

Agenda (26 September 2002)

From [Bertsekas and Gallager]

- Little's Theorem (3.2)
- M/M/1 (3.3.1)
- M/M/m (3.4.1)
- M/M/ ∞ (3.4.2)
- M/M/m/m (3.4.3)
- Multidimensional Markov Chains (3.4.4)

3.2: Little's Theorem

Definitions:

$N(t) \equiv$ Number of customers in system at time t

$\alpha(t) \equiv$ Number of customers who arrived in the interval $[0,t]$

$T_k \equiv$ Time spent in the system by the k^{th} arriving customer

$N_t \equiv$ Typical number of customers in the system up to time t ,
i.e., the time average of $N(\tau)$ up to time t :

$$N_t = \frac{1}{t} \cdot \int_0^t N(\tau) d\tau$$

$\lambda_t \equiv$ Time average arrival rate over the interval $[0,t]$:

$$\lambda_t = \frac{\alpha(t)}{t}$$

$T_t \equiv$ Average time spent in the system per customer up to time t , i.e., the time average of the customer delay up to time t :

$$T_t = \frac{\sum_{k=1}^{\alpha(t)} T_k}{\alpha(t)} \quad (3.1) \quad \text{Note correction to limits of summation}$$

Relating N_t with λ_t and T_t yields:

$$N_t = \lambda_t \cdot T_t \quad (3.2)$$

Ergodic: of or relating to a process in which a sequence or sizable sample is equally representative of the whole (as in regard to statistical parameter)

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$$N \equiv \lim_{t \rightarrow \infty} E[N(t)] = \lim_{t \rightarrow \infty} N_t$$

$$T \equiv \lim_{k \rightarrow \infty} E[T_k] = \lim_{k \rightarrow \infty} \frac{1}{k} \cdot \sum_{j=1}^k T_j = \lim_{t \rightarrow \infty} T_t$$

Lastly, defining $\lambda \equiv \lim_{t \rightarrow \infty} \frac{E[\alpha(t)]}{t} = \lim_{t \rightarrow \infty} \lambda_t$, and combining with N and T yields Little's Theorem:

$$N = \lambda \cdot T$$

3.3.1: M/M/1

Background

Poisson Process

$$P[A(t + \delta) - A(t) = n] = e^{-\lambda \cdot \delta} \cdot \frac{(\lambda \cdot \delta)^n}{n!}, \quad n = 0, 1, \dots \quad (3.10)$$

Define: $\lim_{\delta \rightarrow 0} \frac{o(\delta)}{\delta} = 0$

Taylor Series Expansion:

$$f(\delta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (\delta - a)^n$$

Letting $a = 0 \rightarrow f(\delta) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot \delta^n$

$$\Rightarrow e^{-\lambda \cdot \delta} = 1 - \lambda \cdot \delta + \frac{(\lambda \cdot \delta)^2}{2} - \frac{(\lambda \cdot \delta)^3}{6} + \dots$$

Markov Chain Formulation

For small δ , evaluate $P[A(t + \delta) - A(t) = n]$:

$$P[N = 0] = e^{-\lambda \cdot \delta} = 1 - \lambda \cdot \delta + o(\delta)$$

$$P[N = 1] = \lambda \cdot \delta \cdot e^{-\lambda \cdot \delta} = \lambda \cdot \delta \cdot (1 - \lambda \cdot \delta + o(\delta)) = \lambda \cdot \delta + o(\delta)$$

$$P[N = 2] = \frac{(\lambda \cdot \delta)^2}{2} \cdot e^{-\lambda \cdot \delta} = \frac{(\lambda \cdot \delta)^2}{2} \cdot (1 - \lambda \cdot \delta + o(\delta)) = o(\delta)$$

$$P[N = n \geq 3] = \frac{(\lambda \cdot \delta)^n}{n!} \cdot e^{-\lambda \cdot \delta} = \frac{(\lambda \cdot \delta)^n}{n!} \cdot (1 - \lambda \cdot \delta + o(\delta)) = o(\delta)$$

