitly as a linear combination of noise subspace eigenvectors

$$\mathbf{d} = \sum_{i=0}^{K-1} E_{0i}^* \mathbf{e}_i \tag{14.6.4}$$

where E_{0i}^* the conjugate of the $0i$-th matrix element of E. The function **minorm** computes $\mathbf{d}$ using Eq. (14.6.4). The corresponding pseudospectrum estimate is defined as the inverse magnitude response of the filter $\mathbf{d}$

$$S_{\text{MIN}}(k) = \frac{1}{|D(k)|^2} = \frac{1}{|\mathbf{s}_k^\dagger \mathbf{d}|^2} \tag{14.6.5}$$

The practical implementation of this method requires the same two initial steps as MUSIC; namely, eigenanalysis of $\hat{R}$ and estimation of K. In Fig. 14.6.1, the minimum-norm spectrum was computed by calling the functions **minorm**. The vector $\mathbf{d}$ was normalized to unit norm to make a fair comparison with the MUSIC spectrum. Looking at the spectra is not the best way to evaluate this method because the spurious zeros—even though inside the unit circle—interact with the desired zeros to modify the shape of the spectrum. The minimum-norm method is better judged by comparing the theoretical and empirical zeros of the polynomial $D(z)$, computed from R and $\hat{R}$. They are shown in the following table. The first two zeros are the desired ones.

zeros of $D(z)$			
theoretical		empirical	
$\|z_i\|$	$\arg(z_i)/\pi$	$\|z_i\|$	$\arg(z_i)/\pi$
1.0000	0.2000	0.9989	0.2020
1.0000	0.4000	1.0059	0.4026
0.8162	−0.1465	0.8193	−0.1441
0.7810	−0.4251	0.7820	−0.4227
0.7713	−0.7000	0.7759	−0.6984
0.8162	0.7465	0.8188	0.7481
0.7810	−0.9749	0.7832	−0.9729

The main idea of the minimum-norm method was to separate the desired zeros from the spurious ones by pushing the latter inside the unit circle. In some applications of eigenvector methods, such as pole identification, the desired zeros lie themselves inside the unit circle (being the poles of a stable and causal system) and therefore, cannot be separated from the spurious ones. To separate them, we need a modification of the method that places all the spurious zeros to the outside of the unit circle. This can be

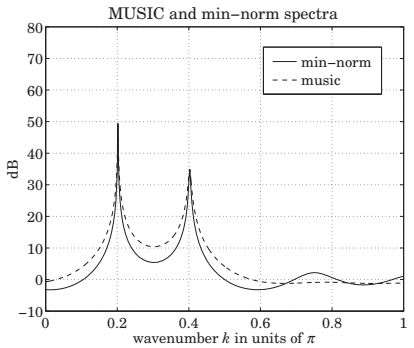


Fig. 14.6.1 MUSIC and min-norm spectra.

done by replacing the vector $\mathbf{f}$ by its reverse $\mathbf{f}^R = J\mathbf{f}^*$, where J is the $K\times K$ reversing matrix. The resulting polynomial will be the reverse of $F(z)$, with all its zeros reflected to the outside of the unit circle. The reverse vector $\mathbf{f}^R$ is the backward prediction filter obtained by minimizing (14.6.2) subject to the constraint that its last element be unity. Using the reversal invariance of R_{aa}, namely, $JR_{aa}J = R_{aa}^*$, we find

$$\mathbf{f}^R = J\mathbf{f}^* = J(R_{aa}^{-1})^*\mathbf{u} = R_{aa}^{-1}J\mathbf{u} = R_{aa}^{-1}\mathbf{v}$$

where $\mathbf{v} = J\mathbf{u} = [0,\ldots,0,1]^T$ is the reverse of $\mathbf{u}$. With the exception of $\mathbf{b}_{K-1}$, the last element of all basis vectors $\mathbf{b}_i$ is zero. Denoting by $\mathbf{v}_0$, the reverse of $\mathbf{u}_0$, it follows that $\mathbf{v}_0^\dagger B = [0,0,\ldots,0,a_L] = a_L\mathbf{v}^\dagger$. Thus, up to a scale factor, $\mathbf{v}$ can be replaced by $B^\dagger\mathbf{v}_0$, and hence, The vector $\mathbf{d}$ becomes

$$\mathbf{d} = B\mathbf{f}^R = B(B^\dagger B)^{-1}B^\dagger\mathbf{v}_0 = E_N E_N^\dagger \mathbf{v}_0 \tag{14.6.6}$$

Up to a scale, this is the minimum-norm vector subject to the constraint that its last element be unity; that is, $\mathbf{v}_0^\dagger\mathbf{d} = d_M = 1$. In terms of the matrix elements of the eigenvector matrix E it reads

$$\mathbf{d} = \sum_{i=0}^{K-1} E_{Mi}^* \mathbf{e}_i \tag{14.6.7}$$

where E_{Mi}^* is the conjugate of the last row of E. The spurious zeros of this vector will lie outside the unit circle. We may refer to this method as the modified minimum-norm method.

14.7 Reduced-Order Method

The basis B of the noise subspace has very special structure, being constructed in terms of the delayed replicas of the same reduced-order vector $\mathbf{a}$. It is evident from Eq. (14.4.11) that $\mathbf{a}$ can be extracted from any column $\mathbf{b}_i$ or B by advancing it by i units. The B basis is linearly related to the orthonormal eigenvector basis by $B = E_N C$ with some $K\times K$

invertible matrix C. Thus, the vector $\mathbf{b}_i$ is expressible as a linear combination of the noise subspace eigenvectors

$$\mathbf{b}_i = \sum_{j=0}^{K-1} \mathbf{e}_j C_{ji}\,, \quad i = 0, 1, \ldots, K-1$$

This vector has a total of $K-1$ vanishing coefficients, namely, the first i and the last $K-1-i$ coefficients. Component-wise, we may write $b_{im} = 0$, for $0 \le m \le i-1$ and for $i+L+1 \le m \le M$. This vector may be specified up to an overall scale factor because we are interested only in the zeros of the reduced-order vector $\mathbf{a}$. Therefore, we may arbitrarily fix one of the coefficients C_{ji} to unity. For example, we may single out the 0th eigenvector:

$$\mathbf{b}_i = \mathbf{e}_0 + \sum_{j=1}^{K-1} \mathbf{e}_j C_{ji} \tag{14.7.1}$$

If $\mathbf{e}_0$ happens to be absent from the sum, we may single out $\mathbf{e}_1$ and so on. The coefficient b_{ii} will no longer be unity, but may be normalized so later. The $K-1$ unknown coefficients C_{ji}, $j = 1, 2, \ldots, K-1$ can be determined by the $K-1$ conditions that the first i and last $K-1-i$ coefficients of $\mathbf{b}_i$ be zero. Written in terms of the matrix elements of the eigenvector matrix E, these conditions read for each $i = 0, 1, \ldots, K-1$:

$$E_{m0} + \sum_{j=1}^{K-1} E_{mj} C_{ji} = 0\,, \quad \text{for} \quad 0 \le m \le i-1 \quad \text{and} \quad i+L+1 \le m \le M \tag{14.7.2}$$

Thus, solving the linear Eqs. (14.7.2) for the coefficients C_{ji} and substituting in Eq. (14.7.1), we obtain $\mathbf{b}_i$ and, advancing it by i units, the reduced-order vector $\mathbf{a}$. Because $B_i(z) = z^{-i}A(z)$, the polynomial $B_i(z)$ has no spurious zeros. In effect, forming the linear combination Eq. (14.7.1) of noise subspace eigenvectors removes the spurious zeros completely by placing them at the *origin* of the z-plane. In a sense, this procedure carries the philosophy of the minimum-norm method further.

When the theoretical R is replaced by the empirical $\hat{R}$ and the corresponding E_N is replaced by the estimated $\hat{E}_N$, it is no longer possible to linearly transform the basis $\hat{E}_N$ to a B basis constructed from a single reduced-order vector $\mathbf{a}$. It is still possible, however, to form linear combinations of the estimated eigenvectors.

$$\hat{\mathbf{b}}_i = \sum_{j=0}^{K-1} \hat{\mathbf{e}}_j C_{ji}\,, \quad i = 0, 1, \ldots, K-1 \tag{14.7.3}$$

such that the resulting vectors $\hat{\mathbf{b}}_i$ will have vanishing first i and last $K-1-i$ coefficients; that is, of the form

$$\hat{\mathbf{b}}_i = [\underbrace{0, \ldots, 0}_{i \text{ zeros}},\ 1,\ a_{i1},\ \ldots,\ a_{iL},\ \underbrace{0, \ldots, 0}_{K-1-i \text{ zeros}}]^T \tag{14.7.4}$$

This can be done by solving Eq. (14.7.2) with E replaced by its estimate, $\hat{E}$, obtained from $\hat{R}$. The resulting K reduced-order vectors $\mathbf{a}_i = [1, a_{i1}, \ldots, a_{iL}]^T$, $i = 0, 1, \ldots, K-1$, will not be the same necessarily. But, each can be considered to be an approximate

estimate of the true reduced-order vector $\mathbf{a}$, and its L zeros will be estimates of the true desired zeros.

It turns out that individually none of the $\mathbf{a}_i$ is a particularly good estimate of $\mathbf{a}$. They may be combined, however, to produce a better estimate. This is analogous to MUSIC, where individual spectra of noise eigenvectors are not good, but combining them by averaging produces a better spectrum. To see how we may best combine the $\mathbf{a}_i$, we form a new basis of the estimated noise subspace in terms of the vectors $\hat{\mathbf{b}}_i$, namely, $\hat{B} = [\hat{\mathbf{b}}_0, \hat{\mathbf{b}}_1, \ldots, \hat{\mathbf{b}}_{K-1}]$. For our 6×4 example, we have

$$\hat{B} = [\hat{\mathbf{b}}_0, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3] = \begin{bmatrix} 1 & 0 & 0 & \\ a_{01} & 1 & 0 & 0 \\ a_{02} & a_{11} & 1 & 0 \\ 0 & a_{12} & a_{21} & 1 \\ 0 & 0 & a_{22} & a_{31} \\ 0 & 0 & 0 & a_{32} \end{bmatrix}$$

The linear transformations (14.7.3) may be written compactly as $\hat{B} = \hat{E}_N C$. Note that $\hat{B}^\dagger \hat{B}$ is no longer Toeplitz and therefore, the LP solution $\mathbf{f}$ of (14.6.2) will not necessarily have minimum phase. Thus, the empirical minimum-norm solution can have spurious zeros outside or near the unit circle. Because the basis $\hat{B}$ is an estimate of the true B, we may try to fit $\hat{B}$ to a matrix of the type B having the special structure (14.4.11) by minimizing the distance between the two matrices according to some matrix norm. For example, we may minimize the Frobenius matrix distance [1166]:

$$\|\hat{B} - B\|^2 = \text{tr}\big[(\hat{B} - B)^\dagger (\hat{B} - B)\big] = \sum_{i=0}^{K-1} \|\hat{\mathbf{b}}_i - \mathbf{b}_i\|^2 = \min$$

Because $\hat{\mathbf{b}}_i$ and $\mathbf{b}_i$ are the delayed versions of the reduced-order vectors $\mathbf{a}_i$ and $\mathbf{a}$, it follows that $\|\hat{\mathbf{b}}_i - \hat{\mathbf{b}}_i\|^2 = \|\mathbf{a}_i - \mathbf{a}\|^2$. Therefore,

$$\|\hat{B} - B\|^2 = \text{tr}\big[(\hat{B} - B)^\dagger (\hat{B} - B)\big] = \sum_{i=0}^{K-1} \|\mathbf{a}_i - \mathbf{a}\|^2 = \min \tag{14.7.5}$$

Minimizing with respect to $\mathbf{a}$ gives the result:

$$\hat{\mathbf{a}} = \frac{1}{K} \sum_{i=0}^{K-1} \mathbf{a}_i\,, \qquad \hat{A}(z) = \frac{1}{K} \sum_{i=0}^{K-1} A_i(z) \tag{14.7.6}$$

that is, the average of the K filters. Thus, we obtain the following *reduced-order* or, *reduced-MUSIC* algorithm [1139]:

1. Solve the eigenproblem for the estimated covariance matrix $\hat{R}$.
2. Using the estimated noise subspace eigenvectors, solve (14.7.2) for the coefficients C_{ji} and using Eq. (14.7.3) obtain the basis vectors $\hat{\mathbf{b}}_i$ and hence the reduced-order vectors $\mathbf{a}_i$, $i = 0, 1, \ldots, K-1$.

3. Use the average (14.7.6) to get an estimate $\hat{A}(z)$ of the reduced-order polynomial $A(z)$. Obtain estimates of the desired zeros by a root-finding procedure on $\hat{A}(z)$, or, by finding the peaks in the pseudospectrum

$$\hat{S}(k) = \frac{1}{|\hat{A}(k)|^2} = \frac{1}{|\mathbf{s}_k^\dagger \hat{\mathbf{a}}|^2} \tag{14.7.7}$$

The MATLAB function **rmusic** implements this algorithm. Fig. 14.7.1 shows a comparison between the reduced-order algorithm and MUSIC for the same example considered in Fig. 14.6.1, where, again, for the purposes of comparison the vector $\hat{\mathbf{a}}$ was normalized to unit norm. As in the case of MUSIC, the spectrum of any individual reduced-order vector $\mathbf{a}_i$ is not good, but the spectrum based on the average $\hat{\mathbf{a}}$ is better. This can be appreciated by comparing the two zeros ($L = 2$) of the six ($K = 6$) individual filters $\hat{A}_i(z)$, $i = 0, 1, \ldots, 5$ with the two zeros of the averaged polynomial $\hat{A}(z)$ and with the theoretical zeros. They are shown in the table below.

zeros	$\hat{A}_0$	$\hat{A}_1$	$\hat{A}_2$	$\hat{A}_3$	$\hat{A}_4$	$\hat{A}_5$	$\hat{A}$	A
$\lvert z_1\rvert$	0.976	1.032	0.964	1.038	0.969	1.025	0.999	1.000
$\arg(z_1)/\pi$	0.197	0.203	0.199	0.199	0.203	0.197	0.201	0.200
$\lvert z_2\rvert$	1.056	0.944	1.115	0.896	1.059	0.947	1.002	1.000
$\arg(z_2)/\pi$	0.393	0.407	0.402	0.402	0.407	0.393	0.399	0.400

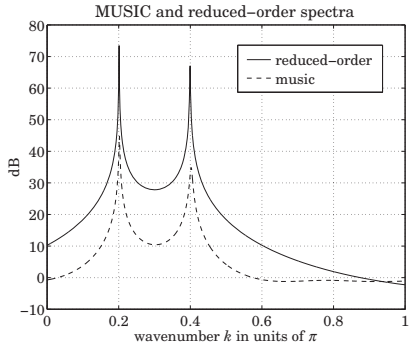


Fig. 14.7.1 MUSIC and reduced-order method.

An alternative method of combining the K estimates is as follows [1163]. Form the $(L+1)\times K$ matrix $A = [\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{K-1}]$ and note that if the $\mathbf{a}_i$ were computed on the basis of the theoretical covariance matrix R, then A would have rank one because each $\mathbf{a}_i$ would be exactly equal to $\mathbf{a}$. But if the empirical matrix $\hat{R}$ is used, then the matrix A will only approximately have rank one, in the sense of its singular value decomposition (SVD). Thus, we may replace A by its rank-one SVD approximant, namely, the rank-one matrix closest to A with respect to the Frobenius or Euclidean matrix norms. This amounts to finding the largest eigenvalue of the $(L+1)\times(L+1)$ matrix

$$AA^\dagger = \sum_{i=0}^{K-1} \mathbf{a}_i \mathbf{a}_i^\dagger \tag{14.7.8}$$

and choosing the corresponding eigenvector to be the estimate of $\mathbf{a}$. This eigenvector is expressible as a weighted sum of the $\mathbf{a}_i$ but with different weights than Eq. (14.7.6). To see this, let σ and $\hat{\mathbf{a}}$ be the largest eigenvalue and eigenvector of $AA^\dagger$. Using $AA^\dagger\hat{\mathbf{a}} = \sigma\hat{\mathbf{a}}$, and defining $\mathbf{w} = \sigma^{-1}A^\dagger\hat{\mathbf{a}}$, we find

$$\hat{\mathbf{a}} = A\mathbf{w} = \sum_{i=0}^{K-1} w_i \mathbf{a}_i \tag{14.7.9}$$

where w_i are the components of $\mathbf{w} = [w_0, w_1, \ldots, w_{K-1}]^T$. The constraint that $\hat{\mathbf{a}}$ and $\mathbf{a}_i$, have first coefficients of unity implies the normalization condition $\sum_{i=0}^{K-1} w_i = 1$.

Even though this method is computationally more complex than Eq. (14.7.6), it allows one to judge the quality of the resulting estimate. This may be done by inspecting the relative magnitudes of the singular values of A, equivalently, the $L+1$ eigenvalues of $AA^\dagger$. Theoretically, all but the maximum eigenvalue must be zero. Applying this method to the above simulation example, we find the estimated zeros:

$$z_1 = 0.9985\, e^{j0.2011\pi}, \quad z_2 = 1.0037\, e^{j0.3990\pi}$$

and the theoretical and empirical SVD values of the matrix A:

theoretical	5.8059	0	0
empirical	5.8139	0.1045	0.0187

14.8 Maximum Likelihood Method

The maximum likelihood method is not, strictly speaking, an eigenvector method; however, some of the ideas we have been discussing apply to it. The method determines the plane wave frequencies and amplitudes by fitting them *directly* to the measured snapshot data using a criterion, such as maximum likelihood or least-squares. Each snapshot is modeled according to Eq. (14.3.10), which can be written compactly as

$$\mathbf{y}(n) = [\mathbf{s}_{k_1}^*, \ldots, \mathbf{s}_{k_L}^*] \begin{bmatrix} A_1(n) \\ \vdots \\ A_L(n) \end{bmatrix} + \mathbf{v}(n) = S^*\mathbf{A}(n) + \mathbf{v}(n) \tag{14.8.1}$$

The unknown amplitudes $\mathbf{A}(n)$ and wavenumbers k_i, $i = 1, 2, \ldots, L$ are treated as deterministic parameters to be fitted to the snapshot data $Y = \{\mathbf{y}(n),\ 0 \le n \le N-1\}$. The maximum likelihood estimates of these parameters are obtained by maximizing the joint density of the snapshots, $p(Y) = \max$. If the wave parameters are deterministic, then the randomness in $\mathbf{y}(n)$ arises only from $\mathbf{v}(n)$. Assuming that $\mathbf{v}(n)$ are complex gaussian (see Problem 14.16) and independent, the joint density of Y is the product of marginal densities:

$$p(Y) = \prod_{n=0}^{N-1} p(\mathbf{v}(n)) = \frac{1}{(\pi\sigma_v^2)^{N(M+1)}} \exp\left[-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} \|\mathbf{v}(n)\|^2\right]$$

$$= \frac{1}{(\pi\sigma_v^2)^{N(M+1)}} \exp\left[-\frac{1}{\sigma_v^2}\sum_{n=0}^{N-1} \|\mathbf{y}(n) - S^*\mathbf{A}(n)\|^2\right]$$

Thus, under gaussian statistics, the maximum likelihood criterion is equivalent to the least-squares minimization criterion:

$$J = \sum_{n=0}^{N-1} \|\mathbf{y}(n) - S^*\mathbf{A}(n)\|^2 = \min \tag{14.8.2}$$

According to the general discussion of [1165], the simultaneous minimization of J with respect to k_i and $\mathbf{A}(n)$ can be done in two steps. First, minimize with respect to the amplitudes $\mathbf{A}(n)$ and then, minimize with respect to the wavenumbers k_i. Setting the gradients with respect to $\mathbf{A}(n)$ to zero, we obtain

$$\frac{\partial J}{\partial \mathbf{A}(n)} = -S^\dagger[\mathbf{y}(n)^* - S\mathbf{A}^*(n)] = 0 \quad \Rightarrow \quad \mathbf{A}(n)^* = (S^\dagger S)^{-1} S^\dagger \mathbf{y}(n)^*$$

Inserting this solution into Eq. (14.8.2), we obtain

$$J = \sum_{n=0}^{N-1} \|\mathbf{y}(n^*) - S\mathbf{A}(n)^*\|^2 = \sum_{n=0}^{N-1} \|[I - S(S^\dagger S)^{-1}S^\dagger]\mathbf{y}(n)^*\|^2$$

Using Eq. (14.4.13), we may rewrite it in terms of the projector onto the noise subspace, namely, $P_N = B(B^\dagger B)^{-1}B^\dagger = I - S(S^\dagger S)^{-1}S^\dagger$

$$J = \sum_{n=0}^{N-1} \|B(B^\dagger B)^{-1}B^\dagger \mathbf{y}(n)^*\|^2 = \sum_{n=0}^{N-1} \|P_N \mathbf{y}(n)^*\|^2$$

Using the projection property $P_N^\dagger P_N = P_N$, and the definition (14.4.14) of the sample covariance matrix, we find

$$J = \sum_{n=0}^{N-1} \mathbf{y}(n)^T P_N \mathbf{y}(n)^* = \operatorname{tr}\left[\sum_{n=0}^{N-1} P_N \mathbf{y}(n)^T \mathbf{y}(n)^*\right] = N \operatorname{tr}[P_N \hat{R}]$$

The minimization of J with respect to the coefficients of the reduced-order vector $\mathbf{a}$ is a highly nonlinear problem. It may be solved, however, iteratively by the solution of a succession of simpler problems, by the following procedure [1141–1143,1159,1161]. Write $\mathbf{y}(n)^T B = [\mathbf{y}(n)^T\mathbf{b}_0, \mathbf{y}(n)^T\mathbf{b}_1, \ldots, \mathbf{y}(n)^T\mathbf{b}_{K-1}]$ and note that $\mathbf{y}(n)^T\mathbf{b}_i = \mathbf{a}^T\mathbf{y}_i(n)$, where $\mathbf{y}_i(n)$ is the $(L+1)$-dimensional portion of $\mathbf{y}(n)$ starting at the ith position, namely,

$$\mathbf{y}_i(n) = [y_i(n), y_{i+1}(n), \ldots, y_{i+L}(n)]^T, \quad i = 0, 1, \ldots, K-1$$

Then, $\mathbf{y}(n)^T B = \mathbf{a}^T[\mathbf{y}_0(n), \mathbf{y}_1(n), \ldots, \mathbf{y}_{K-1}(n)] \equiv \mathbf{a}^T Y(n)$. And, J can be written as

$$J = \sum_{n=0}^{N-1} \mathbf{y}(n)^T B(B^\dagger B)^{-1}B^\dagger \mathbf{y}(n)^* = \mathbf{a}^T\left[\sum_{n=0}^{N-1} Y(n)(B^\dagger B)^{-1}Y(n)^\dagger\right]\mathbf{a}^*$$

The minimization of J is obtained by solving the succession of problems, for $i = 1, 2, \ldots,$

$$J_i = \mathbf{a}_i^T\left[\sum_{n=0}^{N-1} Y(n)(B_{i-1}^\dagger B_{i-1})^{-1}Y(n)^\dagger\right]\mathbf{a}_i^* = \min \tag{14.8.3}$$

where $B_{i-1}^\dagger B_{i-1}$ is constructed from the solution $\mathbf{a}_{i-1}$ of the previous iteration. The iteration is initialized by $\mathbf{a}_0 = [1, 0, \ldots, 0]^T$, which gives $B_0^\dagger B_0 = I_K$. At each iteration, Eq. (14.8.3) is subject to an appropriate constraint on $\mathbf{a}_i$ such as that its first coefficient be unity, or, that its zeros lie on the unit circle. Note that $B^\dagger B$ is Toeplitz and therefore, its inverse can be computed efficiently by the Levinson recursion.

