Agenda (16 October 2001)

Objective: [Bertsekas and Gallager, Section 3.4]

- Review M/M/1 (3.3.1)
- M/M/m (3.4.1)
- M/M/∞ (3.4.2)
- M/M/m/m (3.4.3)
- Multidimensional Markov Chains (3.4.4)

3.3.1: M/M/1

Background

Poisson Process

$$P[A(t+\delta)-A(t)=n]=e^{-\lambda\cdot\delta}\cdot\frac{(\lambda\cdot\delta)^n}{n!},\qquad n=0,1,\ldots\qquad(3.10)$$

Define: $\lim_{\delta\to 0}\frac{o(\delta)}{\delta}=0$

Taylor Series Expansion:

$$f(\delta)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}\cdot(\delta-a)^n$$

Letting $a = 0 \rightarrow f(\delta)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot\delta$

$$\Rightarrow e^{-\lambda\cdot\delta}=1-\lambda\cdot\delta+\frac{(\lambda\cdot\delta)^2}{2}-\frac{(\lambda\cdot\delta)^3}{6}+\ldots$$

Markov Chain Formulation

For small δ, evaluate $P[A(t+\delta)-A(t)=n]$:

$$P[N=0]=e^{-\lambda\cdot\delta}=1-\lambda\cdot\delta+o(\delta)$$
$$P[N=1]=\lambda\cdot\delta\cdot e^{-\lambda\cdot\delta}=\lambda\cdot\delta\cdot(1-\lambda\cdot\delta+o(\delta))=\lambda\cdot\delta+o(\delta)$$
$$P[N=2]=\frac{(\lambda\cdot\delta)^2}{2}\cdot e^{-\lambda\cdot\delta}=\frac{(\lambda\cdot\delta)^2}{2}\cdot(1-\lambda\cdot\delta+o(\delta))=o(\delta)$$
$$P[N=n\geq 3]=\frac{(\lambda\cdot\delta)^n}{n!}\cdot e^{-\lambda\cdot\delta}=\frac{(\lambda\cdot\delta)^n}{n!}\cdot(1-\lambda\cdot\delta+o(\delta))=o(\delta)$$

Consider transition probabilities

$$P_{i,j} = P[N_{k+1} = j \mid N_k = i]$$

Arrival process with rate λ

Server process with rate μ

$$P_{0,0} = e^{-\lambda\cdot\delta} = 1 - \lambda\cdot\delta + o(\delta) \qquad (3.15)$$

$$P_{i,i} = e^{-\lambda\cdot\delta}\cdot e^{-\mu\cdot\delta} = e^{-(\lambda+\mu)\cdot\delta} = 1-(\lambda+\mu)\cdot\delta + o(\delta), \qquad i \geq 1$$
(3.16)

$$P_{i,i+1} = \lambda\cdot\delta\cdot e^{-\lambda\cdot\delta} = \lambda\cdot\delta + o(\delta), \qquad i \geq 0 \qquad (3.17)$$

$$P_{i,i-1} = \mu\cdot\delta\cdot e^{-\lambda\cdot\delta} = \mu\cdot\delta + o(\delta), \qquad i \geq 1 \qquad (3.18)$$

$$P_{i,j} = o(\delta), \qquad j \neq i, i+1, i-1$$

See steady state Markov Chain (Fig. 3.6)

$$p_n\cdot(\lambda\cdot\delta + o(\delta)) = p_{n+1}\cdot(\mu\cdot\delta + o(\delta))$$

$$\Rightarrow \lim_{\delta\to 0}\left\{p_n\cdot\left(\lambda + \frac{o(\delta)}{\delta}\right) = p_{n+1}\cdot\left(\mu + \frac{o(\delta)}{\delta}\right)\right\}$$

$$\Rightarrow p_n\cdot\lambda = p_{n+1}\cdot\mu \qquad (3.20)$$

Letting $\rho = \frac{p_{n+1}}{p_n} = \frac{\lambda}{\mu}$

$$\Rightarrow p_{n+1} = \rho^{n+1}\cdot p_0 \qquad n = 0,1,\ldots \qquad (3.21)$$