Consider transition probabilities

$$P_{i,j} = P[N_{k+1} = j | N_k = i]$$

Arrival process with rate λ

Server process with rate μ

$$P_{0,0} = e^{-\lambda \cdot \delta} = 1 - \lambda \cdot \delta + o(\delta) \quad (3.15)$$

$$P_{i,i} = e^{-\lambda \cdot \delta} \cdot e^{-\mu \cdot \delta} = e^{-(\lambda + \mu) \cdot \delta} = 1 - (\lambda + \mu) \cdot \delta + o(\delta), \quad i \geq 1 \quad (3.16)$$

$$P_{i,i+1} = \lambda \cdot \delta \cdot e^{-\lambda \cdot \delta} = \lambda \cdot \delta + o(\delta), \quad i \geq 0 \quad (3.17)$$

$$P_{i,i-1} = \mu \cdot \delta \cdot e^{-\lambda \cdot \delta} = \mu \cdot \delta + o(\delta), \quad i \geq 1 \quad (3.18)$$

$$P_{i,j} = o(\delta), \quad j \neq i, i+1, i-1$$

See steady state Markov Chain (Fig. 3.6)

$$p_n \cdot (\lambda \cdot \delta + o(\delta)) = p_{n+1} \cdot (\mu \cdot \delta + o(\delta))$$

$$\Rightarrow \lim_{\delta \rightarrow 0} \left\{ p_n \cdot \left(\lambda + \frac{o(\delta)}{\delta} \right) = p_{n+1} \cdot \left(\mu + \frac{o(\delta)}{\delta} \right) \right\}$$

$$\Rightarrow p_n \cdot \lambda = p_{n+1} \cdot \mu \quad (3.20)$$

Letting $\rho = \frac{p_{n+1}}{p_n} = \frac{\lambda}{\mu}$

$$\Rightarrow p_{n+1} = \rho^{n+1} \cdot p_0 \quad n = 0, 1, \dots \quad (3.21)$$

Solving for p_0 :

$$1 = \sum_{n=0}^{\infty} p_n = \sum_{n=0}^{\infty} \rho^n \cdot p_0 = \frac{p_0}{1-\rho} \Rightarrow p_0 = 1-\rho$$

$$\Rightarrow p_n = \rho^n \cdot (1-\rho), \quad n = 0, 1, \dots \quad (3.23)$$

Average number of customers in steady state

$$E[N] = N = \sum_{n=0}^{\infty} n \cdot p_n = \sum_{n=0}^{\infty} n \cdot \rho^n \cdot (1-\rho) = \rho \cdot \sum_{n=0}^{\infty} n \cdot \rho^{n-1} \cdot (1-\rho)$$

Let $q = 1-\rho$:

$$N = (1-q) \cdot \sum_{n=0}^{\infty} n \cdot q \cdot (1-q)^{n-1} = (1-q) \cdot \sum_{n=1}^{\infty} n \cdot q \cdot (1-q)^{n-1}$$

$$\Rightarrow N = (1-q) \cdot \frac{1}{q} = \frac{1-q}{q}$$

$$\Rightarrow N = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda} \quad (3.24)$$

Applying Little's Theorem:

$$T = \frac{N}{\lambda} = \frac{1}{\mu-\lambda}$$

Applying Little's Theorem for the queue: $N_Q = \lambda \cdot W$

$$W = T - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\mu - (\mu - \lambda)}{\mu \cdot (\mu - \lambda)} = \frac{\lambda}{\mu \cdot (\mu - \lambda)} = \frac{\rho}{\mu - \lambda}$$

$$\Rightarrow N_Q = \lambda \cdot \frac{\lambda/\mu}{\mu - \lambda} = \frac{\rho^2}{1 - \rho}$$

3.4.1: M/M/m (See Fig. 3.9)

Balanced equations for steady state probabilities

$$\begin{aligned} \lambda \cdot p_{n-1} &= n \cdot \mu \cdot p_n, & n \leq m \\ \lambda \cdot p_{n-1} &= m \cdot \mu \cdot p_n, & n > m \end{aligned} \quad (3.33)$$

$$\rho = \frac{\lambda}{m \cdot \mu}$$

$$\frac{p_1}{p_0} = \frac{\lambda}{\mu} = m \cdot \rho \Rightarrow p_1 = m \cdot \rho \cdot p_0$$

$$\frac{p_2}{p_1} = \frac{\lambda}{2 \cdot \mu} = \frac{m \cdot \rho}{2} \Rightarrow p_2 = \frac{(m \cdot \rho)^2 \cdot p_0}{2}$$

$$\frac{p_3}{p_2} = \frac{\lambda}{3 \cdot \mu} = \frac{m \cdot \rho}{3} \Rightarrow p_3 = \frac{(m \cdot \rho)^3 \cdot p_0}{3!}$$