14.9 ESPRIT Method

There exist a number of eigenvector methods that employ two or more sets of snapshot measurements obtained from two or more arrays related to each other either by translation or by rotation. Examples are the estimation of signal parameters via rotational invariance techniques (ESPRIT) method [1145], the covariance difference method [1135–1138], and the spatial smoothing method for dealing with coherent signals [1119,1126].

Consider two arrays related to each other by an overall translation by distance Δ along the x-axis. The effect of translation shows up as an overall phase change in each direction vector. For example, the value of a wave on the x-axis with respect to the original and the translated x-axes will be:

$$A_1 e^{-jk_x x} \to A_1 e^{-jk_x(x+\Delta)} = A_1 e^{-jk_x x} e^{-jk_x \Delta}$$

Setting $x_m = md$ and letting $\delta = \Delta/d$ be the displacement in units of d, we obtain at the original and translated mth array elements

$$A_1 e^{-jk_1 m} \to A_1 e^{-jk_1 m} e^{-jk_1 \delta}$$

or, in terms of the direction vectors

$$A_1 \mathbf{s}_1^* \to A_1 \mathbf{s}_1^* e^{-jk_1\delta}$$

It follows that the matrix $S = [\mathbf{s}_{k_1}, \ldots, \mathbf{s}_{k_L}]$ transforms under translation as

$$S \to SD_\delta, \quad D_d = \operatorname{diag}\{e^{jk_1\delta}, e^{jk_2\delta}, \ldots, e^{jk_L\delta}\} \tag{14.9.1}$$

Therefore, the snapshot measurements at the original and translated arrays are

$$\begin{aligned} \mathbf{y}(n) &= S^*\mathbf{A}(n) + \mathbf{v}(n) \\ \mathbf{y}_\delta(n) &= S^* D_\delta^* \mathbf{A}(n) + \mathbf{v}_\delta(n) \end{aligned} \tag{14.9.2}$$

The covariance and cross-covariance matrices are

$$\begin{aligned} R_{yy} &= E[\mathbf{y}(n)^*\mathbf{y}(n)^T] = SPS^\dagger + \sigma_v^2 I \\ R_{y_\delta y_\delta} &= E[\mathbf{y}_\delta(n)^*\mathbf{y}_\delta(n)^T] = SD_\delta P D_\delta^\dagger S^\dagger + \sigma_v^2 I \end{aligned} \tag{14.9.3}$$

$$R_{yy_\delta} = E[\mathbf{y}(n)^*\mathbf{y}_\delta(n)^T] = SPD_\delta^\dagger S^\dagger \tag{14.9.4}$$

where we used $E[\mathbf{v}_\delta(n)^*\mathbf{v}_\delta(n)^T] = E[\mathbf{v}(n)^*\mathbf{v}(n)^T] = \sigma_v^2 I$ and $E[\mathbf{v}(n)^*\mathbf{v}_\delta(n)^T] = 0$.

The ESPRIT method works with the *matrix pencil*, $C(\lambda) = C - \lambda C_\delta$, defined by the pair of matrices

$$C = R_{yy} - \sigma_v^2 I = SPS^\dagger\,, \qquad C_\delta = R_{yy_\delta} = SPD_\delta^\dagger S^\dagger \tag{14.9.5}$$

The generalized eigenvalues of this matrix pencil are, by definition [102], the solutions of $\det(C - \lambda C_\delta) = 0$, and the corresponding generalized eigenvectors satisfy $C\mathbf{e} = \lambda C_\delta \mathbf{e}$. The ESPRIT method is based on the observation that the *nonzero* generalized eigenvalues of $C(\lambda)$ are simply

$$\lambda_i = e^{jk_i\delta}\,, \quad i = 1, 2, \ldots, L \tag{14.9.6}$$

and therefore, the desired wavenumbers k_i can be extracted from the knowledge of the λ_i. Note that $\lambda = 0$ is a generalized eigenvalue because $\det(C) = \det(SPS^\dagger) = 0$. This follows from the fact that $SPS^\dagger$ is an $(M+1)\times(M+1)$ matrix of rank $L < M+1$. The generalized eigenvectors corresponding to $\lambda = 0$ are the vectors in the null space of $SPS^\dagger$; namely, they satisfy $SPS^\dagger\mathbf{e} = 0$, or, equivalently, $S^\dagger\mathbf{e} = 0$. These are the noise subspace eigenvectors of R_{yy}. Next, we show that the only nonzero generalized eigenvalues are those in Eq. (14.9.6). The corresponding generalized eigenvector $\mathbf{e}$ must satisfy

$$SPS^\dagger\mathbf{e} = \lambda SPD_\delta^\dagger S^\dagger\mathbf{e}$$

Multiplying both sides by $S^\dagger$ and removing the common matrix factor $(S^\dagger S)P$, we obtain $S^\dagger\mathbf{e} = \lambda D_\delta^\dagger S^\dagger\mathbf{e}$. Using the fact that $D_\delta^\dagger = D_\delta^{-1}$, and defining the L-dimensional vector $\mathbf{f} = S^\dagger\mathbf{e}$, we obtain

$$D_\delta\mathbf{f} = \lambda\mathbf{f}$$

Clearly, if $\mathbf{e}$ is not in the noise subspace, then $\mathbf{f} = S^\dagger\mathbf{e} \neq 0$; therefore, λ must be an eigenvalue of D_δ, which is already diagonal. This proves Eq. (14.9.6). The eigenvectors of D_δ will be the L-dimensional unit vectors; that is, the columns of the $L\times L$ unit matrix, $\mathbf{f}_i = \mathbf{u}_i$, $i = 1, 2, \ldots, L$. The generalized eigenvectors will be $\mathbf{e}_i = S(S^\dagger S)^{-1}\mathbf{u}_i$. These are obtained by an argument similar to Eq. (14.3.15). Thus, the L columns of the matrix $S(S^\dagger S)^{-1}$ are simply the generalized eigenvectors corresponding to the generalized eigenvalues (14.9.6).

In the practical implementation of the method, we assume we have two sets of snapshots, $\mathbf{y}(n)$ and $\mathbf{y}_\delta(n)$, for $n = 0, 1, \ldots, N-1$, measured at the original and translated arrays. The covariance matrix R_{yy} is estimated by Eq. (14.4.14) and the cross-covariance matrix by

$$\hat{C}_\delta = \hat{R}_{yy_\delta} = \frac{1}{N}\sum_{n=0}^{N-1} \mathbf{y}(n)^*\mathbf{y}_\delta(n)^T$$

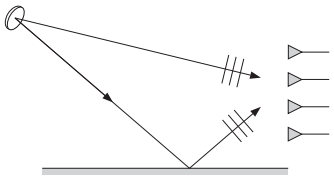
From the eigenproblem of $\hat{R}_{yy}$, we obtain an estimate of $\hat{\sigma}_v^2$, either as the minimum eigenvalue or, as the average of the eigenvalues of the noise subspace. Then, set $\hat{C} = \hat{R}_{yy} - \hat{\sigma}_v^2 I$ and solve the generalized eigenproblem for the pair $\{\hat{C}, \hat{C}_\delta\}$. The L generalized eigenvalues closest to the unit circle are used to extract estimates of the desired wavenumbers k_i by Eq. (14.9.6).

Unlike the minimum-norm and reduced-order methods that require equally spaced linear arrays, the MUSIC and ESPRIT methods can be applied to arrays of arbitrary geometry.

14.10 Spatial Smoothing

Eigenvector methods rely on the property that the noise subspace eigenvectors have at least L zeros on the unit circle at the desired frequency locations. As we saw in Sec. 14.3, this property requires that the $L\times L$ power matrix P have full rank equal to L. To repeat the argument, the condition $R\mathbf{a} = \sigma_v^2\mathbf{a}$ implies that $SPS^\dagger\mathbf{a} = 0$, but what we want is $S^\dagger\mathbf{a} = 0$. Multiplying by $\mathbf{a}^\dagger$, we obtain $(S^\dagger\mathbf{a})^\dagger P(S^\dagger\mathbf{a}) = 0$, but this does not necessarily imply that $S^\dagger\mathbf{a} = 0$ unless P has full rank.

The case of diagonal P corresponds to mutually uncorrelated sources for the L plane waves. The case of a nondiagonal P of full rank implies that the sources are partially correlated. The case of a non-diagonal P with less than full rank implies that some or all of the sources are coherent with each other. This case commonly arises in multipath situations, as shown in the following diagram



To see how eigenvector methods fail if P does not have full rank, consider the worst case when all the sources are coherent, which means that the wave amplitudes $A_i(n)$ are all proportional to each other, say, $A_i(n) = c_i A_1(n)$, $i = 1, 2, \ldots, L$, where the $c_i \neq 0$ (with $c_1 = 1$) are attenuation factors corresponding to the different paths. Compactly, we may write $\mathbf{A}(n) = A_1(n)\mathbf{c}$. Then, the power matrix becomes

$$P = E\big[\mathbf{A}(n)^*\mathbf{A}(n)^T\big] = E\big[|A_1(n)|^2\big]\mathbf{c}^*\mathbf{c}^T = P_1\mathbf{c}^*\mathbf{c}^T \tag{14.10.1}$$

It has rank one. The corresponding covariance matrix is

$$R = SPS^\dagger + \sigma_v^2 I = P_1 S\mathbf{c}^*\mathbf{c}^T S^\dagger + \sigma_v^2 I = P_1\mathbf{s}\mathbf{s}^\dagger + \sigma_v^2 I \tag{14.10.2}$$

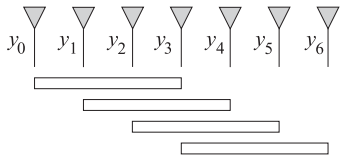
where $\mathbf{s} = S\mathbf{c}^*$. Similarly,

$$\mathbf{y}(n) = A_1(n)S^*\mathbf{c} + \mathbf{v}(n) = A_1(n)\mathbf{s}^* + \mathbf{v}(n)$$

Because R is a rank-one modification of the identity matrix, it will have a one-dimensional signal subspace spanned by $\mathbf{s}$ and a noise subspace of dimension $K = M + 1 - 1 = M$ spanned by the eigenvectors belonging to the minimum eigenvalue σ_v^2. Thus, although we have L different signals, the solution of the eigenproblem will result in a one-dimensional signal subspace. Moreover, the noise eigenvectors, will not necessarily have zeros at the L desired locations. This can be seen as follows. If $R\mathbf{a} = \sigma_v^2\mathbf{a}$, then $P_1\mathbf{s}\mathbf{s}^\dagger\mathbf{a} = 0$, or, $\mathbf{s}^\dagger\mathbf{a} = \mathbf{c}^T S^\dagger\mathbf{a} = 0$, which gives

$$\mathbf{c}^T S^\dagger\mathbf{a} = [c_1, \ldots, c_L]\begin{bmatrix} A(k_1) \\ \vdots \\ A(k_L) \end{bmatrix} = \sum_{i=1}^{L} c_i A(k_i) = 0$$

This does not imply that the individual terms in the sum are zero. One solution to this problem is the method of spatial smoothing [1119,1126], which restores P to full rank, so that the eigenstructure methods can be applied as usual. The method is as follows. The given array of $M+1$ sensors is subdivided into J subarrays each having $\bar{M}+1$ sensors. The first subarray consists of the first $\bar{M}+1$ elements of the given array. Each subsequent subarray is obtained by shifting ahead one array element at a time, as shown in the following diagram



Formally, we define the J subarrays by

$$\bar{\mathbf{y}}_i(n) = [y_i(n), y_{i+1}(n), \ldots, y_{i+\bar{M}}(n)]^T, \quad i = 0, 1, \ldots, J-1 \tag{14.10.3}$$

where the bar indicates that the size of the subarray is $\bar{M}+1$. That is the $(\bar{M}+1)$-dimensional portion of $\mathbf{y}(n)$ starting at the ith array element. Using Eq. (14.9.2), we may write compactly

$$\bar{\mathbf{y}}_i(n) = \bar{S}^* D_i^* \mathbf{A}(n) + \bar{\mathbf{v}}_i(n)$$

where $\bar{S}$ is the same as S but of dimension $\bar{M}+1$. The matrix D_i is given by Eq. (14.9.1) with $\delta = i$, corresponding to translation by i units. The covariance matrix of the ith subarray will be

$$\bar{R}_i = E[\bar{\mathbf{y}}_i(n)^* \bar{\mathbf{y}}_i(n)^T] = \bar{S} D_i P D_i^\dagger \bar{S}^\dagger + \sigma_v^2 \bar{I}$$

where $\bar{I}$ is the $(\bar{M}+1)$-dimensional identity matrix. The average of the subarray covariances is

$$\bar{R} = \frac{1}{J}\sum_{i=0}^{J-1} \bar{R}_i = \bar{S}\bar{P}\bar{S}^\dagger + \sigma_v^2 \bar{I} \tag{14.10.4}$$

where

$$\bar{P} = \frac{1}{J}\sum_{i=0}^{J-1} D_i P D_i^\dagger \tag{14.10.5}$$

To be able to resolve L sources by the $(\bar{M}+1)$-dimensional eigenproblem (14.10.4), we must have $\bar{M} \geq L$, and the rank of $\bar{P}$ must be L. It has been shown [1126] that if the number of subarrays J is greater than the number of signals, $J \geq L$, then, $\bar{P}$ has full rank. If the J subarrays are to fit within the original array of length $M+1$, then we must have $M+1 \geq (\bar{M}+1)+(J-1)$, that is, the length of the first subarray plus the $J-1$ subsequent shifts. Thus, $M+1 \geq \bar{M}+J$. If both J and $\bar{M}$ are greater than L, then we must have $M+1 \geq 2L$. Therefore, the price for restoring the rank of P is that we must use twice as long an array as in the ordinary full-rank case with L sources. A somewhat stronger result is that $J \geq L+1-\rho$, where ρ is the rank of P [1150]; equivalently, we have $J \geq \nu+1$, where $\nu = L-\rho$ is the nullity of P. This would give for the minimum number of array elements, $M+1 \geq 2L+1-\rho$, [1127,1143,1150]. Following [1126], we

derive the condition $J \geq L$ for the worst case, when all the signals are coherent. In that case, P has rank one $(\rho = 1)$ and is given by Eq. (14.10.1); $\bar{P}$ becomes

$$\bar{P} = \frac{P_1}{J}\sum_{i=0}^{J-1} D_i \mathbf{c}^* \mathbf{c}^T D_i^\dagger = \frac{P_1}{J}\sum_{i=0}^{J-1} \mathbf{d}_i \mathbf{d}_i^\dagger, \quad \mathbf{d}_i = D_i \mathbf{c}^*$$

Writing $\sum_{i=0}^{J-1} \mathbf{d}_i \mathbf{d}_i^\dagger = DD^\dagger$, where $D = [\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_{J-1}]$, it follows that the rank of $\bar{P}$ is the same as the rank of D. The matrix element D_{li} is the lth component of the ith column; that is, $D_{li} = (\mathbf{d}_i)_l = c_l^* e^{jk_l i}$. Thus, D can be written as the product, $D = C^* V$, of the diagonal matrix $C^* = \text{diag}\{c_1^*, \ldots, c_L^*\}$ and the $L \times J$ Vandermonde matrix V with matrix elements $V_{li} = e^{jk_l i}$; for example, if $L = 3$ and $J = 4$,

$$V = \begin{bmatrix} 1 & e^{jk_1} & e^{2jk_1} & e^{3jk_1} \\ 1 & e^{jk_2} & e^{2jk_2} & e^{3jk_2} \\ 1 & e^{jk_3} & e^{2jk_3} & e^{3jk_3} \end{bmatrix}$$

The rank of Vandermonde matrices is always full; that is, it is the minimum of the column and row dimensions, $\min(L, J)$. It follows that the rank of $\bar{P}$ is equal to $\min(L, J)$, therefore, it is equal to L only if $J \geq L$.

To appreciate the mechanism by which the rank is restored, let us consider an example with two $(L = 2)$ fully coherent sources. The minimum number of subarrays needed to *decohere* the sources is $J = L = 2$. This implies $\bar{M} = M+1-J = M-1$. The covariance matrix of the full array is

$$R = P_1 [\mathbf{s}_1, \mathbf{s}_2] \begin{bmatrix} c_1^* \\ c_2^* \end{bmatrix} [c_1, c_2] \begin{bmatrix} \mathbf{s}_1^\dagger \\ \mathbf{s}_2^\dagger \end{bmatrix} + \sigma_v^2 I$$

The covariance matrices of the two subarrays are

$$\bar{R}_0 = P_1 [\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2] \begin{bmatrix} c_1^* \\ c_2^* \end{bmatrix} [c_1, c_2] \begin{bmatrix} \bar{\mathbf{s}}_1^\dagger \\ \bar{\mathbf{s}}_2^\dagger \end{bmatrix} + \sigma_v^2 \bar{I}$$

$$\bar{R}_1 = P_1 [\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2] \begin{bmatrix} e^{jk_1} c_1^* \\ e^{jk_2} c_2^* \end{bmatrix} [e^{-jk_1} c_1, e^{-jk_2} c_2] \begin{bmatrix} \bar{\mathbf{s}}_1^\dagger \\ \bar{\mathbf{s}}_2^\dagger \end{bmatrix} + \sigma_v^2 \bar{I}$$

Their average becomes

$$\bar{R} = \frac{1}{2}(\bar{R}_0 + \bar{R}_1) = [\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2] \bar{P} \begin{bmatrix} \bar{\mathbf{s}}_1^\dagger \\ \bar{\mathbf{s}}_2^\dagger \end{bmatrix} + \sigma_v^2 \bar{I}$$

where

$$\bar{P} = \frac{P_1}{2} \begin{bmatrix} c_1^* \\ c_2^* \end{bmatrix} [c_1, c_2] + \frac{P_1}{2} \begin{bmatrix} e^{jk_1} c_1^* \\ e^{jk_2} c_2^* \end{bmatrix} [e^{-jk_1} c_1, e^{-jk_2} c_2]$$

$$= P_1 \begin{bmatrix} c_1^* c_1 & c_1^* c_2 (1 + e^{j(k_1 - k_2)})/2 \\ c_1 c_2^* (1 + e^{j(k_2 - k_1)})/2 & c_2^* c_2 \end{bmatrix}$$

Clearly, $\bar{P}$ is non-singular. The presence of the translation phases makes the two column vectors $[c_1^*, c_2^*]^T$ and $[e^{jk_1} c_1^*, e^{jk_2} c_2^*]^T$ linearly independent. The determinant of $\bar{P}$ is easily found to be

$$\det \bar{P} = |c_1 c_2|^2 \sin^2\left(\frac{k_1 - k_2}{2}\right)$$

Perhaps, an even simpler example is to consider the two quadratic forms

$$Q_0 = (f_1 + f_2)^2 = \mathbf{f}^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1,1]\mathbf{f}\,, \quad \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$Q_1 = f_1^2 = \mathbf{f}^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1,0]\mathbf{f}$$

Separately, they have rank one, but their sum has full rank

$$Q = Q_0 + Q_1 = (f_1 + f_2)^2 + f_1^2 = 2f_1^2 + 2f_1 f_2 + f_2^2 = \mathbf{f}^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{f}$$

where the 2×2 coefficient matrix has rank two, being the sum of two rank-one matrices defined by two linearly independent two-dimensional vectors

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1,1] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1,0]$$

Such quadratic forms can be formed, for example, by $\mathbf{a}^\dagger SPS^\dagger \mathbf{a} = \mathbf{f}^\dagger P \mathbf{f}$, where $\mathbf{f} = S^\dagger \mathbf{a}$. In the practical implementation of the method, the subarray covariances are computed by sample averages over N snapshots; that is,

$$\bar{R}_i = \frac{1}{N} \sum_{n=0}^{N-1} \bar{\mathbf{y}}_i(n)^* \bar{\mathbf{y}}_i(n)^T$$

and then forming the average

$$\bar{R} = \frac{1}{J} \sum_{i=0}^{J-1} \bar{R}_i$$

In addition to spatial smoothing, there exist other methods for dealing with the problem of coherent signal sources [1147,1148,1151,1152].