Solving for p_0:

$$1 = \sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \rho^n \cdot p_0 = \frac{p_0}{1-\rho} \Rightarrow p_0 = 1-\rho$$

$$\Rightarrow p_n = \rho^n \cdot (1-\rho), \qquad n = 0,1,\ldots \qquad (3.23)$$

Average number of customers in steady state

$$E[N] = N = \sum_{n=0}^{\infty} n \cdot p_n = \sum_{n=0}^{\infty} n \cdot p^n \cdot (1-\rho) = \rho \cdot \sum_{n=0}^{\infty} n \cdot \rho^{n-1} \cdot (1-\rho)$$

Let $q = 1-\rho$:

$$N = (1-q) \cdot \sum_{n=0}^{\infty} n \cdot q \cdot (1-q)^{n-1} = (1-q) \cdot \sum_{n=1}^{\infty} n \cdot q \cdot (1-q)^{n-1}$$

$$\Rightarrow N = (1-q) \cdot \frac{1}{q} = \frac{1-q}{q}$$

$$\Rightarrow N = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda} \qquad (3.24)$$

Applying Little's Theorem:

$$T = \frac{N}{\lambda} = \frac{1}{\mu-\lambda}$$

Applying Little's Theorem for the queue: $N_Q = \lambda \cdot W$

$$W = T - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\mu - (\mu - \lambda)}{\mu \cdot (\mu - \lambda)} = \frac{\lambda}{\mu \cdot (\mu - \lambda)} = \frac{\rho}{\mu - \lambda}$$

$$\Rightarrow N_Q = \lambda \cdot \frac{\lambda / \mu}{\mu - \lambda} = \frac{\rho^2}{1 - \rho}$$

3.4.1: M/M/m (See Fig. 3.9)

Balanced equations for steady state probabilities

$$\begin{aligned} \lambda \cdot p_{n-1} &= n \cdot \mu \cdot p_n, \qquad n \le m \\ \lambda \cdot p_{n-1} &= m \cdot \mu \cdot p_n, \qquad n > m \end{aligned} \tag{3.33}$$

$$\rho = \frac{\lambda}{m \cdot \mu}$$

$$\frac{p_1}{p_0} = \frac{\lambda}{\mu} = m \cdot \rho \Rightarrow p_1 = m \cdot \rho \cdot p_0$$

$$\frac{p_2}{p_1} = \frac{\lambda}{2 \cdot \mu} = \frac{m \cdot \rho}{2} \Rightarrow p_2 = \frac{(m \cdot \rho)^2 \cdot p_0}{2}$$

$$\frac{p_3}{p_2} = \frac{\lambda}{3 \cdot \mu} = \frac{m \cdot \rho}{3} \Rightarrow p_3 = \frac{(m \cdot \rho)^3 \cdot p_0}{3!}$$

$$\vdots$$

$$\frac{p_m}{p_{m-1}} = \frac{\lambda}{m \cdot \mu} = \frac{m \cdot \rho}{m} \Rightarrow p_m = \frac{(m \cdot \rho)^m}{m!} \cdot p_0$$

$$\frac{p_{m+1}}{p_m} = \frac{\lambda}{m \cdot \mu} = \rho \Rightarrow p_{m+1} = \frac{m^m \cdot \rho^{m+1}}{m!} \cdot p_0$$

$$\vdots$$

$$\frac{p_n}{p_{n-1}} = \frac{\lambda}{m \cdot \mu} = \rho \Rightarrow p_n = \frac{m^m \cdot \rho^n}{m!} \cdot p_0$$

$$\vdots$$

$$p_n = \begin{cases} \dfrac{(m \cdot \rho)^n}{n!} \cdot p_0, & n \le m \\ \dfrac{m^m \cdot \rho^n}{m!} \cdot p_0, & n > m \end{cases} \tag{3.34}$$

Solving for p_0: $1 = \sum_{n=0}^{\infty} p_n$

$$1 = p_0 \cdot \left[1 + \sum_{n=1}^{m-1} \frac{(m \cdot \rho)^n}{n!} + \sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m!} \cdot \frac{1}{m^{n-m}} \right]$$

$$\Rightarrow p_0 = \left[1 + \sum_{n=1}^{m-1} \frac{(m \cdot \rho)^n}{n!} + \sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m!} \cdot \frac{1}{m^{n-m}} \right]^{-1} \qquad (*)$$