⋮

$$\frac{p_m}{p_{m-1}} = \frac{\lambda}{m \cdot \mu} = \frac{m \cdot \rho}{m} \Rightarrow p_m = \frac{(m \cdot \rho)^m \cdot p_0}{m!}$$

$$\frac{p_{m+1}}{p_m} = \frac{\lambda}{m \cdot \mu} = \rho \Rightarrow p_{m+1} = \frac{m^m \cdot \rho^{m+1}}{m!} \cdot p_0$$

⋮

$$\frac{p_n}{p_{n-1}} = \frac{\lambda}{m \cdot \mu} = \rho \Rightarrow p_n = \frac{m^m \cdot \rho^n}{m!} \cdot p_0$$

⋮

$$p_n = \left\{ \begin{array}{ll} \frac{(m \cdot \rho)^n}{n!} \cdot p_0, & n \leq m \\ \frac{m^m \cdot \rho^n}{m!} \cdot p_0, & n > m \end{array} \right\} \quad (3.34)$$

Solving for p_0 : $1 = \sum_{n=0}^{\infty} p_n$

$$1 = p_0 \cdot \left[1 + \sum_{n=1}^{m-1} \frac{(m \cdot \rho)^n}{n!} + \sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m!} \cdot \frac{1}{m^{n-m}} \right]$$

$$\Rightarrow p_0 = \left[1 + \sum_{n=1}^{m-1} \frac{(m \cdot \rho)^n}{n!} + \sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m!} \cdot \frac{1}{m^{n-m}} \right]^{-1} \quad (*)$$

Evaluating the second summation in (*):

$$\sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m!} \cdot \frac{1}{m^{n-m}} = \frac{1}{m!} \cdot \sum_{n=m}^{\infty} \frac{(m \cdot \rho)^n}{m^{n-m}} = \frac{m^m}{m!} \cdot \sum_{n=m}^{\infty} \rho^n = \frac{(m \cdot \rho)^m}{m!(1-\rho)}$$

$$p_0 = \left[\frac{(m \cdot \rho)^m}{m!(1-\rho)} + \sum_{n=0}^{m-1} \frac{(m \cdot \rho)^n}{n!} \right]^{-1} \quad (3.35)$$

Queuing Probability

$$P_Q = \sum_{n=m}^{\infty} p_n = \sum_{n=m}^{\infty} \frac{p_0 \cdot m^m \cdot \rho^n}{m!} = \frac{p_0 \cdot (m \cdot \rho)^m}{m!} \cdot \sum_{n=m}^{\infty} \rho^{n-m}$$

$$\Rightarrow P_Q = \frac{p_0 \cdot (m \cdot \rho)^m}{m!(1-\rho)} \quad (3.36) \quad \text{Erlang C Formula}$$

Q: What is P_Q for M/M/1 system?

Number of customers in queue:

$$\begin{aligned}
 N_Q &= \sum_{n=0}^{\infty} n \cdot p_{m+n} = \sum_{n=0}^{\infty} n \cdot p_0 \cdot \frac{m^m \cdot \rho^{m+n}}{m!} = \frac{p_0 \cdot (m \cdot \rho)^m}{m!} \cdot \sum_{n=0}^{\infty} n \cdot \rho^n \\
 &\Rightarrow N_Q = \frac{p \cdot (m \cdot \rho)^m}{m!} \cdot \frac{1}{1-\rho} \cdot \sum_{n=0}^{\infty} n \cdot \rho^n \cdot (1-\rho) \\
 &\Rightarrow N_Q = \frac{p_0 \cdot (m \cdot \rho)^m}{m! \cdot (1-\rho)} \cdot \rho \cdot \sum_{n=1}^{\infty} n \cdot \rho^{n-1} \cdot (1-\rho) \\
 &\Rightarrow N_Q = P_Q \cdot \frac{\rho}{1-\rho}, \quad \rho = \frac{\lambda}{m \cdot \mu} \quad (3.37)
 \end{aligned}$$