14.11 Asymptotic Properties

Statistically, the sample covariance matrix $\hat{R}$ approximates the theoretical R, and therefore, the linear predictor based on $\hat{R}$ will approximate the one based on R. Similarly, the eigenstructure of $\hat{R}$ will approximate that of R. In this section, we derive the asymptotic statistical properties that justify such approximations [1179–1205].

The basic technique for deriving asymptotic results is to perform a linearization of the empirical solution about the theoretical one and then use the asymptotic statistical properties of $\hat{R}$. In Sec. 1.6, we obtained the asymptotic covariance of $\hat{R}$ for a large number of snapshots N:

$$E[\Delta R_{ij} \Delta R_{kl}] = \frac{1}{N} (R_{ik} R_{jl} + R_{il} R_{jk}) \tag{14.11.1}$$

where $\Delta R = \hat{R} - R$ is the deviation of $\hat{R}$ from its mean. This was valid in the real-valued case; the complex-valued version will be considered shortly. The normal equations of linear prediction based on $\hat{R}$ and R are

$$\hat{R}\hat{\mathbf{a}} = \hat{E}\mathbf{u}_0\,, \quad \hat{\mathbf{a}} = \begin{bmatrix} 1 \\ \hat{\boldsymbol{\alpha}} \end{bmatrix} \quad \text{and} \quad R\mathbf{a} = E\mathbf{u}_0\,, \quad \mathbf{a} = \begin{bmatrix} 1 \\ \boldsymbol{\alpha} \end{bmatrix}$$

where $\hat{E}$ and E are the minimized values of the mean-square prediction errors given by $\hat{E} = \hat{\mathbf{a}}^T \hat{R} \hat{\mathbf{a}}$ and $E = \mathbf{a}^T R \mathbf{a}$. Setting $\hat{\mathbf{a}} = \mathbf{a} + \Delta\mathbf{a}$ and $\hat{E} = E + \Delta E$, we obtain

$$(R + \Delta R)(\mathbf{a} + \Delta\mathbf{a}) = (E + \Delta E)\mathbf{u}_0 \quad \Rightarrow \quad R(\Delta\mathbf{a}) + (\Delta R)\mathbf{a} = (\Delta E)\mathbf{u}_0 \tag{14.11.2}$$

where we kept only the first-order terms. Because $\hat{\mathbf{a}}$ and $\mathbf{a}$ have first coefficient of unity, $\Delta\mathbf{a} = \hat{\mathbf{a}} - \mathbf{a}$ will have zero first coefficient, that is, $\mathbf{u}_0^T(\Delta\mathbf{a}) = 0$. Multiplying both sides of Eq. (14.11.2) by $\mathbf{a}^T$, we obtain $\mathbf{a}^T R(\Delta\mathbf{a}) + \mathbf{a}^T(\Delta R)\mathbf{a} = \Delta E$. Using the normal equations for $\mathbf{a}$, we have $\mathbf{a}^T R(\Delta\mathbf{a}) = E\mathbf{u}_0^T(\Delta\mathbf{a}) = 0$. Thus, $\Delta E = \mathbf{a}^T(\Delta R)\mathbf{a}$. Solving Eq. (14.11.2) for $\Delta\mathbf{a}$ and using $R^{-1}\mathbf{u}_0 = E^{-1}\mathbf{a}$, we find

$$\Delta\mathbf{a} = E^{-1}(\Delta E)\mathbf{a} - R^{-1}(\Delta R)\mathbf{a}\,, \quad \Delta E = \mathbf{a}^T(\Delta R)\mathbf{a} \tag{14.11.3}$$

For the purpose of computing the asymptotic covariances of $\Delta\mathbf{a}$ and ΔE, it proves convenient to express Eq. (14.11.3) in terms of the vector $\delta\mathbf{a} \equiv (\Delta R)\mathbf{a}$. Then,

$$\Delta\mathbf{a} = E^{-1}(\Delta E)\mathbf{a} - R^{-1}(\delta\mathbf{a})\,, \quad \Delta E = \mathbf{a}^T(\delta\mathbf{a}) \tag{14.11.4}$$

Using Eq. (14.11.1), we find for the covariance of $\delta\mathbf{a}$

$$E[\delta a_i \delta a_k] = E\Big[\sum_j \Delta R_{ij} a_j \sum_l \Delta R_{kl} a_l\Big] = \sum_{jl} E[\Delta R_{ij} \Delta R_{kl}] a_j a_l$$

$$= \frac{1}{N} \sum_{jl} (R_{ik} R_{jl} + R_{jk} R_{il}) a_j a_l = \frac{1}{N} [R_{ik} (\mathbf{a}^T R \mathbf{a}) + (R\mathbf{a})_i (\mathbf{a}^T R)_k]$$

or,

$$E[\delta\mathbf{a}\,\delta\mathbf{a}^T] = \frac{1}{N} [ER + R\mathbf{a}\,\mathbf{a}^T R] \tag{14.11.5}$$

Writing $\Delta E = \delta\mathbf{a}^T\mathbf{a}$, we find

$$E[\delta\mathbf{a}\,\Delta E] = E[\delta\mathbf{a}\,\delta\mathbf{a}^T]\mathbf{a} = \frac{1}{N} [ER + R\mathbf{a}\,\mathbf{a}^T R]\mathbf{a} = \frac{1}{N} [ER\mathbf{a} + R\mathbf{a}(\mathbf{a}^T R\mathbf{a})] = \frac{2E}{N} R\mathbf{a}$$

Using this result, we find for the asymptotic variance of $\hat{E}$:

$$E[(\Delta E)^2] = \mathbf{a}^T E[\delta\mathbf{a}\,\Delta E] = \frac{2E}{N} \mathbf{a}^T R\mathbf{a} = \frac{2E^2}{N} \tag{14.11.6}$$

This generalizes Eq. (1.16.4). Similarly, we find for the cross-covariance between $\hat{E}$ and $\hat{\mathbf{a}}$:

$$E[\Delta\mathbf{a}\,\Delta E] = E[(E^{-1}\Delta E\mathbf{a} - R^{-1}\delta\mathbf{a})\Delta E] = E^{-1} E[(\Delta E)^2]\mathbf{a} - R^{-1} E[\delta\mathbf{a}\,\Delta E]\,, \quad \text{or,}$$

$$E[\Delta\mathbf{a}\,\Delta E] = E^{-1}\frac{2E^2}{N}\mathbf{a} - R^{-1}\left(\frac{2E}{N}R\mathbf{a}\right) = 0 \tag{14.11.7}$$

Finally, we find for the covariance of the predictor $\hat{\mathbf{a}}$

$$E[\Delta\mathbf{a}\,\Delta\mathbf{a}^T] = E\left[\Delta\mathbf{a}\,(E^{-1}\Delta E\mathbf{a}^T - \delta\mathbf{a}R^{-1})\right] = -E[\Delta\mathbf{a}\,\delta\mathbf{a}^T]R^{-1}$$

$$= -E\left[(E^{-1}\mathbf{a}\,\Delta E - R^{-1}\delta\mathbf{a})\,\delta\mathbf{a}^T\right]R^{-1} = -\left[E^{-1}\mathbf{a}\frac{2E}{N}\mathbf{a}^T R - R^{-1}\frac{1}{N}(ER + R\mathbf{a}\,\mathbf{a}^T R)\right]R^{-1}$$

$$= \frac{E}{N}(R^{-1} - E^{-1}\mathbf{a}\,\mathbf{a}^T) = \frac{E}{N}\begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \tilde{R}^{-1} \end{bmatrix}$$

where we used Eq. (12.9.16) or (1.8.35), and $\tilde{R}$ is the lower-order portion of R. Such result was expected because $\Delta\mathbf{a}$ is of the form $\Delta\mathbf{a} = \begin{bmatrix} 0 \\ \Delta\boldsymbol{\alpha} \end{bmatrix}$. Thus,

$$E[\Delta\boldsymbol{\alpha}\,\Delta\boldsymbol{\alpha}^T] = \frac{E}{N}\tilde{R}^{-1} \tag{14.11.8}$$

This is a well-known result, and although we obtained it for sample covariance matrices of the type (1.6.21), where the snapshots $\mathbf{y}(n)$ were assumed to be independent, it can be proved in the case of autoregressive models where $\hat{R}$ is built out of the sample autocorrelation function [1171,1181–1191].

It can also be shown that asymptotically $\hat{E}$ and $\hat{\boldsymbol{\alpha}}$ are the *maximum likelihood* estimates of the LP parameters E and $\boldsymbol{\alpha}$, having all the good properties of such estimates, namely, asymptotic unbiasedness, consistency, efficiency, and gaussian distribution about the theoretical values with covariances given by Eqs. (14.11.6)–(14.11.8), which are none other than the Cramér-Rao bounds of these parameters. It is instructive to use the general formula (1.18.17) to derive these bounds, where the parameter vector is defined as $\boldsymbol{\lambda} = [E, \boldsymbol{\alpha}^T]^T$. We must determine the dependence of R on these parameters and then compute the derivatives $\partial R/\partial E$ and $\partial R/\partial\boldsymbol{\alpha}$. We write the UL factorization of R in the form of Eq. (1.8.33):

$$R = \begin{bmatrix} \rho_a & \mathbf{r}_a^T \\ \mathbf{r}_a & \tilde{R} \end{bmatrix} = U^{-1}D_a U^{-T} = \begin{bmatrix} 1 & \boldsymbol{\alpha}^T \\ \mathbf{0} & \tilde{U} \end{bmatrix}^{-1}\begin{bmatrix} E & \mathbf{0}^T \\ \mathbf{0} & \tilde{D} \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{\alpha} & \tilde{U}^T \end{bmatrix}$$

The parametrization of R on the parameters E and $\boldsymbol{\alpha}$ is shown explicitly. It is evident that the entries ρ_a and $\mathbf{r}_a$ depend on E and $\boldsymbol{\alpha}$, whereas $\tilde{R}$ does not. We have

$$\mathbf{r}_a = -\tilde{R}\boldsymbol{\alpha}\,, \quad \rho_a = E - \boldsymbol{\alpha}^T\mathbf{r}_a = E + \boldsymbol{\alpha}^T\tilde{R}\boldsymbol{\alpha}$$

Working with differentials, we find $d\mathbf{r}_a = -\tilde{R}d\boldsymbol{\alpha}$ and $d\rho_a = dE + 2\boldsymbol{\alpha}^T\tilde{R}d\boldsymbol{\alpha}$. Differentiating R entry-by-entry and using Eq. (1.8.35) for R^{-1}, we find

$$R^{-1}dR = E^{-1}\begin{bmatrix} dE + \boldsymbol{\alpha}^T\tilde{R}d\boldsymbol{\alpha} & -d\boldsymbol{\alpha}^T\tilde{R} \\ (dE + \boldsymbol{\alpha}^T\tilde{R}d\boldsymbol{\alpha})\,\boldsymbol{\alpha} - Ed\boldsymbol{\alpha} & -\boldsymbol{\alpha}d\boldsymbol{\alpha}^T\tilde{R} \end{bmatrix} \tag{14.11.9}$$

Writing a similar expression for a second differential $R^{-1}\delta R$, multiplying the two, and taking the trace, we find

$$\operatorname{tr}(R^{-1}dRR^{-1}\delta R) = E^{-2}dE\delta E + 2E^{-1}d\boldsymbol{\alpha}^T\tilde{R}\delta\boldsymbol{\alpha} \tag{14.11.10}$$

This gives for the matrix elements of the Fisher information matrix

$$J_{EE} = \frac{1}{2}N\operatorname{tr}\left[R^{-1}\frac{\partial R}{\partial E}R^{-1}\frac{\partial R}{\partial E}\right] = \frac{N}{2E^2}$$

$$J_{\alpha E} = \frac{1}{2}N\operatorname{tr}\left[R^{-1}\frac{\partial R}{\partial\boldsymbol{\alpha}}R^{-1}\frac{\partial R}{\partial E}\right] = 0$$

$$J_{\alpha\alpha} = \frac{1}{2}N\operatorname{tr}\left[R^{-1}\frac{\partial R}{\partial\boldsymbol{\alpha}}R^{-1}\frac{\partial R}{\partial\boldsymbol{\alpha}^T}\right] = \frac{N}{E}\tilde{R}$$

As we know, the inverse of the information matrix is the Cramér-Rao bound for unbiased estimates. This inverse agrees with Eqs. (14.11.6)-(14.11.8).

Following the discussion of [1186,1192], we may also derive the asymptotic covariances of the reflection coefficients. The forward and backward Levinson recursion establishes a one-to-one correspondence between the prediction coefficients $\boldsymbol{\alpha}$ and the vector of reflection coefficients $\boldsymbol{\gamma}$. Therefore, we have the differential correspondence $\Delta\boldsymbol{\gamma} = \Gamma\Delta\boldsymbol{\alpha}$, where Γ is the matrix of partial derivatives $\Gamma_{ij} = \partial\gamma_i/\partial\alpha_j$. It follows that the asymptotic covariance of $\boldsymbol{\gamma}$ will be

$$E[\Delta\boldsymbol{\gamma}\Delta\boldsymbol{\gamma}^T] = \Gamma E[\Delta\boldsymbol{\alpha}\Delta\boldsymbol{\alpha}^T]\Gamma^T = \frac{E}{N}\Gamma\tilde{R}^{-1}\Gamma^T \tag{14.11.11}$$

Example 14.11.1: For the first-order case, we have $\tilde{R} = [R(0)]$ and $E_1 = (1-\gamma_1^2)R(0)$, where $\gamma_1 = -a_{11}$. Thus, we obtain Eq. (1.16.4) as a special case

$$E[(\Delta a_{11})^2] = E[(\Delta\gamma_1)^2] = \frac{1-\gamma_1^2}{N}$$

For the second-order case, $\Delta\boldsymbol{\alpha} = [\Delta a_{12}, \Delta a_{22}]^T$, and we have $E_2 = R(0)(1-\gamma_1^2)(1-\gamma_2^2)$ and $\tilde{R}$ is the order-one autocorrelation matrix. Thus, we find

$$E[\Delta\boldsymbol{\alpha}\,\Delta\boldsymbol{\alpha}^T] = \frac{E_2}{N}\tilde{R}^{-1} = \frac{E_2}{N}\begin{bmatrix} R(0) & R(1) \\ R(1) & R(0) \end{bmatrix}^{-1}$$

$$= \frac{(1-\gamma_1^2)(1-\gamma_2^2)}{N(1-\gamma_1^2)}\begin{bmatrix} 1 & -\gamma_1 \\ -\gamma_1 & 1 \end{bmatrix} = \frac{1-\gamma_2^2}{N}\begin{bmatrix} 1 & -\gamma_1 \\ -\gamma_1 & 1 \end{bmatrix}$$

From the Levinson recursion, we find for the second-order predictor $a_{22} = -\gamma_1(1-\gamma_2)$ and $a_{22} = -\gamma_2$. Differentiating, we have

$$d\boldsymbol{\alpha} = \begin{bmatrix} da_{12} \\ da_{22} \end{bmatrix} = \begin{bmatrix} -(1-\gamma_2) & \gamma_1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} d\gamma_1 \\ d\gamma_2 \end{bmatrix}$$

Inverting, we find

$$d\boldsymbol{\gamma} = \begin{bmatrix} d\gamma_1 \\ d\gamma_2 \end{bmatrix} = \frac{1}{1-\gamma_2}\begin{bmatrix} -1 & -\gamma_1 \\ 0 & -(1-\gamma_2) \end{bmatrix}d\boldsymbol{\alpha} = \Gamma d\boldsymbol{\alpha}$$

Forming the product $\Gamma\tilde{R}^{-1}\Gamma^T$, we finally find

$$E[\Delta\boldsymbol{\gamma}\,\Delta\boldsymbol{\gamma}^T] = \frac{1}{N}\frac{1-\gamma_2^2}{(1-\gamma_2)^2}\begin{bmatrix} 1-\gamma_1^2 & 0 \\ 0 & (1-\gamma_2)^2 \end{bmatrix}$$

which gives component-wise

$$E[(\Delta\gamma_1)^2] = \frac{1}{N}\frac{(1+\gamma_2)(1-\gamma_1^2)}{1-\gamma_2}, \quad E[\Delta\gamma_1\Delta\gamma_2] = 0, \quad E[(\Delta\gamma_2)^2] = \frac{1-\gamma_2^2}{N}$$

Setting $\gamma_2 = 0$, the variance of γ_1 becomes equal to that of the first-order case and $E[(\Delta\gamma_2)^2] = 1/N$. More generally, for an autoregressive process of order M, all reflection coefficients of order greater than M vanish, but their asymptotic variances are equal to $1/N$, that is, $E[(\Delta\gamma_p)^2] = 1/N$, for $p > M$, [1186,1192]. □

Next, we consider the asymptotic properties of the eigenstructure of $\hat{R}$ [1197–1205]. In the complex-valued case $\hat{R}$ is given by Eq. (14.4.14), and Eq. (14.11.1) is replaced by

$$E[\Delta R_{ij}\,\Delta R_{kl}] = \frac{1}{N}R_{il}R_{kj} \tag{14.11.12}$$

where again $\Delta R = \hat{R} - R$. This can be shown in the same way as Eq. (1.6.23) using the following expression for the expectation value of the product of four complex gaussian random variables arising from the (independent) snapshots $\mathbf{y}(n)$ and $\mathbf{y}(m)$:

$$E[y_i(n)^* y_j(n) y_k(m)^* y_l(m)] = R_{ij}R_{kl} + \delta_{nm}R_{il}R_{kj}$$

Equation (14.11.12) may be written more conveniently in the form

$$E[(\mathbf{a}^\dagger\Delta R\mathbf{b})(\mathbf{c}^\dagger\Delta R\mathbf{d})] = \frac{1}{N}(\mathbf{a}^\dagger R\mathbf{d})(\mathbf{c}^\dagger R\mathbf{b}) \tag{14.11.13}$$

for any four $(M+1)$-dimensional vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$. In particular, we may apply it to four eigenvectors of R. Let $\mathbf{e}_i$ denote the orthonormal eigenvectors of R, $R\mathbf{e}_i = \lambda_i\mathbf{e}_i$, with eigenvalues arranged in increasing order. Then,

$$E[(\mathbf{e}_i^\dagger\Delta R\mathbf{e}_j)(\mathbf{e}_k^\dagger\Delta R\mathbf{e}_l)] = \frac{1}{N}(\mathbf{e}_i^\dagger R\mathbf{e}_l)(\mathbf{e}_k^\dagger R\mathbf{e}_j) = \frac{1}{N}\lambda_i\lambda_j\delta_{il}\delta_{kj}$$

where we used $(\mathbf{e}_i^\dagger R)\mathbf{e}_l = \lambda_i\mathbf{e}_i^\dagger\mathbf{e}_l = \lambda_i\delta_{il}$. Arranging the eigenvectors into the eigenvector matrix $E = [\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_M]$, we recognize that the quantities $\mathbf{e}_i\Delta R\mathbf{e}_j$, are the matrix elements of ΔR in the E basis; that is, the elements of the matrix $\Delta V = E^\dagger\Delta R E$. Thus, we obtain the *diagonalized* version of Eq. (14.11.12)

$$E[\Delta V_{ij}\,\Delta V_{kl}] = \frac{1}{N}\lambda_i\lambda_j\delta_{il}\delta_{kj} \tag{14.11.14}$$

The asymptotic properties of the eigenstructure of $\hat{R}$ are obtained by using Eq. (14.11.14) and standard first-order perturbation theory. The eigenproblems for R and $\hat{R}$ are,

$$RE = E\Lambda \quad \text{and} \quad \hat{R}\hat{E} = \hat{E}\hat{\Lambda} \tag{14.11.15}$$

where $\hat{E}, E$ are the eigenvector matrices and $\hat{\Lambda}, \Lambda$ the diagonal matrices of the eigenvalues. Because the eigenvectors E form a complete set, it follows that the eigenvectors $\hat{E}$ can be expanded as linear combinations of the former; that is, $\hat{E} = EF$. The orthonormality and completeness of $\hat{E}$ and E require that F be a unitary matrix, satisfying $F^\dagger F = FF^\dagger = I$. This is easily shown; for example, $I = \hat{E}^\dagger\hat{E} = F^\dagger E^\dagger EF = F^\dagger IF = F^\dagger F$.

In carrying out the first-order perturbation analysis, we shall assume initially that all the eigenvalues of R are distinct. This corresponds to the Pisarenko case, where the noise subspace is one-dimensional and thus, $L = M$.