Evaluating the second summation in (*):

$$\sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m!} \cdot \frac{1}{m^{n-m}} = \frac{1}{m!} \cdot \sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m^{n-m}} = \frac{m^m}{n!} \cdot \sum_{n=m}^{\infty} \rho^n = \frac{(m \cdot \rho)^m}{m! \cdot (1 - \rho)}$$

$$p_0 = \left[\frac{(m \cdot \rho)^m}{m! \cdot (1 - \rho)} + \sum_{n=0}^{m-1} \frac{(m \cdot \rho)^n}{n!} \right]^{-1} \qquad (3.35)$$

Queuing Probability

$$P_Q = \sum_{n=m}^{\infty} p_n = \sum_{n=m}^{\infty} \frac{p_0 \cdot m^m \cdot \rho^n}{m!} = \frac{p_0 \cdot (m \cdot \rho)^m}{m!} \cdot \sum_{n=m}^{\infty} \rho^{n-m}$$

$$\Rightarrow P_Q = \frac{p_0 \cdot (m \cdot \rho)^m}{m! \cdot (1 - \rho)} \qquad (3.36)$$ **Erlang C Formula**

Q: What is P_Q for M/M/1 system?

Number of customers in queue:

$$N_Q = \sum_{n=0}^{\infty} n \cdot p_{m+n} = \sum_{n=0}^{\infty} n \cdot p_0 \cdot \frac{m^m \cdot \rho^{m+n}}{m!} = \frac{p_0 \cdot (m \cdot \rho)^m}{m!} \cdot \sum_{n=0}^{\infty} n \cdot \rho^n$$

$$\Rightarrow N_Q = \frac{p \cdot (m \cdot \rho)^m}{m!} \cdot \frac{1}{1-\rho} \cdot \sum_{n=0}^{\infty} n \cdot \rho^n \cdot (1-\rho)$$

$$\Rightarrow N_Q = \frac{p_0 \cdot (m \cdot \rho)^m}{m! \cdot (1-\rho)} \cdot \rho \cdot \sum_{n=1}^{\infty} n \cdot \rho^{n-1} \cdot (1-\rho)$$

$$\Rightarrow N_Q = P_Q \cdot \frac{\rho}{1-\rho}, \qquad \rho = \frac{\lambda}{m \cdot \mu} \tag{3.37}$$

Applying Little's Theorem

$$W = \frac{N_Q}{\lambda} = \frac{\rho \cdot P_Q}{\lambda \cdot (1-\rho)}$$

Use queuing delay W to obtain total delay T:

$$T = \frac{1}{\mu} + W = \frac{1}{\mu} + \frac{\rho \cdot P_Q}{\lambda \cdot (1-\rho)} = \frac{1}{\mu} + \frac{P_Q}{m \cdot \mu - \lambda} \tag{3.38}$$

$$\Rightarrow N = \lambda \cdot T = \frac{\lambda}{\mu} + \frac{\lambda \cdot P_Q}{m \cdot \mu - \lambda} = m \cdot \rho + \frac{\rho \cdot P_Q}{1-\rho}$$

Note, here $N_Q = \lambda \cdot W$ is used to obtain $N = \lambda \cdot T$, opposite of presentation for M/M/1 case.

3.4.2: M/M/∞

Recall (3.33):
$$\lambda \cdot p_{n-1} = n \cdot \mu \cdot p_n, \qquad n \le m$$
$$\lambda \cdot p_{n-1} = m \cdot \mu \cdot p_n, \qquad n > m$$

Since $m = \infty$:

$$\Rightarrow \lambda \cdot p_{n-1} = n \cdot \mu \cdot p_n, \qquad n = 1,2,\ldots$$

Thus, $\dfrac{p_n}{p_{n-1}} = \dfrac{\lambda}{n \cdot \mu}, \qquad n = 1,2,\ldots$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \cdot p_0, \qquad n = 1,2,\ldots$$