The assumption of distinct eigenvalues means that, under a perturbation, $\hat{R} = R + \Delta R$, each eigenvector changes by a small correction of the form $\hat{E} = E + \Delta E$. By the completeness of the basis E we may write $\Delta E = E\Delta C$ so that $\hat{E} = E(I + \Delta C) = EF$. The unitarity of the matrix $F = I + \Delta C$ requires that ΔC be anti-hermitian; that is, $\Delta C + \Delta C^\dagger = 0$. This follows from the first-order approximation $F^\dagger F = I + \Delta C + \Delta C^\dagger$. The perturbation changes the eigenvalues by $\hat{\lambda}_i = \lambda_i + \Delta\lambda_i$, or, $\hat{\Lambda} = \Lambda + \Delta\Lambda$. To determine the first-order corrections we use Eq. (14.11.15)

$$(R + \Delta R)(E + \Delta E) = (E + \Delta E)(\Lambda + \Delta\Lambda) \quad \Rightarrow \quad (\Delta R)E + R(\Delta E) = (\Delta E)\Lambda + E(\Delta\Lambda)$$

where we kept only the first-order terms. Multiplying both sides by $E^\dagger$ and using $E^\dagger RE = \Lambda$ and the definition $\Delta V = E^\dagger(\Delta R)E$, we obtain

$$\Delta V + \Lambda(\Delta C) = (\Delta C)\Lambda + \Delta\Lambda \quad \Rightarrow \quad \Delta\Lambda + (\Delta C)\Lambda - \Lambda(\Delta C) = \Delta V$$

or, component-wise

$$\Delta\lambda_i\delta_{ij} + (\lambda_j - \lambda_i)\Delta C_{ij} = \Delta V_{ij}$$

Setting $i = j$ and then $i \neq j$, we find

$$\Delta\lambda_i = \Delta V_{ii}, \quad \Delta C_{ij} = -\frac{\Delta V_{ij}}{\lambda_i - \lambda_j}, \quad \text{for} \quad i \neq j \tag{14.11.16}$$

Using Eq. (14.11.14), we obtain the asymptotic variances of the eigenvalues

$$E[(\Delta\lambda_i)^2] = E[\Delta V_{ii}\,\Delta V_{ii}] = \frac{\lambda_i^2}{N} \tag{14.11.17}$$

For the eigenvectors, we write

$$\Delta\mathbf{e}_i = \hat{\mathbf{e}}_i - \mathbf{e}_i = \sum_{j\neq i}\mathbf{e}_j\Delta C_{ji}$$

and their covariances are

$$E[\Delta\mathbf{e}_i\Delta\mathbf{e}_i^\dagger] = \sum_{j\neq i}\sum_{k\neq i}\mathbf{e}_j\mathbf{e}_k^\dagger E[\Delta C_{ji}\Delta C_{ki}^*]$$

Using the anti-hermiticity of ΔC and Eq. (14.11.14), we find

$$E[\Delta C_{ji}\Delta C_{ki}^*] = -\frac{E[\Delta V_{ji}\Delta V_{ik}]}{(\lambda_j - \lambda_i)(\lambda_i - \lambda_k)} = \frac{1}{N}\frac{\lambda_i\lambda_j}{(\lambda_i - \lambda_j)^2}\delta_{jk}$$

which gives

$$E[\Delta\mathbf{e}_i\Delta\mathbf{e}_i^\dagger] = \frac{1}{N}\sum_{j\neq i}\frac{\lambda_i\lambda_j}{(\lambda_i - \lambda_j)^2}\mathbf{e}_j\mathbf{e}_j^\dagger \tag{14.11.18}$$

Separating out the minimum eigenvalue λ_0 and eigenvector $\mathbf{e}_0$, and denoting the remaining signal subspace eigenvectors and eigenvalues by $E_S = [\mathbf{e}_1, \ldots, \mathbf{e}_M]$ and $\Lambda_S = \text{diag}\{\lambda_1, \ldots, \lambda_M\}$, we may write Eq. (14.11.18) compactly

$$E[\Delta\mathbf{e}_0 \Delta\mathbf{e}_0^\dagger] = \frac{\lambda_0}{N} E_S \Lambda_S (\Lambda_S - \lambda_0 I_M)^{-2} E_S^\dagger \tag{14.11.19}$$

where I_M is the M-dimensional unit matrix. The zeros of the polynomial $\mathbf{e}_0$ contain the desired frequency information. The asymptotic variances for the zeros can be obtained by writing

$$\Delta z_i = \left(\frac{\partial z_i}{\partial \mathbf{e}_0}\right)^T \Delta\mathbf{e}_0$$

which gives

$$E\big[|\Delta z_i|^2\big] = \left(\frac{\partial z_i}{\partial \mathbf{e}_0}\right)^T E[\Delta\mathbf{e}_0 \Delta\mathbf{e}_0^\dagger] \left(\frac{\partial z_i}{\partial \mathbf{e}_0}\right)^* \tag{14.11.20}$$

Example 14.11.2: In the $L = M = 1$ Example 14.3.1, we have for the eigenvalues and orthonormal eigenvectors of R

$$\lambda_0 = \sigma_v^2, \quad \lambda_1 = \sigma_v^2 + 2P_1, \quad \mathbf{e}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -e^{jk_1} \end{bmatrix}, \quad \mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{jk_1} \end{bmatrix}$$

It follows from Eq. (14.11.19) that

$$E[\Delta\mathbf{e}_0 \Delta\mathbf{e}_0^\dagger] = \frac{1}{N} \mathbf{e}_1 \mathbf{e}_1^\dagger \frac{\lambda_1 \lambda_0}{(\lambda_1 - \lambda_0)^2}$$

Using the general formula for the sensitivities of zeros with respect to the coefficients of a polynomial [12].

$$\frac{\partial z_i}{\partial a_m} = -\frac{1}{a_0} \frac{z_i^{M-m}}{\prod_{j \neq i}(z_i - z_j)}$$

we find for the zero $z_1 = e^{jk_1}$ of $\mathbf{e}_0$

$$\frac{\partial z_1}{\partial \mathbf{e}_0} = -\sqrt{2} \begin{bmatrix} z_1 \\ 1 \end{bmatrix}$$

Using this into Eq. (14.11.20), we find

$$E\big[|\Delta z_i|^2\big] = \frac{1}{N} \frac{4\lambda_1 \lambda_0}{(\lambda_1 - \lambda_0)^2} = \frac{1}{N} \frac{1 + 2SNR}{SNR^2}, \quad SNR = \frac{P_1}{\sigma_v^2}$$

This implies that the quality of the estimated zero improves either by increasing the number of snapshots N or the signal to noise ratio. For low SNR, the denominator $(\lambda_1 - \lambda_0)^2$ becomes small and the variance of z_1 increases, resulting in degradation of performance. For a given level of quality there is a tradeoff between the number of snapshots and SNR. In general, the signal subspace eigenvalues Λ_S will be separated from $\lambda_0 = \sigma_v^2$ by a term that depends on the signal powers, say, $\Lambda_S = \lambda_0 I_M + P_S$. Then,

$$\lambda_0 \Lambda_S (\Lambda_S - \lambda_0 I_M)^{-2} = (I_M + P_S/\sigma_v^2)(P_S/\sigma_v^2)^{-2}$$

and Eq. (14.11.19) implies that the estimate of $\mathbf{e}_0$ becomes better for higher SNRs. □

When the noise subspace has dimension $K = M + 1 - L$ and the minimum eigenvalue λ_0 has K-fold degeneracy, the first-order perturbation analysis becomes somewhat more complicated. The eigenproblem for R is divided into its noise and signal subspace parts

$$RE_N = \lambda_0 E_N, \quad RE_S = E_S \Lambda_S$$

where E_N consists of the K degenerate eigenvectors belonging to the minimum eigenvalue $\lambda_0 = \sigma_v^2$ and E_S consists of the remaining L signal subspace eigenvectors. Under a perturbation $\hat{R} = R + \Delta R$, the degeneracy of E_N is lifted and the noise subspace eigenvalues become unequal $\hat{\lambda}_i = \lambda_0 + \Delta\lambda_i$, $i = 0, 1, \ldots, K-1$, or, $\hat{\Lambda}_N = \lambda_0 I_K + \Delta\Lambda_N$. Similarly, the signal subspace eigenvalues change to $\hat{\Lambda}_S = \Lambda_S + \Delta\Lambda_S$.

The signal subspace eigenvectors, belonging to distinct eigenvalues, change in the usual way; namely, each eigenvector changes by receiving small contributions from all other eigenvectors. The noise subspace eigenvectors, however, being degenerate, are mixed up by the perturbation into linear combinations of themselves, and in addition, they receive small corrections from the signal subspace eigenvectors. Thus, the eigenproblem for the perturbed matrix $\hat{R}$ is

$$\hat{R}\hat{E}_N = \hat{E}_N \hat{\Lambda}_N, \quad \hat{R}\hat{E}_S = \hat{E}_S \hat{\Lambda}_S \tag{14.11.21}$$

where the corrections of the eigenvectors are of the form

$$\hat{E}_N = E_N C + E_S \Delta C, \quad \hat{E}_S = E_S + E_S \Delta B + E_N \Delta D \tag{14.11.22}$$

In absence of the perturbation ΔR, the choice of the degenerate basis E_N is arbitrary and can be replaced by any linear combination $E_N C$. The presence of the perturbation fixes this particular linear combination by the requirement that the change in the eigenvectors be small. Combining the two equations into the full eigenvector matrices, we have

$$\hat{E} = [\hat{E}_N, \hat{E}_S] = [E_N, E_S] \begin{bmatrix} C & \Delta D \\ \Delta C & I_L + \Delta B \end{bmatrix} = EF$$

The orthonormality and completeness requirements for $\hat{E}$ imply that $F^\dagger F = FF^\dagger = I$. To first order, these conditions are equivalent to

$$C^\dagger C = I_K, \quad \Delta C + \Delta D^\dagger C = 0, \quad \Delta B + \Delta B^\dagger = 0 \tag{14.11.23}$$

Thus, C must be unitary. Inserting Eq. (14.11.22) into the first term of (14.11.21) and using (14.11.23), we find

$$(R + \Delta R)(E_N C - E_S \Delta D^\dagger C) = (E_N C - E_S \Delta D^\dagger C)(\lambda_0 I_K + \Delta\Lambda_N)$$

and equating first-order terms,

$$\Delta R E_N C - E_S \Lambda_S \Delta D^\dagger C = E_N C \Delta\Lambda_N - E_S \Delta D^\dagger C \lambda_0$$

Multiplying both sides first by $E_N^\dagger$ and then by $E_S^\dagger$ and using the orthonormality properties (14.4.3), we obtain

$$\Delta V_{NN} C = C \Delta\Lambda_N \tag{14.11.24}$$

where $\Delta V_{NN} = E_N^\dagger \Delta R E_N$, and

$$\Delta V_{SN} C - \Lambda_S \Delta D^\dagger C = -\Delta D^\dagger C \lambda_0$$

where $\Delta V_{SN} = E_S^\dagger \Delta R E_N$, and solving for $\Delta D^\dagger$

$$\Delta D^\dagger = (\Lambda_S - \lambda_0 I_L)^{-1} \Delta V_{SN} \tag{14.11.25}$$

Similarly, from the second term of Eq. (14.11.21), we find for ΔB

$$\Delta\Lambda_S + \Delta B \Lambda_s - \Lambda_s \Delta B = \Delta V_{SS}\,, \quad \Delta V_{SS} = E_S^\dagger \Delta R E_S \tag{14.11.26}$$

which can be solved as in Eq. (14.11.16). To summarize, the corrections to the noise subspace eigenvalues $\Delta\Lambda_N$ and the unitary matrix C are obtained from the solution of the $K\times K$ eigenproblem (14.11.24), ΔD constructed by (14.11.25), then ΔC is constructed by (14.11.23), and ΔB by (14.11.26).

Because the corrections to the signal subspace eigenvectors are obtained from the non-degenerate part of the perturbation analysis, it follows that (14.11.18) is still valid for the signal eigenvectors. More specifically, because we index the noise subspace eigenvectors for $0 \le i \le K-1$ and the signal subspace eigenvectors for $K \le i \le M$, we may split the sum over the noise and signal subspace parts

$$E[\Delta \mathbf{e}_i \Delta \mathbf{e}_i^\dagger] = \frac{1}{N} \frac{\lambda_0 \lambda_i}{(\lambda_0 - \lambda_i)^2} \sum_{j=0}^{K-1} \mathbf{e}_j \mathbf{e}_j^\dagger + \frac{1}{N} \sum_{\substack{j\neq i \\ j=K}}^{M} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \mathbf{e}_j \mathbf{e}_j^\dagger$$

where we used the fact that all noise subspace eigenvalues are equal to λ_0. The first term is recognized as the projector onto the noise subspace. Thus, for $K \le i \le M$,

$$E[\Delta \mathbf{e}_i \Delta \mathbf{e}_i^\dagger] = \frac{1}{N} \frac{\lambda_0 \lambda_i}{(\lambda_0 - \lambda_i)^2} E_N E_N^\dagger + \frac{1}{N} \sum_{\substack{j\neq i \\ j=K}}^{M} \frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \mathbf{e}_j \mathbf{e}_j^\dagger \tag{14.11.27}$$

Because most eigenvector methods can also be formulated in terms of the signal subspace eigenvectors, it is enough to consider only the asymptotic covariances of these eigenvectors. For example, in the reduced-order method of Sec. 14.7, the reduced-order polynomials $\mathbf{a}_i$ may alternatively be computed by requiring that the corresponding shifted vectors $\mathbf{b}_i$ be orthogonal to the signal subspace [1139]; namely, $E_S^\dagger \mathbf{b}_i = 0$, $i = 0, 1, \ldots, K-1$, and similarly, for the empirical quantities $\hat{E}_S^\dagger \hat{\mathbf{b}}_i = 0$. If we denote by G_i the part of E_S consisting of $L+1$ rows starting with the ith row, then, these conditions become $G_i^\dagger \mathbf{a}_i = 0$. Because the first coefficient of $\mathbf{a}_i$ is unity, these give rise to L linear equations for the L last coefficients $\mathbf{a}_i$. It follows that $\mathbf{a}_i$ can be constructed as a function of the signal eigenvectors, and thus, one can obtain the corresponding covariance of $\mathbf{a}_i$ using Eq. (14.11.27). An example will illustrate this remark.

Example 14.11.3: Consider the case of one plane wave ($L = 1$) and arbitrary M. The covariance matrix $R = \sigma_v^2 I + P_1 \mathbf{s}_{k_1} \mathbf{s}_{k_1}^\dagger$ has a one-dimensional signal subspace so that $E_S = [\mathbf{e}_M]$, Its eigenvalue is $\lambda_M = \sigma_v^2 + (M+1)P_1$. The matrix G_i is formed by row i to row $i+L = i+1$, that is,

$$G_i = \begin{bmatrix} e_{M,i} \\ e_{M,i+1} \end{bmatrix} = \frac{1}{\sqrt{M+1}} \begin{bmatrix} e^{jk_1 i} \\ e^{jk_1 (i+1)} \end{bmatrix}$$

The equation $G_i^\dagger \mathbf{a}_i = 0$ becomes for the first-order filters $\mathbf{a}_i$,

$$G_i^\dagger \mathbf{a}_i = \frac{1}{\sqrt{M+1}} [e^{-jk_1 i},\, e^{-jk_1 (i+1)}] \begin{bmatrix} 1 \\ a_{i1} \end{bmatrix} = 0 \quad \Rightarrow \quad a_{i1} = -e^{jk_1}$$

and hence, all the reduced-order polynomials are equal to the theoretical one, $A_i(z) = 1 - e^{jk_1} z^{-1}$. Now, if the empirical $\hat{\mathbf{e}}_M$ is used , then a similar calculation gives $a_{i1} = -e_{M,i}^* / e_{M,i+1}^*$, and therefore, the estimated zero will be $\hat{z}_1 = e_{M,i}^* / e_{M,i+1}^*$. Differentiating, we obtain $d\hat{z}_1 = de_{M,i}^* / e_{M,i+1}^* - e_{M,i}^* de_{M,i+1}^* / e_{M,i+1}^{*2}$; therefore, its covariance will be

$$E[|\Delta z_1|^2] = \frac{1}{|e_{M,i+1}|^2} E[|\Delta e_{M,i}|^2] + \frac{|e_{M,i}|^2}{|e_{M,i+1}|^4} E[|\Delta e_{M,i+1}|^2] - 2\,\mathrm{Re}\left[\frac{e_{M,i}^*}{e_{M,i+1} e_{M,i+1}^{*2}} E[\Delta e_{M,i} \Delta e_{M,i+1}^*] \right]$$

This simplifies to

$$E[|\Delta z_1|^2] = (M+1)\left[E[|\Delta e_{M,i}|^2] + E[|\Delta e_{M,i+1}|^2] - 2\,\mathrm{Re}\left(e^{jk_1} E[\Delta e_{M,i} \Delta e_{M,i+1}^*]\right) \right]$$

Because the signal subspace is one-dimensional, the second term in Eq. (14.11.27) is absent. The noise-subspace projector can be expressed in terms of the signal-subspace projector $E_N E_N^\dagger = I - E_S E_S^\dagger$. Thus, Eq. (14.11.27) gives

$$E[\Delta \mathbf{e}_0 \Delta \mathbf{e}_0^\dagger] = \frac{1}{N} \frac{\lambda_M \lambda_0}{(\lambda_M - \lambda_0)^2} \left(I - \frac{1}{M+1} \mathbf{s}_{k_1} \mathbf{s}_{k_1}^\dagger \right)$$

Extracting the ith and $(i+1)$st components, we get for the variance of the estimated zero

$$E[|\Delta z_1|^2] = \frac{1}{N} \frac{2(M+1)\lambda_M \lambda_0}{(\lambda_M - \lambda_0)^2} = \frac{1}{N} \frac{2[1 + (M+1)\,SNR]}{(M+1)\,SNR^2}$$

where $SNR = P_1 / \sigma_v^2$. Setting $M = 1$, we recover the result of Example 14.11.2. □

14.12 Computer Project – LCMV Beamforming and GSC

This computer project, divided into separate parts, deals with the theory of linearly-constrained minimum-variance (LCMV) beamforming and its equivalence to the generalized sidelobe canceler [1209–1221]. The problem also has application in linearly-constrained time-series Wiener filtering, and other applications, such as optimum minimum-variance Markowitz portfolios in finance (those are discussed in the following project). The following topics are included,

- Linearly-constrained Wiener filtering problem.
- Linearly-constrained minimum-variance beamforming.

- Retrodirective beams towards multiple interferers.
- Quiescent pattern control with linear constraints.
- Equivalence of LCMV and the generalized sidelobe canceler.

1. *Linearly-constrained Wiener filtering problem.* Let R be a given $M\times M$ positive-definite Hermitian matrix and $\mathbf{r}$ be a given $M\times 1$ complex-valued vector. It is desired to find the weight vector $\mathbf{a}$ that minimizes:

$$\mathcal{E} = \mathbf{a}^\dagger R\mathbf{a} - \mathbf{r}^\dagger\mathbf{a} - \mathbf{a}^\dagger\mathbf{r} = \min \tag{14.12.1}$$

subject to the K linear constraints:

$$C^\dagger\mathbf{a} = \mathbf{g} \tag{14.12.2}$$

where $K < M$, and C is a $M\times K$ matrix with linearly independent columns, and $\mathbf{g}$ is a given $K\times 1$ vector of "gains". Component-wise, Eq. (14.12.2) reads $\mathbf{c}_i^\dagger\mathbf{a} = g_i$, $i = 1, 2, \ldots, K$, where $\mathbf{c}_i$ is the ith column of C, and g_i the ith component of $\mathbf{g}$.

(a) Show that the unconstrained minimization of Eq. (14.12.1) gives the solution:

$$\mathbf{a}_u = R^{-1}\mathbf{r} \tag{14.12.3}$$

(b) Introduce a K-dimensional complex-valued vector of Lagrange multipliers $\boldsymbol{\lambda}$ and minimize the modified performance index:

$$J = \mathbf{a}^\dagger R\mathbf{a} - \mathbf{r}^\dagger\mathbf{a} - \mathbf{a}^\dagger\mathbf{r} + \boldsymbol{\lambda}^\dagger(\mathbf{g} - C^\dagger\mathbf{a}) + (\mathbf{g}^\dagger - \mathbf{a}^\dagger C)\boldsymbol{\lambda} = \min$$

Show that the solution the solution of this problem is the solution of the constrained problem of Eqs. (14.12.1) and (14.12.2) can be expressed in terms of $\mathbf{a}_u$ as follows:

$$\mathbf{a} = \mathbf{a}_u + R^{-1}C(C^\dagger R^{-1}C)^{-1}(\mathbf{g} - C^\dagger\mathbf{a}_u) \tag{14.12.4}$$

Many of the subsequent questions are special cases of this result.

2. *LCMV Beamforming.* Consider an array of M antennas equally-spaced at distance d (in units of the wavelength λ) along the x-axis. The array response is to be designed to achieve K prescribed gain values g_i at the directions θ_i corresponding to the steering vectors:

$$\mathbf{s}_i = \mathbf{s}_{k_i}\,, \quad k_i = 2\pi d\,\sin\theta_i\,, \quad i = 1, 2, \ldots, K \tag{14.12.5}$$

where

$$\mathbf{s}_k = \begin{bmatrix} 1 \\ e^{jk} \\ e^{2jk} \\ \vdots \\ e^{(M-1)jk} \end{bmatrix}$$

Thus, the constraint matrix C and gain vector are:

$$C = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_K]\,, \quad \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_K \end{bmatrix}$$

We assume that L plane waves are incident on the array from L directions which may or may not coincide with the above constraint angles. Let S be the $M\times L$ steering matrix of the incident plane waves and let P be their $L\times L$ power matrix, assumed to have full rank. The $M\times M$ autocorrelation matrix of the array output in the presence of uncorrelated noise is:

$$R = \sigma_v^2 I + SPS^\dagger \tag{14.12.6}$$

(a) Using the results of the previous question, show that the optimum array weights that minimize the output power subject to the K constraints:

$$\boxed{\mathcal{E} = \mathbf{a}^\dagger R\mathbf{a} = \min\,, \quad \text{subject to} \quad C^\dagger\mathbf{a} = \mathbf{g}} \tag{14.12.7}$$

are given by:

$$\boxed{\mathbf{a} = R^{-1}C(C^\dagger R^{-1}C)^{-1}\mathbf{g}} \quad \text{(LCMV)} \tag{14.12.8}$$

This is known as the *linearly-constrained minimum variance* (LCMV) beamformer. See Frost [1209] for its LMS adaptive implementation. The corresponding array response towards an angle θ is defined as follows in absolute units and in dB

$$A(\theta) = |\mathbf{s}_k^\dagger\mathbf{a}|\,, \quad A_{\text{dB}}(\theta) = 20\log_{10}A(\theta)\,, \quad k = 2\pi d\sin\theta \tag{14.12.9}$$

(b) With R given by Eq. (14.12.6), show that if $C = S$, then, Eq. (14.12.8) reduces to

$$\mathbf{a} = S(S^\dagger S)^{-1}\mathbf{g} \tag{14.12.10}$$

which is recognized as the minimum-norm solution of the equation $S^\dagger\mathbf{a} = \mathbf{g}$. Being the minimum-norm solution implies that it minimizes $\mathbf{a}^\dagger\mathbf{a}$. How is this reconciled with the fact that $\mathbf{a}$ minimizes Eq. (14.12.7)?

(c) *Retrodirective Beamforming towards Multiple Interferers.* An example of retrodirective beamforming was given in Eq. (14.3.27). In the present notation, this corresponds to the case $C = [\mathbf{s}_1]$, $\mathbf{g} = [1]$, and $S = [\mathbf{s}_1, \mathbf{s}_2]$ with $\mathbf{s}_1$ being the desired look-direction and $\mathbf{s}_2$ representing a jammer.

Suppose that the incident signals are divided into two groups, the desired signals S_1 and the interferers S_2, so that $S = [S_1, S_2]$, where S_1 and S_2 have L_1 and L_2 columns, respectively, such that $L_1 + L_2 = L$, and let us assume that the corresponding power matrices are P_1 and P_2 and that the S_1 and S_2 are spatially uncorrelated so that the full power matrix is block-diagonal so that R is expressed in the form:

$$R = \sigma_v^2 I + S_2P_2S_2^\dagger + S_1P_1S_1^\dagger \equiv R_n + S_1P_1S_1^\dagger$$

where $R_n = \sigma_v^2 I + S_2 P_2 S_2^\dagger$ is the noise-plus-interference covariance matrix. Using the matrix inversion lemma, show the identity:

$$R^{-1}S_1(S_1^\dagger R^{-1}S_1)^{-1} = R_n^{-1}S_1(S_1^\dagger R_n^{-1}S_1)^{-1}$$

Thus, if we choose $C = S_1$ and $\mathbf{g}$ being an arbitrary vector of responses towards the desired look directions, the optimum weights will be given by:

$$\mathbf{a} = R^{-1}C(C^\dagger R^{-1}C)^{-1}\mathbf{g} = R_n^{-1}S_1(S_1^\dagger R_n^{-1}S_1)^{-1}\mathbf{g}$$

Using the matrix inversion lemma on R_n, show the following results:

$$R_n^{-1}S_1 = \frac{1}{\sigma_v^2}[S_1 - S_2(\sigma_v^2 P_2^{-1} + S_2^\dagger S_2)^{-1}S_2^\dagger S_1]$$

$$S_2^\dagger R_n^{-1}S_1 = P_2^{-1}(\sigma_v^2 P_2^{-1} + S_2^\dagger S_2)^{-1}S_2^\dagger S_1$$

Thus, the array gain steered towards the interferers, $S_2^\dagger \mathbf{a}$, becomes smaller with increasing interferer power P_2.

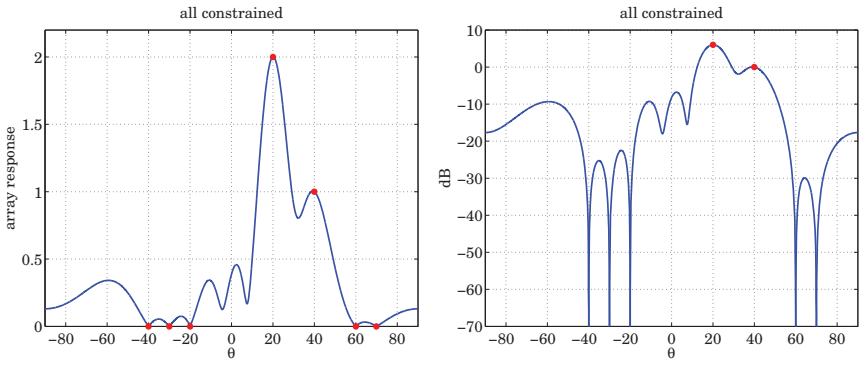
(d) Consider an array of 10 antennas equally-spaced at half-wavelength spacings ($d = 0.5$). The array is designed to receive two desired signals from angles $\theta_1 = 20^o$ and $\theta_2 = 40^o$ and reject five interferers coming in from the directions:

$$\{\theta_3, \theta_4, \theta_5, \theta_6, \theta_7\} = \{-40^o, -30^o, -20^o, 60^o, 70^o\}$$

The response towards θ_1 is to be double that towards θ_2, while the responses towards the interferers must be zero. Thus, the constraint matrix and gain vector must be chosen as:

$$C = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5, \mathbf{s}_6, \mathbf{s}_7], \quad \mathbf{g} = [2, 1, 0, 0, 0, 0, 0]^T$$

Assume that all plane waves have 10-dB power levels (and P is diagonal). Design the optimum weight vector and calculate and plot the array response of Eq. (14.12.9) in absolute units over the angle range $-90^o \le \theta \le 90^o$. Indicate on the graph the values of the gain vector $\mathbf{g}$. Plot it also in dB with a vertical scales ranging from $[-70, 10]$ dB.

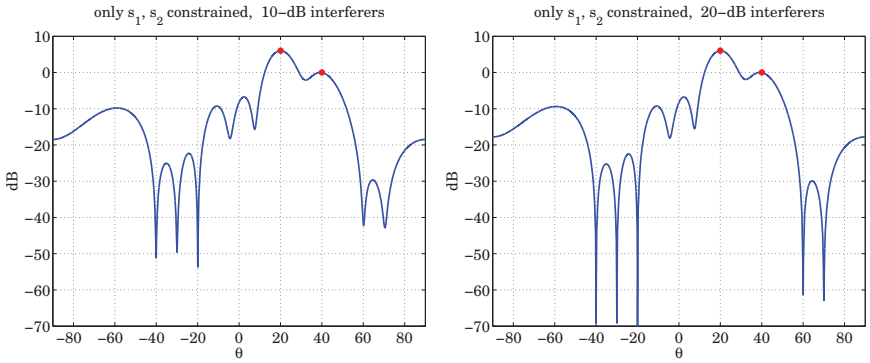


(e) To see the retrodirective action of the optimum beamformer, redesign the optimum weights based only on the desired signal constraints:

$$C = [\mathbf{s}_1, \mathbf{s}_2], \quad \mathbf{g} = [2, 1]^T$$

and plot the array response in dB using the same vertical dB scales as before. Note the nulls at the interferer directions.

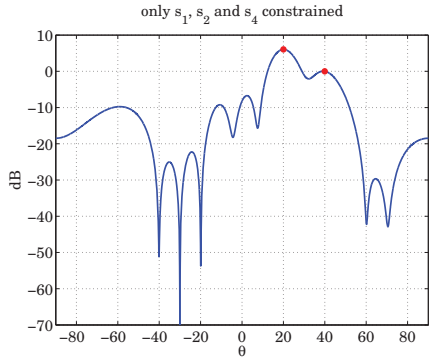
Repeat when the interferer powers are increased to 20 dB. In this case, you should expect the interferer nulls to be deeper.



(f) Repeat when the interferer at θ_4 is to be nulled exactly and the other interferers nulled approximately by the retrodirective action of the beamformer, that is, choose

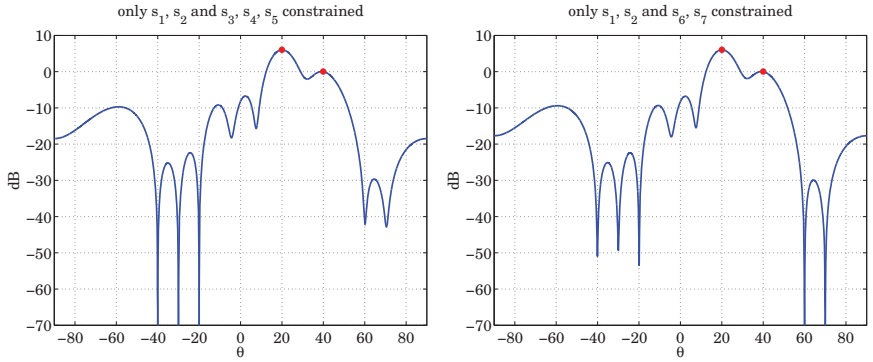
$$C = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_4], \quad \mathbf{g} = [2, 1, 0]^T$$

Plot the array gain in dB using the same vertical scales as before. Assume 10-dB power levels for all signals.



(g) Repeat the design and the gain plot for the following two cases. When interferes 3,4,5 are to be nulled exactly, and when 6,7 are to be nulled, that is, for the two cases defined by:

$$C = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5], \quad \mathbf{g} = [2,1,0,0,0]^T$$
$$C = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_6, \mathbf{s}_7], \quad \mathbf{g} = [2,1,0,0]^T$$



Note that the construction of the steering matrix S and constraint matrix C can be done with the help of the MATLAB function `steermat`, and the array gain can be calculated with the help of the `dtft` function, both of which may be found in the collection `osp-toolbox`. For example, in the very last case, we may define,

```
Pdb = [10 10 10 10 10 10 10]; P = diag(10.^(Pdb/10));
S = steermat(M-1, exp(j*[k1, k2, k3, k4, k5, k6, k7]));
C = steermat(M-1, exp(j*[k1, k2, k6, k7]));
R = eye(M) + S*P*S';    % assume unit-variance noise
```

where $k_i = 2\pi d \sin\theta_i$. For the gain computation, you may use:

```
th = linspace(-90,90,1001); thr = pi*th/180; d=0.5;
k = 2*pi*d*sin(thr);
A = abs(dtft(a,k));          % array response, |A(theta)|
plot(th,A);
```

3. *Quiescent Pattern Control.* Next, we consider the proper design of the quiescent response of an array that achieves a desired shape and respects the beam constraints. The method is discussed by Griffiths and Buckley [1213].

Consider an antenna array defined by Eqs. (14.12.6)–(14.12.9). The quiescent weights $\mathbf{a}_q$ correspond to the case when the incident signals are absent and only noise is present, that is, when $R = \sigma_v^2 I$. In this case, show that the optimum weights (14.12.8) are given by

$$\mathbf{a}_q = C(C^\dagger C)^{-1}\mathbf{g} \tag{14.12.11}$$

They correspond to the minimum-norm solution of the constraints, $C^\dagger \mathbf{a} = \mathbf{g}$.

(a) Suppose that the array is to be steered towards a desired look-direction θ_1 corresponding to a steering vector $\mathbf{s}_1$. If we choose $C = \mathbf{s}_1$ for the constraint matrix and $\mathbf{g} = [1]$, that is, $\mathbf{s}_1^\dagger \mathbf{a} = 1$ for the constraint, then show that the quiescent weights are:

$$\mathbf{a}_q = \frac{1}{M}\mathbf{s}_1 \tag{14.12.12}$$

and that the array response becomes

$$A(\theta) = |\mathbf{s}_k^\dagger \mathbf{a}_q| = \frac{1}{M}\,|W(k-k_1)| \tag{14.12.13}$$

where $k = 2\pi d \sin\theta$, $k_1 = 2\pi d \sin\theta_1$, and $W(k)$ is the response of the rectangular window of length M.

(b) If $C = \mathbf{s}_1$ as above, but only the desired signal is present, that is, $R = \sigma_v^2 I + P_1 \mathbf{s}_1 \mathbf{s}_1^\dagger$, then show that the optimum weights are again given by Eq. (14.12.12).

(c) Consider an array of 21 antennas spaced at half-wavelength intervals. The desired signal comes from the direction $\theta_1 = 30^\circ$ and has 0-dB signal to noise ratio.

Plot the quiescent response (14.12.13) versus angle using dB scales with a vertical range of $[-80, 5]$ dB. Notice the usual 13-dB level of the highest sidelobes. This corresponds to a steered uniform array.

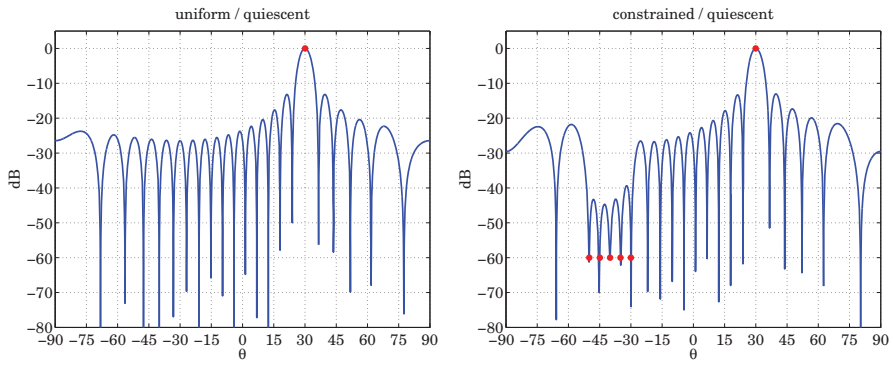
(d) It is anticipated that one or more interferers may come on in the angular range of $-50^\circ \le \theta \le -30^\circ$. To mitigate their effect, we try to force the array gain to be -60 dB at the following angles:

$$\{\theta_2, \theta_3, \theta_4, \theta_5, \theta_6\} = \{-50^\circ, -45^\circ, -40^\circ, -35^\circ, -30^\circ\}$$

Thus, we may choose the constraint matrix and gain vector to be:

$$C = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5, \mathbf{s}_6], \quad \mathbf{g} = [1, g, g, g, g, g]^T \tag{14.12.14}$$

where $g = 10^{-60/20} = 10^{-3}$. Compute the corresponding quiescent weights using Eq. (14.12.11) and plot the corresponding array angular pattern in dB as above. Place the points θ_i on the graph.



(e) The quiescent responses of the uniform array considered above are not very good because of their substantial sidelobe level. As an example of a better response, consider the design of a 21-element Dolph-Chebyshev array that is steered towards the angle $\theta_1 = 30^o$ and has a sidelobe level of -40 dB. Such an array can be designed with the MATLAB function `dolph`. The MATLAB code is as follows:

```
d = 0.5; th1 = 30; k1 = 2*pi*d*sin(pi*th1/180);
s1 = steermat(M-1, exp(j*k1));
a = dolph(d, 90-th1, M, 40)';
ad = a/(s1'*a);
```

where the calling convention of `dolph`, and the conjugation implied by the prime operation, have to do with the conventions used in that toolbox. The last line normalizes the designed array vector $\mathbf{a}_d$ so that $\mathbf{s}_1^\dagger \mathbf{a}_d = 1$ and have unity gain towards $\mathbf{s}_1$.

Plot the array response of the desired weights $\mathbf{a}_d$ using the same scales as the above two graphs. Note the -40 dB sidelobe level and the gains at the five constraint points θ_2-θ_6, with the constraints yet to be enforced.

(f) The main idea of Ref. [1213] is to find that weight vector $\bar{\mathbf{a}}$ that satisfies the constraints $C^\dagger \bar{\mathbf{a}} = \mathbf{g}$ and is closest to the desired weight $\mathbf{a}_d$ with respect to the Euclidean norm, that is, find $\bar{\mathbf{a}}$ that is the solution of the minimization problem:

$$J = (\bar{\mathbf{a}} - \mathbf{a}_d)^\dagger (\bar{\mathbf{a}} - \mathbf{a}_d) = \min, \quad \text{subject to} \quad C^\dagger \bar{\mathbf{a}} = \mathbf{g} \tag{14.12.15}$$

Using the results of Eq. (14.12.4), show that the optimum solution is

$$\bar{\mathbf{a}} = \mathbf{a}_d + C(C^\dagger C)^{-1}(\mathbf{g} - C^\dagger \mathbf{a}_d) \tag{14.12.16}$$

which can be written in the form:

$$\bar{\mathbf{a}} = [I - C(C^\dagger C)^{-1} C^\dagger]\mathbf{a}_d + C(C^\dagger C)^{-1}\mathbf{g} \equiv \mathbf{a}_\perp + \mathbf{a}_q \tag{14.12.17}$$

Note that $C^\dagger \mathbf{a}_\perp = 0$ and that $\mathbf{a}_q$ and $\mathbf{a}_\perp$ are orthogonal, $\mathbf{a}_q^\dagger \mathbf{a}_\perp = 0$. Show that $\bar{\mathbf{a}}$ can also be written in the form:

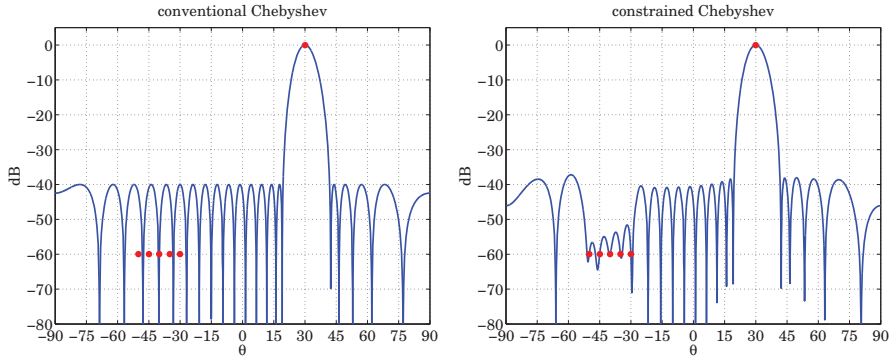
$$\bar{\mathbf{a}} = \mathbf{a}_\perp + C(C^\dagger C)^{-1}\mathbf{g} = [C, \mathbf{a}_\perp] \begin{bmatrix} (C^\dagger C)^{-1} & 0 \\ 0 & (\mathbf{a}_\perp^\dagger \mathbf{a}_\perp)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{g} \\ \mathbf{a}_\perp^\dagger \mathbf{a}_\perp \end{bmatrix}$$

Using the orthogonality property $C^\dagger \mathbf{a}_\perp = 0$, show that the above expression can be written in the form:

$$\bar{\mathbf{a}} = \bar{C}(\bar{C}^\dagger \bar{C})^{-1}\bar{\mathbf{g}}, \quad \bar{C} = [C, \mathbf{a}_\perp], \quad \bar{\mathbf{g}} = \begin{bmatrix} \mathbf{g} \\ \mathbf{a}_\perp^\dagger \mathbf{a}_\perp \end{bmatrix} \tag{14.12.18}$$

Therefore, the modified weights $\bar{\mathbf{a}}$ may be thought of as the quiescent weights with respect to the constraints $\bar{C}^\dagger \bar{\mathbf{a}} = \bar{\mathbf{g}}$, which involve one more constraint equation appended to the old ones.

(g) Using the constraints defined by Eq. (14.12.14) and the conventional Chebyshev weights $\mathbf{a}_d$ computed in part (e), construct the new quiescent "constrained Chebyshev" weights according to Eq. (14.12.16) and plot the corresponding array pattern using the same scales as before. Place the constrained points on the graph.



(h) The previous question dealt with the quiescent pattern. Here, assume that there are actually four incident plane waves, one from the desired look-direction θ_1 and three interferers from the directions $\theta_2, \theta_3, \theta_4$ as defined in part (d). Thus, the steering matrix used to construct the covariance matrix R will be $S = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4]$. All four SNRs are assumed to be 20 dB.

However in this part, we wish to clamp down the three interferers at the -70 dB level. The constraint matrix C remains the same as in Eq. (14.12.14), but the gain vector $\mathbf{g}$ needs to be modified appropriately so that its entries 2–4 reflect the -70 dB requirement.

Construct the extended constraint set $\bar{C}, \bar{\mathbf{g}}$ as in Eq. (14.12.18) and then construct the corresponding optimum weights using Eq. (14.12.8) (with $\bar{C}, \bar{\mathbf{g}}$ in place of $C, \mathbf{g}$). Plot the corresponding angular pattern using the same scales as before, and place the constraint points on the graph.