Solving for p_0: $\displaystyle\sum_{n=0}^{\infty} p_n = 1$

$$\Rightarrow p_0 \cdot \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} = 1$$

$$\Rightarrow p_0 = \left[1 + \frac{\lambda}{\mu} + \frac{1}{2} \cdot \left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{6} \cdot \left(\frac{\lambda}{\mu}\right)^3 + \ldots\right]^{-1}$$

$$\Rightarrow p_0 = e^{-\lambda/\mu}$$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{e^{-\lambda/\mu}}{n!}, \qquad n = 0,1,\ldots$$

Little's Theorem:

$$E[N] = N = \frac{\lambda}{\mu} \text{ and } T = \frac{1}{\mu}$$

3.4.3: M/M/m/m

Recall (3.33):

$$\lambda \cdot p_{n-1} = n \cdot \mu \cdot p_n, \qquad n \le m$$
$$\lambda \cdot p_{n-1} = m \cdot \mu \cdot p_n, \qquad n > m$$

And considering Fig. 3.10 yields the balanced equations:

$$\lambda \cdot p_{n-1} = n \cdot \mu \cdot p_n, \qquad n = 1,2,\ldots,m$$

$$\Rightarrow \frac{p_n}{p_{n-1}} = \frac{\lambda}{n \cdot \mu}, \qquad n = 1,2,\ldots,m$$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \cdot p_0, \qquad n = 0,1,\ldots,m$$

$$\sum_{n=0}^{m} p_n = 1 \Rightarrow p_0 \cdot \sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} = 1$$

$$\Rightarrow p_0 = \left[\sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right) \cdot \frac{1}{n!}\right]^{-1}$$

Erlang B Formula:

$$\Rightarrow P_B = P_m = \frac{\left(\frac{\lambda}{\mu}\right)^m}{m! \cdot \sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!}}$$

3.4.4: Multidimensional Markov Chains

Key assumptions for tractable analysis

1) Detailed balance equations hold:

$$\lambda \cdot P[n_1, n_2, \ldots n_{i-1}, n_i, n_{i+1}, \ldots, n_K] = \mu \cdot P[n_1, n_2, \ldots n_{i-1}, 1 + n_i, n_{i+1}, \ldots, n_K]$$

2) Stationary distribution can be expressed in product form:

$$P[n_1, n_2, \ldots, n_K] = P_1[n_1] \cdot P_2[n_2] \ldots P_K[n_K]$$

Truncated Markov Chains **(M/M/1)**

Recall for M/M/1

$$p_n = \rho^n \cdot p_0 = \rho^n \cdot (1 - \rho), \qquad \rho = \frac{\lambda}{\mu}$$

Thus,

$$P[n_1, n_2, \ldots, n_K] = P_1[n_1] \cdot P_2[n_2] \ldots P_K[n_K] \text{ — property (2)}$$

with $P_i[n_i] = \rho_i^{n_i} \cdot (1 - \rho_i), \qquad \rho_i = \frac{\lambda_i}{\mu_i}$

Normalizing,

$$P[n_1, n_2, \ldots, n_K] = \frac{\rho_1^{n_1} \cdot \rho_2^{n_2} \ldots \rho_K^{n_K}}{G} \qquad (3.39)$$

$$G = \sum_{(n_1, n_2, \ldots, n_K) \in S} \rho_1^{n_1} \cdot \rho_2^{n_2} \ldots \rho_K^{n_K} \qquad (3.40)$$

Truncated Chains **(M/M/∞)**

$$P[n_1, n_2, \ldots, n_k] = \frac{\frac{\rho_1^{n_1}}{n_1!} \cdot \frac{\rho_2^{n_2}}{n_2!} \cdots \frac{\rho_K^{n_K}}{n_K!}}{G}$$

$$G = \sum_{(n_1, n_2, \ldots, n_K) \in S} \frac{\rho_1^{n_1}}{n_1!} \cdot \frac{\rho_2^{n_2}}{n_2!} \cdots \frac{\rho_K^{n_K}}{n_K!}$$

where $\rho_i = \frac{\lambda_i}{\mu_i}$.

Note: $p_{0,i} = 1 - \rho_i$ is omitted from (3.39) and (3.40) as it would otherwise cancel exactly. Similarly, explicit recognition of $p_{0,i} = e^{-\rho_i}$ is omitted in the analysis of the M/M/∞ case.