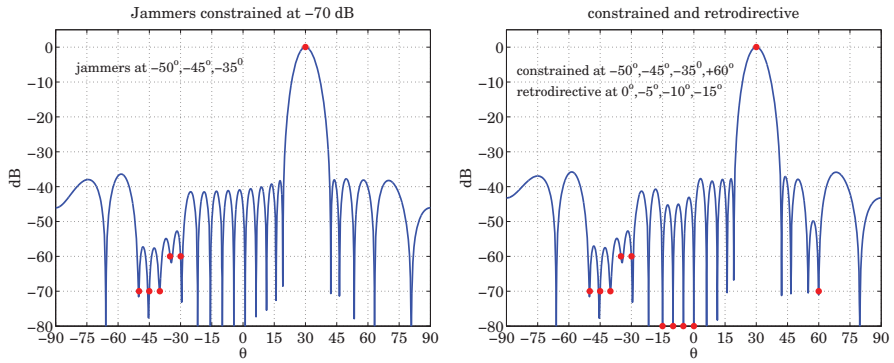
(i) In this part, in addition to the four plane waves of the part (h), there are also incident the following interferers:

$$\{\theta_7, \theta_8, \theta_9, \theta_{10}, \theta_{11}\} = \{-15^o, -10^o, -5^o, 0^o, +60^o\}$$

Again, all SNRs are 20 dB. We wish to meet the following requirements: (1) $\mathbf{s}_1$ is the desired incident signal with unity gain, (2) $\mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ are incident waves to be clamped down at -70 dB, (3) $\mathbf{s}_5, \mathbf{s}_6$ whose angles were defined in part (d) are constraint directions clamped at -60 dB but they do not correspond to incident waves, (4) $\mathbf{s}_7, \mathbf{s}_8, \mathbf{s}_9, \mathbf{s}_{10}$ are incident and will be nulled by the retrodirective action of the array, and (5) $\mathbf{s}_{11}$ is incident and also to be clamped at -70 dB.

Construct the covariance matrix R and the new constraints $C, \mathbf{g}$. Then, construct the extended constraint set $\bar{C}, \bar{\mathbf{g}}$, calculate the optimum weights from

Eq. (14.12.8), and plot the corresponding array pattern using the same scales as before. Indicate the constraint points on the graph. Label the four retrodirective beam points at the bottom of the graph, that is, at the −80 dB level.



4. *Equivalence of LCMV and the Generalized Sidelobe Canceler.* Finally, we look at the equivalence of the LCMV beamformer and the generalized sidelobe canceler (GSC). Consider the LCMV problem defined by Eq. (14.12.7) and its solution, Eq. (14.12.8). The constraints $C^\dagger\mathbf{a} = \mathbf{g}$ can be regarded as a full-rank under-determined system of equations.

   (a) Show that the pseudoinverse of the constraint matrix $C^\dagger$ is:

$$(C^\dagger)^+ = C(C^\dagger C)^{-1}$$

   (b) Show that the most general solution of $C^\dagger\mathbf{a} = \mathbf{g}$ can be expressed in the form:

$$\mathbf{a} = \mathbf{a}_q - B\mathbf{c} \tag{14.12.19}$$

   where $\mathbf{a}_q = (C^\dagger)^+\mathbf{g} = C(C^\dagger C)^{-1}\mathbf{g}$ is the minimum-norm solution and B is an $M\times(M-K)$ matrix that forms a basis for the $(M-K)$-dimensional null space of $C^\dagger$, that is, the space $N(C^\dagger)$, and $\mathbf{c}$ is an arbitrary $(M-K)$-dimensional vector of coefficients. Thus, B must satisfy

$$C^\dagger B = 0 \tag{14.12.20}$$

   For example, B may be constructed from the full SVD, $C = U\Sigma V^\dagger$, where the $M\times M$ unitary matrix U can be decomposed into its $M\times K$ and $M\times(M-K)$ parts, $U = [U_1, U_2]$. One can choose then $B = U_2$, or more generally, $B = U_2F$, where F is any $(M-K)\times(M-K)$ invertible matrix.

   (c) Because $\mathbf{a} = \mathbf{a}_q - B\mathbf{c}$ already satisfies the constraints, the LCMV problem (14.12.7) can be replaced by the following *unconstrained* minimization problem for the determination of the coefficients $\mathbf{c}$:

$$J = \mathbf{a}^\dagger R\mathbf{a} = (\mathbf{a}_q - B\mathbf{c})^\dagger R(\mathbf{a}_q - B\mathbf{c}) = \min \tag{14.12.21}$$

Show that the optimum $\mathbf{c}$ is given by

$$\mathbf{c} = (B^\dagger RB)^{-1}B^\dagger R\mathbf{a}_q \tag{14.12.22}$$

and hence the optimum $\mathbf{a}$ can be written in the form:

$$\mathbf{a} = \mathbf{a}_q - B\mathbf{c} = [I - B(B^\dagger RB)^{-1}B^\dagger R]\mathbf{a}_q \tag{14.12.23}$$

The solution (14.12.23) must be the same as that of Eq. (14.12.8).

(d) Show that the orthogonality condition (14.12.20) and the fact that the two matrices C, B together form a basis for $\mathbb{C}^M$, imply that B and C must satisfy the following relationships:

$$C(C^\dagger C)^{-1}C^\dagger + B(B^\dagger B)^{-1}B^\dagger = I \tag{14.12.24}$$

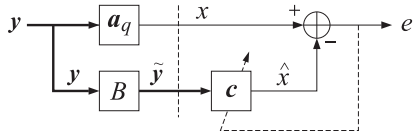
and

$$R^{-1}C(C^\dagger R^{-1}C)^{-1}C^\dagger + B(B^\dagger RB)^{-1}B^\dagger R = I \tag{14.12.25}$$

Hints: For the first one, use the full SVD of C and the fact that $B = U_2F$. For the second one, work with the matrices $\bar{C} = R^{-1/2}C$ and $\bar{B} = R^{1/2}B$, where $R^{1/2}$ is a hermitian positive square root of R, and then apply the same argument as for the first part.

(e) Using Eq. (14.12.25) show the equivalence of Eqs. (14.12.8) and (14.12.23).

(f) The above results lead to the following block diagram realization of the LCMV, known as the generalized sidelobe canceler:



where $\mathbf{y}$ is the overall input vector at the M antennas, and the other variables are defined as follows:

$$\boxed{\begin{aligned} x &= \mathbf{a}_q^T\mathbf{y} \\ \tilde{\mathbf{y}} &= B^T\mathbf{y} \\ \hat{x} &= \mathbf{c}^T\tilde{\mathbf{y}} \\ e &= x - \hat{x} = \mathbf{a}_q^T\mathbf{y} - \mathbf{c}^TB^T\mathbf{y} = (\mathbf{a}_q - B\mathbf{c})^T\mathbf{y} = \mathbf{a}^T\mathbf{y} \end{aligned}} \tag{14.12.26}$$

The portion of the block diagram to the right of the vertical dividing line may be thought of as an ordinary Wiener filtering problem of estimating the signal x from the vector $\tilde{\mathbf{y}}$. This follows by noting that the output power minimization of e is equivalent to the estimation problem:

$$J = \mathbf{a}^T R\mathbf{a} = E[|e|^2] = E[|x - \hat{x}|^2] = \min$$

where $R = E[\mathbf{y}^*\mathbf{y}^T]$. Let $\tilde{R} = E[\tilde{\mathbf{y}}^*\tilde{\mathbf{y}}^T]$ and $\tilde{\mathbf{r}} = E[x\tilde{\mathbf{y}}^*]$. Then, show that the optimum $\mathbf{c}$ of Eq. (14.12.22) is indeed the Wiener solution:

$$\mathbf{c} = \tilde{R}^{-1}\tilde{\mathbf{r}} \tag{14.12.27}$$

The great advantage of the GSC is that this unconstrained Wiener filtering part can be implemented adaptively using any adaptive method such as LMS or RLS, applied to the signals $x(n), \tilde{\mathbf{y}}(n)$. For example, the complete LMS algorithm would be as follows. At each time n, the input $\mathbf{y}(n)$ and weights $\mathbf{c}(n)$ are available, then,

$$\begin{aligned} x(n) &= \mathbf{a}_q^T\mathbf{y}(n) \\ \tilde{\mathbf{y}}(n) &= B^T\mathbf{y}(n) \\ \hat{x}(n) &= \mathbf{c}^T(n)\tilde{\mathbf{y}}(n) \\ e(n) &= x(n) - \hat{x}(n) \\ \mathbf{c}(n+1) &= \mathbf{c}(n) + \mu e(n)\tilde{\mathbf{y}}^*(n) \end{aligned} \qquad \text{(adaptive GSC)}$$

with the vector $\mathbf{a}(n) = \mathbf{a}_q - B\mathbf{c}(n)$ converging to the desired optimum constrained solution of Eq. (14.12.8).

14.13 Computer Project – Markowitz Portfolio Theory

This project, divided into separate questions, deals with Markowitz's optimum mean-variance portfolio theory. [1222–1233]. It can be considered to be a special case of the linearly-constrained quadratic optimization problem of the previous project. The project develops the following topics from financial engineering:

- optimum mean-variance Markowitz portfolios
- efficient frontier between risk and return
- quantifying risk aversion
- two mutual fund theorem
- inequality-constrained portfolios without short selling
- risk-free assets and tangency portfolio
- capital asset line and the Sharp ratio
- market portfolios and capital market line
- stock's beta, security market line, risk premium
- capital asset pricing model (CAPM)
- minimum-variance with multiple constraints

1. *Mean-Variance Portfolio Theory.* The following constrained optimization problem finds application in investment analysis and optimum portfolio selection in which one tries to balance return versus risk.[†]

[†]Harry Markowitz received the Nobel prize in economics for this work.

Suppose one has identified M stocks or assets $\mathbf{y} = [y_1, y_2, \ldots, y_M]^T$ into which to invest. From historical data, the expected returns of the individual stocks are assumed to be known, $E[\mathbf{y}] = \mathbf{m} = [m_1, m_2, \ldots, m_M]^T$, as are the cross-correlations between the assets, $R_{ij} = E[(y_i - m_i)(y_j - m_j)]$, or, $R = E[(\mathbf{y} - \mathbf{m})(\mathbf{y} - \mathbf{m})^T]$, assumed to have full rank. The variance $\sigma_i^2 = R_{ii}$ is a measure of the volatility, or risk, of the ith asset.

A portfolio is selected by choosing the percentage a_i to invest in the ith asset y_i. The portfolio is defined by the random variable:

$$y = \sum_{i=1}^{M} a_i y_i = [a_1, a_2, \ldots, a_M]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = \mathbf{a}^T\mathbf{y}$$

where the weights a_i must add up to unity (negative weights are allowed, describing so-called "short sells"):

$$\sum_{i=1}^{M} a_i = [a_1, a_2, \ldots, a_M]\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{a}^T\mathbf{u} = 1$$

The expected return of the portfolio and its variance or risk are given by:

$$\mu = E[y] = E[\mathbf{a}^T\mathbf{y}] = \mathbf{a}^T\mathbf{m}$$

$$\sigma^2 = E[(y - \mu)^2] = \mathbf{a}^T R\mathbf{a}$$

An optimum portfolio may be defined by finding the weights that minimize the risk σ^2 for a given value of the return μ, that is,

$$\sigma^2 = \mathbf{a}^T R\mathbf{a} = \min, \quad \text{subject to} \quad \mathbf{a}^T\mathbf{m} = \mu, \quad \mathbf{a}^T\mathbf{u} = 1 \tag{14.13.1}$$

(a) Incorporate the constraints by means of two Lagrange multipliers, λ_1, λ_2, and minimize the modified performance index:

$$J = \frac{1}{2}\mathbf{a}^T R\mathbf{a} + \lambda_1(\mu - \mathbf{a}^T\mathbf{m}) + \lambda_2(1 - \mathbf{a}^T\mathbf{u}) = \min$$

Show that the quantities $\{\mathbf{a}, \lambda_1, \lambda_2\}$ can be obtained from the solution of the $(M+2)\times(M+2)$ linear system of equations:

$$\begin{bmatrix} R & -\mathbf{m} & -\mathbf{u} \\ \mathbf{m}^T & 0 & 0 \\ \mathbf{u}^T & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mu \\ 1 \end{bmatrix}$$

where the invertibility of this matrix requires that the vectors $\mathbf{m}$ and $\mathbf{u}$ be not collinear.

(b) Show that the solution of this system for $\mathbf{a}$ takes the form:

$$\mathbf{a} = \lambda_1 R^{-1}\mathbf{m} + \lambda_2 R^{-1}\mathbf{u} \tag{14.13.2}$$

and that λ_1, λ_2 can be obtained from the reduced 2×2 linear system:

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \lambda_1 &= \frac{\mu C - B}{D} \\ \lambda_2 &= \frac{A - \mu B}{D} \end{aligned}$$

where A, B, C, D are defined in terms of $\mathbf{m}, R$ by

$$\begin{aligned} A &= \mathbf{m}^T R^{-1}\mathbf{m} \\ B &= \mathbf{m}^T R^{-1}\mathbf{u} \quad \text{and} \quad D = AC - B^2 \\ C &= \mathbf{u}^T R^{-1}\mathbf{u} \end{aligned}$$

(c) Show that the quantities A, C, D are non-negative.

(d) Show that the minimized value of the risk $\sigma^2 = \mathbf{a}^T R\mathbf{a}$ can be written in the form:

$$\sigma^2 = \mu\lambda_1 + \lambda_2 = \frac{C\mu^2 - 2B\mu + A}{D} \tag{14.13.3}$$

Thus, the dependence of the variance σ^2 on the return μ has a parabolic shape, referred to as the *efficient frontier*.

(e) The apex of this parabola is obtained by minimizing Eq. (14.13.3) with respect to μ. Setting $\partial\sigma^2/\partial\mu = 0$, show that the absolute minimum is reached for the following values of the return, risk, and weights:

$$\mu_0 = \frac{B}{C}, \quad \sigma_0^2 = \frac{1}{C}, \quad \mathbf{a}_0 = \frac{R^{-1}\mathbf{u}}{\mathbf{u}^T R^{-1}\mathbf{u}} \tag{14.13.4}$$

(f) Show that Eq. (14.13.3) can be re-expressed as

$$\sigma^2 = \sigma_0^2 + \frac{C}{D}(\mu - \mu_0)^2 \tag{14.13.5}$$

which can be solved for μ in terms of σ^2, as is common in practice:

$$\mu = \mu_0 \pm \sqrt{\frac{D}{C}}\sqrt{\sigma^2 - \sigma_0^2} \tag{14.13.6}$$

Of course, only the upper sign corresponds to the efficient frontier because it yields higher return for the same risk. (See some example graphs below.)

(g) Show that the optimum portfolio of Eq. (14.13.2) can be written in the form:

$$\mathbf{a} = \mathbf{a}_0 + \frac{C}{D}(\mu - \mu_0)\, R^{-1}(\mathbf{m} - \mu_0\,\mathbf{u}) \tag{14.13.7}$$

where $\mathbf{a}_0$ was defined in Eq. (14.13.4). Thus, as expected, $\mathbf{a} = \mathbf{a}_0$, if $\mu = \mu_0$.

(h) Show that the optimum portfolio of Eq. (14.13.2) can be written in the form

$$\mathbf{a} = \mu\,\mathbf{g} + \mathbf{h}$$

where $\mathbf{g}, \mathbf{h}$ depend only on the asset statistics $\mathbf{m}, R$ and are independent of μ. Moreover, show that $\mathbf{m}^T\mathbf{g} = 1$ and $\mathbf{m}^T\mathbf{h} = 0$, and that $\mathbf{u}^T\mathbf{g} = 0$ and $\mathbf{u}^T\mathbf{h} = 1$. In particular, show that $\mathbf{g}, \mathbf{h}$ are given by,

$$\begin{aligned} \mathbf{g} &= \frac{C}{D}R^{-1}\mathbf{m} - \frac{B}{D}R^{-1}\mathbf{u} \\ \mathbf{h} &= \frac{A}{D}R^{-1}\mathbf{u} - \frac{B}{D}R^{-1}\mathbf{m} \end{aligned}$$

(i) Consider two optimal portfolios $\mathbf{a}_1$ and $\mathbf{a}_2$ having return-risk values that lie on the efficient frontier, μ_1, σ_1 and μ_2, σ_2, satisfying Eq. (14.13.5), and assume that $\mu_1 < \mu_2$. Using the results of the previous question, show that any other optimum portfolio with return-risk pair μ, σ, such that $\mu_1 < \mu < \mu_2$, can be constructed as a linear combination of $\mathbf{a}_1, \mathbf{a}_2$ as follows, with positive weights p_1, p_2, such that, $p_1 + p_2 = 1$,

$$\mathbf{a} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2\,, \quad p_1 = \frac{\mu_2 - \mu}{\mu_2 - \mu_1}\,, \quad p_2 = \frac{\mu - \mu_1}{\mu_2 - \mu_1}$$

Thus, the investor need only invest in the two standard portfolios $\mathbf{a}_1$ and $\mathbf{a}_2$ in the proportions p_1 and p_2, respectively. This is known as the *two mutual fund theorem*. The restriction $\mu_1 < \mu < \mu_2$ can be relaxed if short selling of the funds is allowed.

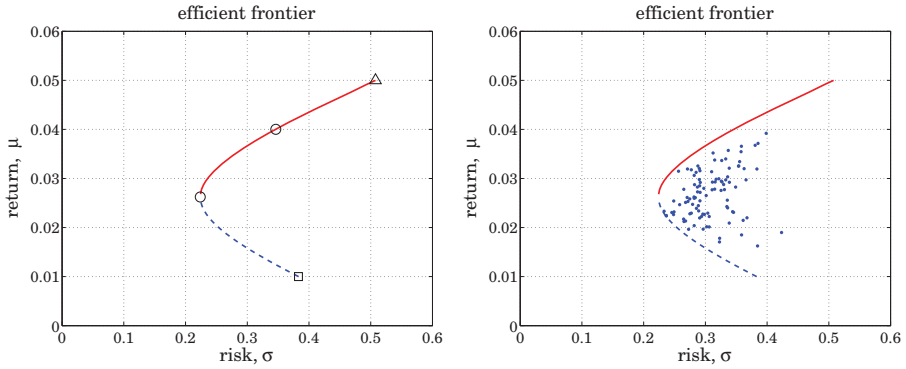
(j) Consider a portfolio of four assets having return (given as annual rate) and covariance matrix:

$$\mathbf{m} = \begin{bmatrix} 0.02 \\ 0.03 \\ 0.01 \\ 0.05 \end{bmatrix}, \quad R = \begin{bmatrix} 0.10 & -0.02 & 0.04 & -0.01 \\ -0.02 & 0.20 & 0.05 & 0.02 \\ 0.04 & 0.05 & 0.30 & 0.03 \\ -0.01 & 0.02 & 0.03 & 0.40 \end{bmatrix}$$

Make a plot of the efficient frontier of μ versus σ according to Eq. (14.13.6). To do so, choose 51 equally-spaced μs in the interval $[\min(\mathbf{m}), \max(\mathbf{m})]$ and calculate the corresponding σs.

Compute the risk σ and weight vector $\mathbf{a}$ that would achieve a return of $\mu = 0.04$ and indicate that point on the efficient frontier. Why wouldn't an investor want to put all his/her money in stock y_4 since it has a higher return of $m_4 = 0.05$?

(k) Generate 100 weight vectors $\mathbf{a}$ randomly (but such that $\mathbf{a}^T\mathbf{u} = 1$), compute the values of the quantities $\mu = \mathbf{a}^T\mathbf{m}$ and $\sigma^2 = \mathbf{a}^T R\mathbf{a}$ and make a scatterplot of the points (μ, σ) to see that they lie on or below the efficient frontier.



2. *Risk aversion.* A somewhat different minimization criterion is often chosen for the portfolio selection problem, that is,

$$J = \frac{1}{2}\mathbf{a}^T R\mathbf{a} - \gamma \mathbf{a}^T\mathbf{m} = \min, \quad \text{subject to } \mathbf{u}^T\mathbf{a} = 1 \tag{14.13.8}$$

where γ is a positive parameter that quantifies the investor's aversion for risk versus return—small γ emphasizes low risk, larger γ, high return.

For various values of γ, the optimum weights $\mathbf{a}$ are determined and then the corresponding return $\mu = \mathbf{a}^T\mathbf{m}$ and risk $\sigma^2 = \mathbf{a}^T\Sigma\mathbf{a}$ are calculated. The resulting plot of the pairs (μ, σ) is the efficient frontier in this case.

Given the parameters $\gamma, \mathbf{m}, R$, incorporate the constraint, $\mathbf{a}^T\mathbf{u} = 1$, with a Lagrange multiplier and work with the modified performance index:

$$J = \frac{1}{2}\mathbf{a}^T R\mathbf{a} - \gamma \mathbf{a}^T\mathbf{m} + \lambda_2(1 - \mathbf{u}^T\mathbf{a}) = \min$$

Show that one obtains exactly the same solution for $\mathbf{a}$ as in the previous problem and that the correspondence between the assumed γ and the realized return μ is given by

$$\mu = \mu_0 + \frac{D}{C}\gamma$$

3. *Portfolio with inequality constraints.* If short sales are not allowed, the portfolio weights a_i must be restricted to be in the interval $0 \le a_i \le 1$. In this case, the following optimization problem must be solved:

$$\sigma^2 = \mathbf{a}^T R\mathbf{a} = \min, \quad \text{subject to } \begin{cases} \mathbf{a}^T\mathbf{m} = \mu \\ \mathbf{a}^T\mathbf{u} = 1 \\ 0 \le a_i \le 1, \quad i = 1, 2, \ldots, M \end{cases} \tag{14.13.9}$$

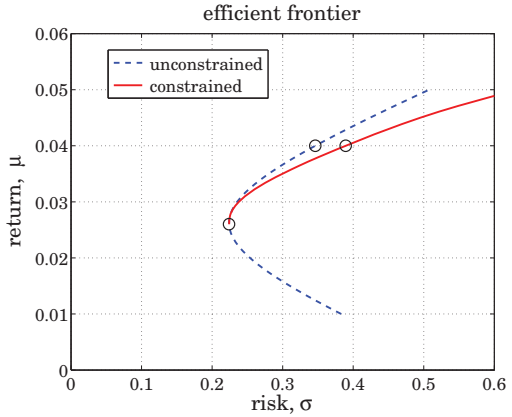
This is a convex optimization problem that can be solved conveniently using the CVX package.[†] For example, given a desired value for μ, the following CVX code will solve for $\mathbf{a}$:

[†] http://cvxr.com/cvx


```
% define M, m, R, mu
cvx_begin
   variable a(M)
   minimize( a'*R*a );
   subject to
      a'*u == 1;
      a'*m == mu;
      a <= ones(M,1);
     -a <= zeros(M,1);
cvx_end
```

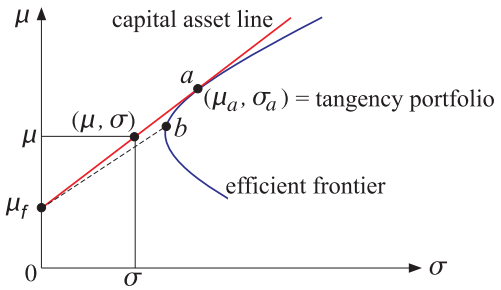
(a) For the numerical example of Question (1.j), choose 51 equally-spaced μs in the interval $[\min(\mathbf{m}), \max(\mathbf{m})]$ and calculate the corresponding $\mathbf{a}$ and σs, and plot the efficient frontier, that is the pairs (μ, σ). Superimpose on this graph the efficient frontier of the unconstrained case from Question 1.

(b) Compute the risk σ and weight vector $\mathbf{a}$ that would achieve a return of $\mu = 0.04$ and indicate that point on the efficient frontier of part (a). Place on the graph also the unconstrained solution for the same μ, and explain why the inequality-constrained case is slightly worse.



4. *Capital Asset Line.* A variation of the mean-variance portfolio theory was considered by W. F. Sharpe, who in addition to the collection of risky assets, allowed the presence of a *risk-free* asset, such as a Treasury T-bill, that has a fixed guaranteed return, say $\mu = \mu_f$, and no risk, $\sigma_f = 0$.

Let us assume again that we also have M risky assets $\mathbf{y} = [y_1, y_2, \ldots, y_M]^T$ with expected returns $E[\mathbf{y}] = \mathbf{m} = [m_1, m_2, \ldots, m_M]^T$, and known covariance matrix $R = E[(\mathbf{y} - \mathbf{m})(\mathbf{y} - \mathbf{m})^T]$. Clearly, we must assume that $\mu_f < m_i$, $i = 1, 2, \ldots, M$, otherwise, we would put all our money into the risk-free asset. We form an optimum portfolio of risky assets $y = \mathbf{a}^T\mathbf{y}$ by choosing a point on the efficient frontier, for example, the indicated point b on the figure below.



Then, we join with a straight line the point b to the risk-free point μ_f (the dashed line on the figure), and we allocate a fraction w_f of our total funds to the risk-free asset and hence a fraction $(1-w_f)$ to the portfolio y, that is, the combined portfolio will be:

$$y_{\text{tot}} = w_f\mu_f + (1-w_f)y = w_f\mu_f + (1-w_f)\mathbf{a}^T\mathbf{y}$$

with mean and variance:

$$\begin{aligned}\mu &= E[y_{\text{tot}}] = w_f\mu_f + (1-w_f)\mathbf{a}^T\mathbf{m}\\ \sigma^2 &= E\big[(y_{\text{tot}}-\mu)^2\big] = (1-w_f)^2\mathbf{a}^T R\mathbf{a}\end{aligned} \tag{14.13.10}$$

The lowest location for b is at the apex of the frontier, for which we have $\mathbf{a} = \mathbf{a}_0$. It should be evident from the figure that if we move the point b upwards along the efficient frontier, increasing the slope of the straight line, we would obtain a better portfolio having larger μ.

We may increase the slope until the line becomes tangent to the efficient frontier, say at the point a, which would correspond to an optimum portfolio $\mathbf{a}$ with mean and variance μ_a, σ_a.

This portfolio is referred to as the *tangency portfolio* and the tangent line (red line) is referred to as the *capital asset line* and its maximum slope, say β, as the *Sharpe ratio*. Eqs. (14.13.10) become now:

$$\begin{aligned}\mu &= w_f\mu_f + (1-w_f)\mu_a\\ \sigma &= (1-w_f)\sigma_a\end{aligned} \tag{14.13.11}$$

where $\mu_a = \mathbf{a}^T\mathbf{m}$ and $\sigma_a^2 = \mathbf{a}^T R\mathbf{a}$. Solving the second of Eq. (14.13.11) for w_f, we find $1-w_f = \sigma/\sigma_a$, and substituting in the first, we obtain the equation for the straight line on the (μ,σ) plane:

$$\mu = \mu_f + \beta\sigma\,, \quad \beta = \frac{\mu_a-\mu_f}{\sigma_a} = \text{slope} \tag{14.13.12}$$

This line is the *efficient frontier* for the combined portfolio. Next we impose the condition that the point a be a tangency point on the efficient frontier. This will fix μ_a, σ_a. We saw that the frontier is characterized by the parabolic curve of

Eq. (14.13.5) with the optimum weight vector given by (14.13.7). Applying these to the pair (μ_a, σ_a), we have:

$$\begin{aligned}\sigma_a^2 &= \sigma_0^2 + \frac{C}{D}(\mu_a-\mu_0)^2\\ \mathbf{a} &= \mathbf{a}_0 + \frac{C}{D}(\mu_a-\mu_0)\,R^{-1}(\mathbf{m}-\mu_0\,\mathbf{u})\end{aligned} \tag{14.13.13}$$

The slope of the tangent at the point a is given by the derivative $d\mu_a/d\sigma_a$, and it must agree with the slope β of the line (14.13.12):

$$\frac{d\mu_a}{d\sigma_a} = \beta = \frac{\mu_a-\mu_f}{\sigma_a} \tag{14.13.14}$$

(a) Using condition (14.13.14) and Eq. (14.13.13), show the following relationships:

$$\begin{aligned}\frac{C}{D}(\mu_a-\mu_0)(\mu_0-\mu_f) &= \frac{1}{C}\\ \sigma_a^2 &= \frac{C}{D}(\mu_a-\mu_0)(\mu_a-\mu_f)\\ \beta &= \sigma_a C(\mu_0-\mu_f)\end{aligned} \tag{14.13.15}$$

The first can be solved for μ_a, and show that it can be expressed as:

$$\mu_a = \frac{A-\mu_f B}{C(\mu_0-\mu_f)} \tag{14.13.16}$$

(b) Using Eqs. (14.13.13) and (14.13.15), show that the optimum weights are given by

$$\mathbf{a} = \frac{1}{C(\mu_0-\mu_f)}\,R^{-1}(\mathbf{m}-\mu_f\mathbf{u}) \tag{14.13.17}$$

and verify that they satisfy the constraints $\mathbf{a}^T\mathbf{m} = \mu_a$ and $\mathbf{a}^T\mathbf{u} = 1$.

(c) Show that the slope β can also be expressed by:

$$\beta^2 = \left(\frac{\mu_a-\mu_f}{\sigma_a}\right)^2 = (\mathbf{m}-\mu_f\mathbf{u})^T R^{-1}(\mathbf{m}-\mu_f\mathbf{u}) \tag{14.13.18}$$

(d) Define $\mathbf{w} = (1-w_f)\mathbf{a}$ to be the effective weight for the total portfolio:

$$y_{\text{tot}} = w_f\mu_f + (1-w_f)\mathbf{a}^T\mathbf{y} = w_f\mu_f + \mathbf{w}^T\mathbf{y} \tag{14.13.19}$$

Show that $\mathbf{w}$ is given in terms of the return $\mu = w_f\mu_f + (1-w_f)\mu_a$ as follows:

$$\mathbf{w} = (1-w_f)\mathbf{a} = \frac{\mu-\mu_f}{\beta^2}\,R^{-1}(\mathbf{m}-\mu_f\mathbf{u}) \tag{14.13.20}$$

(e) The optimality of the capital asset line and the tangency portfolio can also be derived directly by considering the following optimization problem. Let w_f and $\mathbf{w}$ be the weights to be assigned to the risk-free asset μ_f and the risky assets $\mathbf{y}$. Then the total portfolio, its mean μ, and variance σ^2 are given by:

$$\begin{aligned} y_{\text{tot}} &= w_f \mu_f + \mathbf{w}^T \mathbf{y} \\ \mu &= w_f \mu_f + \mathbf{w}^T \mathbf{m} \\ \sigma^2 &= \mathbf{w}^T R \mathbf{w} \end{aligned} \tag{14.13.21}$$

where the weights must add up to unity: $w_f + \mathbf{w}^T \mathbf{u} = 1$. Given μ, we wish to determine the weights $w_f, \mathbf{w}$ to minimize σ^2. Incorporating the constraints by two Lagrange multipliers, we obtain the performance index:

$$J = \frac{1}{2}\mathbf{w}^T R \mathbf{w} + \lambda_1 (\mu - w_f \mu_f - \mathbf{w}^T \mathbf{m}) + \lambda_2 (1 - w_f - \mathbf{w}^T \mathbf{u}) = \min$$

Show that the minimization of J with respect to $w_f, \mathbf{w}$ results in:

$$\mathbf{w} = \lambda_1 R^{-1} (\mathbf{m} - \mu_f \mathbf{u}) \,, \quad \lambda_2 = -\lambda_1 \mu_f$$

(f) Imposing the constraints show that,

$$\mathbf{w}^T (\mathbf{m} - \mu_f \mathbf{u}) = \mu - \mu_f \,, \quad \lambda_1 = \frac{\mu - \mu_f}{\beta^2}$$

where β^2 is given as in Eq. (14.13.18),

$$\beta^2 = (\mathbf{m} - \mu_f \mathbf{u})^T R^{-1} (\mathbf{m} - \mu_f \mathbf{u})$$

and hence, show that $\mathbf{w}$ is given by Eq. (14.13.20)

$$\mathbf{w} = \frac{\mu - \mu_f}{\beta^2} R^{-1} (\mathbf{m} - \mu_f \mathbf{u})$$

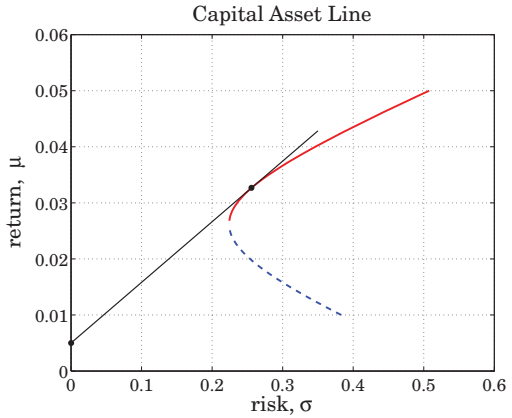
(g) Show that $\sigma^2 = \mathbf{w}^T R \mathbf{w}$ is given by,

$$\sigma^2 = \left(\frac{\mu - \mu_f}{\beta} \right)^2$$

which corresponds to the straight-line frontier on the μ, σ plane:

$$\mu = \mu_f + \beta \sigma$$

(h) For the numerical example of Question (1.j) and for a fixed-asset return of $\mu_f = 0.005$, connect the points (μ_a, σ_a) and $(\mu_f, 0)$ by a straight line and plot it together with the parabolic efficient frontier of the risky assets, and verify the tangency of the line.



5. *Security market line and CAPM.* When the collection of risky stocks of the previous problem are taken to be representative of the market, such as for example, the stocks in the S&P 500 index, the tangency portfolio at point a is referred to as the *market portfolio*, and the capital asset line, as the *capital market line*.

Let $y_a = \mathbf{a}^T \mathbf{y}$ be the linear combination of stocks for the market portfolio, and consider another stock y_i with return and risk μ_i, σ_i. The stock y_i is arbitrary and not necessarily belonging to those that make up the representative market. Define the *beta* for the stock as the ratio of the following covariances relative to the market portfolio:

$$\beta_i = \frac{R_{ia}}{R_{aa}}$$

where $R_{ia} = E[(y_i - \mu_i)(y_a - \mu_a)]$ and $R_{aa} = \sigma_a^2 = E[(y_a - \mu_a)^2]$.

Let us now make a portfolio y consisting of a percentage w of the stock y_i and a percentage $(1 - w)$ of the market y_a. Then, y and its mean and variance will be given as follows, where $R_{ii} = \sigma_i^2$.

$$\begin{aligned} y &= w y_i + (1 - w) y_a \\ \mu &= w \mu_i + (1 - w) \mu_a \\ \sigma^2 &= w^2 R_{ii} + 2w(1 - w) R_{ia} + (1 - w)^2 R_{aa} \end{aligned} \tag{14.13.22}$$

As w varies, the pair (μ, σ) varies over its own parabolic efficient frontier.

The *capital asset pricing model* (CAPM) asserts that the tangent line at the market point $y = y_a$ on this frontier obtained when $w = 0$, coincides with the capital market line between the risk-free asset μ_f and the market portfolio. This establishes the following relationship to be proved below:

$$\mu_i - \mu_f = \beta_i (\mu_a - \mu_f) \tag{14.13.23}$$

so that the excess return above μ_f, called the *risk premium*, is proportional to the stock's beta. The straight line of μ_i vs. β_i is referred to as the *security market line*.

(a) Using the differentiation rule,

$$\frac{d\sigma}{d\mu} = \frac{d\sigma}{dw}\cdot\frac{dw}{d\mu}$$

show that the slope at the market point, i.e., at $w = 0$, is given by

$$\left.\frac{d\mu}{d\sigma}\right|_{w=0} = \frac{\mu_i - \mu_a}{(\beta_i - 1)\,\sigma_a} \tag{14.13.24}$$

(b) Then, derive Eq. (14.13.23) by equating (14.13.24) to the slope (14.13.12).

6. *Minimum-variance with multiple constraints.* A generalization of the portfolio constrained minimization problem involves more that two constraints:

$$J = \frac{1}{2}\mathbf{a}^T R\,\mathbf{a} - \mathbf{b}^T\mathbf{a} = \min$$

subject to K linear constraints:

$$\mathbf{c}_i^T\mathbf{a} = \mu_i\,, \quad i = 1, 2, \ldots, K$$

where R is an $M\times M$ positive definite symmetric matrix, $\mathbf{b}$ is a given $M\times 1$ vector, and we assume that $K < M$ and that the $M\times 1$ vectors $\mathbf{c}_i$ are linearly independent. Defining the $M\times K$ matrix of constraints C and the K-dimensional vector of "returns" $\boldsymbol{\mu}$,

$$C = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_K]\,, \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_K \end{bmatrix}$$

the above minimization problem can be cast compactly in the form:

$$J = \frac{1}{2}\mathbf{a}^T R\mathbf{a} - \mathbf{b}^T\mathbf{a} = \min\,, \quad \text{subject to} \quad C^T\mathbf{a} = \boldsymbol{\mu}$$

(a) Introduce a K-dimensional vector of Lagrange multipliers $\boldsymbol{\lambda}$ and replace the performance index by:

$$J = \frac{1}{2}\mathbf{a}^T R\mathbf{a} - \mathbf{b}^T\mathbf{a} + \boldsymbol{\lambda}^T(\boldsymbol{\mu} - C^T\mathbf{a}) = \min$$

Show that the quantities $\mathbf{a}, \boldsymbol{\lambda}$ may be obtained as the solution of the $(M+K)\times(M+K)$ linear system of equations:

$$\begin{bmatrix} R & -C \\ C^T & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{a} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \boldsymbol{\mu} \end{bmatrix}$$

(b) Show that the above matrix has the following inverse:

$$\left[\begin{array}{c|c} R^{-1} - R^{-1}C(C^TR^{-1}C)^{-1}C^TR^{-1} & R^{-1}C(C^TR^{-1}C)^{-1} \\ \hline -(C^TR^{-1}C)^{-1}C^TR^{-1} & (C^TR^{-1}C)^{-1} \end{array}\right]$$

and explain why the assumed full rank of C guarantees the existence of the matrix inverse $(C^TR^{-1}C)^{-1}$.

(c) Show that the solution for $\mathbf{a}$ and $\boldsymbol{\lambda}$ can be obtained by:

$$\boldsymbol{\lambda} = (C^TR^{-1}C)^{-1}[\boldsymbol{\mu} - C^TR^{-1}\mathbf{b}]$$

$$\mathbf{a} = R^{-1}[\mathbf{b} + C\boldsymbol{\lambda}]$$

(d) Show that the "variance" $\sigma^2 = \mathbf{a}^T R\mathbf{a}$ is parabolic in the "returns", like Eq. (14.13.3), thus defining a generalized "efficient frontier":

$$\sigma^2 = \sigma_0^2 + \boldsymbol{\mu}^T(C^TR^{-1}C)^{-1}\boldsymbol{\mu}$$

where the constant σ_0^2 is defined by:

$$\sigma_0^2 = \mathbf{b}^T[R^{-1} - R^{-1}C(C^TR^{-1}C)^{-1}C^TR^{-1}]\mathbf{b}$$

Note that if additional inequality constraints are included, such as for example $a_i > 0$ for the weights, then this becomes a much harder problem that must be solved with *quadratic programming* techniques. The CVX package or MATLAB's function `quadprog` from the optimization toolbox solves such problems. The antenna or sensor array version of this problem is LCMV beamforming.

14.14 Problems

14.1 *Computer Experiment.* A fourth-order autoregressive process is defined by the difference equation

$$y_n + a_1y_{n-1} + a_2y_{n-2} + a_3y_{n-3} + a_4y_{n-4} = \epsilon_n$$

where ϵ_n is zero-mean, unit-variance, white gaussian noise. The filter parameters $\{a_1, a_2, a_3, a_4\}$ are chosen such that the prediction error filter

$$A(z) = 1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + a_4z^{-4}$$

has zeros at the locations

$$0.99\exp(\pm 0.2\pi j) \quad \text{and} \quad 0.99\exp(\pm 0.4\pi j)$$

(a) Determine $\{a_1, a_2, a_3, a_4\}$.

(b) Using a random number generator for ϵ_n, generate a realization of y_n consisting of 50 samples. To avoid transient effects, be sure to let the filter run for a while. For instance, discard the first 500 or 1000 outputs and keep the last 50.

(c) Compute the sample autocorrelation of y_n based on the above block of data.

(d) Solve the normal equations by means of Levinson's algorithm to determine the Yule-Walker estimates of the model parameters $\{a_1, a_2, a_3, a_4; \sigma_\epsilon^2\}$ and compare them with the exact values.

(e) Compute the corresponding Yule-Walker spectrum and plot it together with the exact autoregressive spectrum versus frequency. Be sure to allow for a sufficiently dense grid of frequencies to be able to resolve the narrow peaks of this example. Plot all spectra in decibels.

(f) Using the same finite block of y_n data, determine estimates of the model parameters $\{a_1, a_2, a_3, a_4; \sigma_\epsilon^2\}$ using Burg's method, and compare them with the Yule-Walker estimates and with the exact values.

(g) Compute the corresponding Burg spectrum and plot it together with the exact spectrum versus frequency.

(h) Using the same block of y_n data, compute the ordinary periodogram spectrum and plot it together with the exact spectrum.

(i) Window the y_n data with a Hamming window and then compute the corresponding periodogram spectrum and plot it together with the exact spectrum.

(j) Repeat parts (b) through (i) using a longer realization of length 100.

(k) Repeat parts (b) through (i) using a length-200 realization of y_n.

(l) Evaluate the various results of this experiment.

14.2 Show that the classical Bartlett spectrum of Eq. (14.2.6) can be written in the compact matrix form of Eq. (14.2.7).

14.3 Show that in the limit of large M, the first sidelobe of the smearing function $W(\omega)$ of Eq. (14.2.10) is approximately 13 dB down from the main lobe.

14.4 *Computer Experiment.* (a) Reproduce the spectra shown in Figs. 14.2.1 and 14.2.2.

(b) For the AR case, let $M = 6$, and take the SNRs of both sinusoids to be 6 dB, but change the sinusoid frequencies to

$$\omega_1 = 0.5 + \Delta\omega\,, \quad \omega_2 = 0.5 - \Delta\omega$$

where $\Delta\omega$ is variable. Study the dependence of bias of the spectral peaks on the frequency separation $\Delta\omega$ by computing and plotting the spectra for various values of $\Delta\omega$. (Normalize all spectra to 0 dB at the sinusoid frequency ω_1).

14.5 Derive Equation (14.2.30).

14.6 Let

$$R = \sigma_v^2 I + \sum_{i=1}^{L} P_i \mathbf{s}_{\omega_i} \mathbf{s}_{\omega_i}^\dagger$$

be the autocorrelation matrix of Eq. (14.2.8). Show that the inverse R^{-1} can be computed recursively as follows:

$$R_k^{-1} = R_{k-1}^{-1} - \frac{R_{k-1}^{-1} \mathbf{s}_{\omega_k} \mathbf{s}_{\omega_k}^\dagger R_{k-1}^{-1}}{\mathbf{s}_{\omega_k}^\dagger R_{k-1}^{-1} \mathbf{s}_{\omega_k} + P_k^{-1}}$$

for $k = 1, 2, \ldots, L$, initialized by $R_0 = \sigma_v^2 I$.

14.7 Consider the case of one sinusoid ($L = 1$) in noise and arbitrary filter order $M > 2$, so that the $(M+1)\times(M+1)$ autocorrelation matrix is

$$R = \sigma_v^2 I + P_1 \mathbf{s}_{\omega_1} \mathbf{s}_{\omega_1}^\dagger$$

(a) Show that the $(L = 1)$-dimensional signal subspace is spanned by the eigenvector

$$\mathbf{e}_M = \mathbf{s}_{\omega_1}$$

and determine the corresponding eigenvalue.

(b) Show that the $M + 1 - L = M$ dimensional noise subspace is spanned by the M linearly independent eigenvectors, all belonging to the minimum eigenvalue σ_v^2:

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ -e^{j\omega_1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ -e^{j\omega_1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -e^{j\omega_1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_{M-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -e^{j\omega_1} \end{bmatrix}$$

(c) Show that the eigenpolynomial $A(z)$ corresponding to an arbitrary linear combination of the M noise eigenvectors

$$\mathbf{a} = \mathbf{e}_0 + c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 + \cdots + c_{M-1} \mathbf{e}_{M-1}$$

can be factored in the form

$$A(z) = (1 - e^{j\omega_1} z^{-1})\,(1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_{M-1} z^{-(M-1)})$$

exhibiting one zero at the desired sinusoid frequency $e^{j\omega_1}$ on the unit circle, and $M - 1$ additional spurious zeros with arbitrary locations that depend on the particular choice of the coefficients c_i.

14.8 The constraint (14.2.31) can be incorporated into the performance index (14.2.32) by means of a Lagrange multiplier

$$\mathcal{E} = \mathbf{a}^\dagger R \mathbf{a} + \lambda\,(1 - \mathbf{a}^\dagger \mathbf{a})$$

Show that the minimization of $\mathcal{E}$ is equivalent to the Pisarenko eigenvalue problem of Eq. (14.2.29), with the multiplier λ playing the role of the eigenvalue. Show that the minimum of $\mathcal{E}$ is the minimum eigenvalue.

14.9 Show Eq. (14.3.11).

14.10 Consider a singular $(M+1)\times(M+1)$ autocorrelation matrix R having non-singular principal submatrices, and let $\mathbf{a}$ be the symmetric or antisymmetric order-M prediction filter satisfying $R\mathbf{a} = 0$, as discussed in Sec. 12.5. First, argue that the M zeros of this filter lie on the unit circle $z_i = e^{j\omega_i}$, $i = 1, 2, \ldots, M$. Then, consider the eigenvalue decomposition of this matrix in the form $R = E\Lambda E^\dagger$, where Λ is the diagonal matrix of the M nonzero eigenvalues of R and E is the $(M+1)\times M$ matrix whose columns are the M corresponding eigenvectors. Let $S = [\mathbf{s}_{\omega_1}, \mathbf{s}_{\omega_2}, \ldots, \mathbf{s}_{\omega_M}]$ be the matrix of phasing vectors defined by the zeros of $\mathbf{a}$. Argue that E is linearly related to S and that R can be written in the form $R = SPS^\dagger$, where P is an $M\times M$ positive-definite matrix. Finally, show that the requirement that R be Toeplitz implies that P must be diagonal, and therefore, R admits the sinusoidal representation

$$R = \sum_{i=1}^{M} P_i \mathbf{s}_{\omega_i} \mathbf{s}_{\omega_i}^\dagger\,, \quad \text{with } P_i > 0$$

14.11 *Computer Experiment.* To simulate Eq. (14.3.7), the amplitudes $A_i(n)$ may be generated by

$$A_i(n) = A_i e^{j\phi_{in}}$$

where ϕ_{in} are independent random phases distributed uniformly over the interval $[0, 2\pi]$, and A_i are deterministic amplitudes related to the assumed signal to noise ratios (SNR) in units of decibels by

$$SNR_i = 10 \log_{10} \left[\frac{|A_i|^2}{\sigma_v^2} \right]$$

(a) Consider one plane wave incident on an array of seven sensors from an angle $\theta_1 = 30^o$. The sensors are equally spaced at half-wavelength spacings; i.e., $d = \lambda/2$. For each of the following values of the SNR of the wave

$$SNR = 0 \text{ dB}, \quad 10 \text{ dB}, \quad 20 \text{ dB}$$

generate $N = 1000$ snapshots of Eq. (14.3.7) and compute the empirical spatial correlation matrix across the array by

$$\hat{R} = \frac{1}{N} \sum_{n=0}^{N-1} \mathbf{y}(n)^* \mathbf{y}(n)^T$$

Compute and plot on the same graph the three spatial spectra: Bartlett, autoregressive (AR), and maximum likelihood (ML), versus wavenumber k.

(b) Repeat for two plane waves incident from angles $\theta_1 = 25^o$ and $\theta_2 = 35^o$, and with equal powers of 30 dB.

(c) Repeat part (b) for angles $\theta_1 = 28^o$ and $\theta_2 = 32^o$.

(d) Repeat part (c) by gradually decreasing the (common) SNR of the two plane waves to the values of 20 dB, 10 dB, and 0 dB.

(e) For parts (a) through (d), also plot all the theoretical spectra.

14.12 Consider L plane waves incident on a linear array of $M+1$ sensors ($L \le M$) in the presence of spatially coherent noise. As discussed in Sec. 14.3, the corresponding covariance matrix is given by

$$R = \sigma_v^2 Q + \sum_{i=1}^{L} P_i \mathbf{s}_{k_i} \mathbf{s}_{k_i}^\dagger$$

where the waves are assumed to be mutually uncorrelated.

(a) Show that the generalized eigenvalue problem

$$R\mathbf{a} = \lambda Q \mathbf{a}$$

has (1) an $(M+1-L)$-dimensional noise subspace spanned by $M+1-L$ linearly independent degenerate eigenvectors, all belonging to the eigenvalue $\lambda = \sigma_v^2$, and (2) an L-dimensional signal subspace with L eigenvalues greater than σ_v^2.

(b) Show that any two eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ belonging to distinct eigenvalues λ_1 and λ_2 are orthogonal to each other with respect to the inner product defined by the matrix Q, that is, show that $\mathbf{a}_1^\dagger Q \mathbf{a}_2 = 0$.

(c) Show that the L-dimensional signal subspace is spanned by the L vectors

$$Q^{-1}\mathbf{s}_{k_i}, \quad i = 1, 2, \ldots, L$$

(d) Show that any vector $\mathbf{a}$ in the noise subspace corresponds to a polynomial $A(z)$ that has L of its M zeros on the unit circle at locations

$$z_i = e^{jk_i}, \quad i = 1, 2, \ldots, L$$

The remaining $M - L$ zeros can have arbitrary locations.

14.13 The previous problem suggests the following approach to the problem of "selectively nulling" some of the sources and not nulling others. Suppose L_1 of the sources are not to be nulled and have known SNRs and directions of arrival, and L_2 of the sources are to be nulled. The total number of sources is then $L = L_1 + L_2$, and assuming incoherent background noise, the incident field will have covariance matrix

$$R = \sigma_v^2 I + \sum_{i=1}^{L_1} P_i \mathbf{s}_{k_i} \mathbf{s}_{k_i}^\dagger + \sum_{i=L_1+1}^{L_1+L_2} P_i \mathbf{s}_{k_i} \mathbf{s}_{k_i}^\dagger$$

Define Q by

$$\sigma_v^2 Q = \sigma_v^2 I + \sum_{i=1}^{L_1} P_i \mathbf{s}_{k_i} \mathbf{s}_{k_i}^\dagger$$

so that we may write R as follows

$$R = \sigma_v^2 Q + \sum_{i=L_1+1}^{L_1+L_2} P_i \mathbf{s}_{k_i} \mathbf{s}_{k_i}^\dagger$$

Then, the nulling of the L_2 sources at wavenumbers k_i, $i = L_1+1, \ldots, L_1+L_2$, can be effected by the $(M+1-L_2)$-dimensional noise subspace of the generalized eigenvalue problem

$$R\mathbf{a} = \lambda Q \mathbf{a}$$

having minimum eigenvalue equal to σ_v^2.

(a) As an example, consider the case $M = 2$, $L_1 = L_2 = 1$. Then,

$$R = \sigma_v^2 Q + P_2 \mathbf{s}_{k_2} \mathbf{s}_{k_2}^\dagger, \quad \sigma_v^2 Q = \sigma_v^2 I + P_1 \mathbf{s}_{k_1} \mathbf{s}_{k_1}^\dagger$$

Show that the $(M+1-L_2 = 2)$-dimensional noise subspace is spanned by the two eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ -e^{jk_2} \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ -e^{jk_2} \end{bmatrix}$$

(b) Show that an arbitrary linear combination

$$\mathbf{a} = \mathbf{e}_1 + \rho \mathbf{e}_2$$

corresponds to a filter $A(z)$ having one zero at the desired location $z_2 = e^{jk_2}$, and a spurious zero with arbitrary location.

(c) Show that the $(L_2 = 1)$-dimensional signal subspace is spanned by the vector

$$\mathbf{e}_3 = Q^{-1}\mathbf{s}_{k_2}$$

and that the corresponding generalized eigenvalue is

$$\lambda = \sigma_v^2 + P_2 \mathbf{s}_{k_2}^\dagger Q^{-1} \mathbf{s}_{k_2}$$

(d) Verify the orthogonality properties $\mathbf{e}_i^\dagger Q\mathbf{e}_3 = 0$, $i = 1,2$, for the three eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ defined in parts (a) and (c).

(e) As another example, consider the case $M = 3$ and $L_1 = L_2 = 1$. Show that the $(M + 1 - L_2 = 3)$-dimensional noise subspace is spanned by the three eigenvectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ -e^{jk_2} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ -e^{jk_2} \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -e^{jk_2} \end{bmatrix}$$

and the signal eigenvector is $\mathbf{e}_4 = Q^{-1}\mathbf{s}_{k_2}$. Generalize this part and part (a), to the case of arbitrary M and $L_1 = L_2 = 1$.

(f) As a final example that corresponds to a unique noise eigenvector, consider the case $M = 2$, $L_1 = 1$, and $L_2 = 2$, so that

$$R = \sigma_v^2 Q + P_2\mathbf{s}_{k_2}\mathbf{s}_{k_2}^\dagger + P_3\mathbf{s}_{k_3}\mathbf{s}_{k_3}^\dagger, \quad \sigma_v^2 Q = \sigma_v^2 I + P_1\mathbf{s}_{k_1}\mathbf{s}_{k_1}^\dagger$$

with k_2 and k_3 to be nulled. Show that the $(M + 1 - L_2 = 1)$-dimensional noise subspace is spanned by

$$\mathbf{a} = \mathbf{e}_1 = \begin{bmatrix} 1 \\ -(e^{jk_2} + e^{jk_3}) \\ e^{jk_2}e^{jk_3} \end{bmatrix}$$

and that the corresponding polynomial $A(z)$ factors into the two desired zeros

$$A(z) = (1 - e^{jk_2}z^{-1})(1 - e^{jk_3}z^{-1})$$

14.14 *Computer Experiment.* Consider a nine-element $(M = 8)$ linear array with half-wavelength spacing and two mutually uncorrelated incident plane waves with wavenumbers $k_1 = 0.3\pi$, $k_2 = 0.5\pi$ and equal powers of 20 dB. The background noise is incoherent with variance $\sigma_v^2 = 1$.

(a) Construct the theoretical matrix R of Eq. (14.3.13) and solve its eigenproblem determining the nine eigenvectors and eigenvalues. Using a root finder (see e.g., [1206]), compute the eight zeros of each of the seven noise subspace eigenvectors and verify that the desired zeros lie on the unit circle.

(b) Generate $N = 100$ snapshots, construct the sample covariance matrix R of Eq. (14.4.14), solve its eigenproblem, use the AIC and MDL criteria to check the dimension of the noise subspace, but regardless of these criteria take that dimension to be seven. Compare the empirical eigenvalues with the theoretical ones found above. Compute the zeros of the noise subspace eigenvectors and decide if the desired zeros are among them and if any spurious ones lie close to the unit circle. Also, compute the zeros of the Min-Norm vector $\mathbf{d}$.

(c) On the same graph, plot in dB the pseudospectra of a few of the noise subspace eigenvectors, say, the first three. On a separate graph, but using the same vertical scales as the previous one, plot the MUSIC and Min-Norm spectra.

(d) Using the same set of snapshots, repeat parts (b,c) for the symmetrized sample covariance matrix of Eq. (14.4.15).

(e) For fixed SNR, repeat parts (b,c,d) for the following choices of number of snapshots: $N = 20, 50, 150, 200, 500$.

(f) With the number of snapshots fixed at $N = 100$, repeat parts (a,b,c,d) for the following values of the signal to noise ratio: $SNR = -10, -5, 0, 5, 10, 30$ dB.

(g) Repeat parts (a–f) for three 20-dB plane waves with $k_1 = 0.3\pi$, $k_2 = 0.4\pi$, $k_3 = 0.5\pi$.

14.15 Show Eqs. (14.11.9) and (14.11.10).

14.16 Consider an M-dimensional complex random vector $\mathbf{y}$ with real and imaginary parts $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, so that $\mathbf{y} = \boldsymbol{\xi} + j\boldsymbol{\eta}$. With the complex vector $\mathbf{y}$ we associate a $(2M)$-dimensional real random vector $\bar{\mathbf{y}} = \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{bmatrix}$. The corresponding covariance matrices are defined by

$$R = E[\mathbf{y}^*\mathbf{y}^T], \quad \bar{R} = E[\bar{\mathbf{y}}\bar{\mathbf{y}}^T]$$

(a) Show that the conditions $E[\boldsymbol{\xi}\boldsymbol{\xi}^T] = E[\boldsymbol{\eta}\boldsymbol{\eta}^T]$ and $E[\boldsymbol{\xi}\boldsymbol{\eta}^T] = -E[\boldsymbol{\eta}\boldsymbol{\xi}^T]$ are equivalent to the condition $E[\mathbf{y}\mathbf{y}^T] = 0$, and that in this case the covariance matrices can be written as follows:

$$R = 2(A + jB), \quad \bar{R} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}, \quad A = E[\boldsymbol{\xi}\boldsymbol{\xi}^T], \quad B = E[\boldsymbol{\xi}\boldsymbol{\eta}^T]$$

The matrix A is symmetric and B antisymmetric. Show the equality of the quadratic forms

$$\mathbf{y}^T R^{-1}\mathbf{y}^* = \frac{1}{2}\bar{\mathbf{y}}^T\bar{R}^{-1}\bar{\mathbf{y}}$$

Also, show the relationship between the determinants $\det R = 2^M(\det\bar{R})^{1/2}$.
Hint: Apply a correlation canceling transformation on $\bar{R}$ and use the matrix identity $A + BA^{-1}B = (A + jB)A^{-1}(A - jB)$.

(b) A complex gaussian random vector $\mathbf{y}$ is defined by the requirement that the corresponding real vector $\bar{\mathbf{y}}$ be gaussian [1207,1208]. Equating the elemental probabilities $p(\mathbf{y})d^{2M}\mathbf{y} = p(\bar{\mathbf{y}})d^{2M}\bar{\mathbf{y}}$ and using the results of part (a), show that if $p(\bar{\mathbf{y}})$ is an ordinary (zero-mean) gaussian with covariance $\bar{R}$, then the density of $\mathbf{y}$ is

$$p(\bar{\mathbf{y}}) = \frac{1}{(2\pi)^M(\det\bar{R})^{1/2}}\exp\left(-\frac{1}{2}\bar{\mathbf{y}}^T\bar{R}^{-1}\bar{\mathbf{y}}\right) \quad \Rightarrow \quad p(\mathbf{y}) = \frac{1}{\pi^M\det R}\exp(-\mathbf{y}^T R^{-1}\mathbf{y}^*)$$

(c) Using this density show for any four components of $\mathbf{y}$

$$E[y_i^* y_j y_k^* y_l] = R_{ij}R_{kl} + R_{il}R_{kj}$$

(d) Use this result to prove Eq. (14.11.12)

14.17 Show that the log-likelihood function based on N independent complex gaussian snapshots is given by (up to a constant)

$$\ln p = -N\,\mathrm{tr}\left[\ln R + R^{-1}\hat{R}\right]$$

where $\hat{R}$ is given byEq. (14.4.14). Note that it differs by a factor of two from the real-valued case. From the discussion of Sec. 1.18, it follows that $\hat{R}$ is the maximum likelihood estimate of R. Moreover, the trace formula for the Fisher information matrix also differs by a factor of two, namely,

$$J_{ij} = N\,\mathrm{tr}\left[R^{-1}\frac{\partial R}{\partial\lambda_i}R^{-1}\frac{\partial R}{\partial\lambda_j}\right]$$

14.18 Using Eq. (14.11.12), show that the covariances of the LP parameters E and $\mathbf{a}$ are in the complex-valued case:

$$E[(\Delta E)^2] = \frac{E^2}{N}, \quad E[\Delta\mathbf{a}\,\Delta E] = 0, \quad E[\Delta\mathbf{a}\,\Delta\mathbf{a}^\dagger] = \frac{E}{N}(R^{-1} - E^{-1}\mathbf{a}\,\mathbf{a}^\dagger)$$

14.19 Let $S(k) = \mathbf{s}_k^\dagger R \mathbf{s}_k$ be the Bartlett spectrum. Using Eq. (14.11.13), show that its variance is

$$E[(\Delta S(k))^2] = \frac{1}{N} S(k)^2$$

Show that the variance of the ML spectrum $S(k) = 1/\mathbf{s}_k^\dagger R^{-1}\mathbf{s}_k$ is also given by a similar formula.

14.20 (a) Let $A(k) = \mathbf{s}_k^\dagger \mathbf{a}$ be the frequency response of the LP polynomial in the complex-valued case. Using the results of Problem 14.18, show that its variance is

$$E[|\Delta A(k)|^2] = \frac{E}{N}[\mathbf{s}_k^\dagger R^{-1}\mathbf{s}_k - E^{-1}|A(k)|^2]$$

Use the kernel representation of Problem 12.17 to argue that the right-hand side is positive. Alternatively, show that it is positive by writing $A(k) = E(\mathbf{s}_k^\dagger R^{-1}\mathbf{u}_0)$ and $E = (\mathbf{u}_0^\dagger R^{-1}\mathbf{u}_0)^{-1}$, and using the Schwarz inequality.

(b) In the complex case, show that $E[\Delta\mathbf{a}\,\Delta\mathbf{a}^T] = 0$. Then, show that the variance of the AR spectrum $S(k) = E/|A(k)|^2$ is given by

$$E[(\Delta S(k))^2] = \frac{1}{N} S(k)^2 [2S(k)(\mathbf{s}_k^\dagger R^{-1}\mathbf{s}_k) - 1]$$

and show again that the right-hand side is positive.

15

SVD and Signal Processing

15.1 Vector and Matrix Norms

The three most widely used vector norms [1234,1235] are the L_2 or Euclidean norm, the L_1 and the L_∞ norms, defined for a vector $\mathbf{x} \in \mathbb{R}^N$ by:

$$\begin{aligned} \|\mathbf{x}\|_2 &= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_N|^2} = \sqrt{\mathbf{x}^T\mathbf{x}} \\ \|\mathbf{x}\|_1 &= |x_1| + |x_2| + \cdots + |x_N| \\ \|\mathbf{x}\|_\infty &= \max(|x_1|, |x_2|, \ldots, |x_N|) \end{aligned} \qquad \text{where} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \tag{15.1.1}$$

All vector norms satisfy the *triangle inequality*:

$$\|\mathbf{x} + \mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|, \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^N \tag{15.1.2}$$

Unless otherwise specified, from now on the notation $\|\mathbf{x}\|$ will denote the Euclidean norm. The *Cauchy-Schwarz inequality* for the Euclidean norm reads:

$$|\mathbf{x}^T\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\| \tag{15.1.3}$$

where equality is achieved when $\mathbf{y}$ is any scalar multiple of $\mathbf{x}$, that is, $\mathbf{y} = c\mathbf{x}$. The "angle" between the two vectors $\mathbf{x}, \mathbf{y}$ is defined through:

$$\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} \tag{15.1.4}$$

An $N \times M$ matrix A is a linear mapping from $\mathbb{R}^M$ to $\mathbb{R}^N$, that is, for each $\mathbf{x} \in \mathbb{R}^M$, the vector $\mathbf{y} = A\mathbf{x}$ is in $\mathbb{R}^N$. For each vector norm, one can define a corresponding *matrix norm* through the definition:

$$\|A\| = \sup_{\|\mathbf{x}\| \neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \sup_{\|\mathbf{x}\| = 1} \|A\mathbf{x}\| \tag{15.1.5}$$

We will see later that the Euclidean matrix norm $\|A\|_2$ is equal to the *largest singular value* of the SVD decomposition of A, or equivalently, the square-root of the largest