Applying Little's Theorem

$$W = \frac{N_Q}{\lambda} = \frac{\rho \cdot P_Q}{\lambda \cdot (1-\rho)}$$

Use queuing delay W to obtain total delay T :

$$\begin{aligned}
 T &= \frac{1}{\mu} + W = \frac{1}{\mu} + \frac{\rho \cdot P_Q}{\lambda \cdot (1-\rho)} = \frac{1}{\mu} + \frac{P_Q}{m \cdot \mu - \lambda} \quad (3.38) \\
 &\Rightarrow N = \lambda \cdot T = \frac{\lambda}{\mu} + \frac{\lambda \cdot P_Q}{m \cdot \mu - \lambda} = m \cdot \rho + \frac{\rho \cdot P_Q}{1-\rho}
 \end{aligned}$$

Note, here $N_Q = \lambda \cdot W$ is used to obtain $N = \lambda \cdot T$, opposite of presentation for M/M/1 case.

3.4.2: M/M/∞

$$\text{Recall (3.33): } \begin{aligned} \lambda \cdot p_{n-1} &= n \cdot \mu \cdot p_n, & n \leq m \\ \lambda \cdot p_{n-1} &= m \cdot \mu \cdot p_n, & n > m \end{aligned}$$

Since $m = \infty$:

$$\Rightarrow \lambda \cdot p_{n-1} = n \cdot \mu \cdot p_n, \quad n = 1, 2, \dots$$

$$\text{Thus, } \frac{p_n}{p_{n-1}} = \frac{\lambda}{n \cdot \mu}, \quad n = 1, 2, \dots$$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \cdot p_0, \quad n = 1, 2, \dots$$

$$\text{Solving for } p_0: \sum_{n=0}^{\infty} p_n = 1$$

$$\Rightarrow p_0 \cdot \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} = 1$$

$$\Rightarrow p_0 = \left[1 + \frac{\lambda}{\mu} + \frac{1}{2} \cdot \left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{6} \cdot \left(\frac{\lambda}{\mu}\right)^3 + \dots \right]^{-1}$$

$$\Rightarrow p_0 = e^{-\lambda/\mu}$$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{e^{-\lambda/\mu}}{n!}, \quad n = 0, 1, \dots$$

Little's Theorem:

$$E[N] = N = \frac{\lambda}{\mu} \quad \text{and} \quad T = \frac{1}{\mu}$$

3.4.3: M/M/m/m

Recall (3.33):

$$\begin{aligned} \lambda \cdot p_{n-1} &= n \cdot \mu \cdot p_n, & n \leq m \\ \lambda \cdot p_{n-1} &= m \cdot \mu \cdot p_n, & n > m \end{aligned}$$

And considering Fig. 3.10 yields the balanced equations:

$$\lambda \cdot p_{n-1} = n \cdot \mu \cdot p_n, \quad n = 1, 2, \dots, m$$

$$\Rightarrow \frac{p_n}{p_{n-1}} = \frac{\lambda}{n \cdot \mu}, \quad n = 1, 2, \dots, m$$

$$\Rightarrow p_n = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \cdot p_0, \quad n = 0, 1, \dots, m$$

$$\sum_{n=0}^m p_n = 1 \Rightarrow p_0 \cdot \sum_{n=0}^m \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} = 1$$

$$\Rightarrow p_0 = \left[\sum_{n=0}^m \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} \right]^{-1}$$

Erlang B Formula:

$$\Rightarrow P_B = P_m = \frac{\left(\frac{\lambda}{\mu}\right)^m}{m! \cdot \sum_{n=0}^m \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!}}$$

3.4.4: Multidimensional Markov Chains

Key assumptions for tractable analysis

1) Detailed balance equations hold:

$$\lambda_i \cdot P[n_1, n_2, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_K] = \mu_i \cdot P[n_1, n_2, \dots, n_{i-1}, 1 + n_i, n_{i+1}, \dots, n_K]$$

2) Stationary distribution can be expressed in product form:

$$P[n_1, n_2, \dots, n_K] = P_1[n_1] \cdot P_2[n_2] \dots P_K[n_K]$$

Truncated Markov Chains (**M/M/1**) — (Figure 3.14)

Recall for M/M/1

$$p_n = \rho^n \cdot p_0 = \rho^n \cdot (1 - \rho), \quad \rho = \frac{\lambda}{\mu}$$

Thus,

$$P[n_1, n_2, \dots, n_K] = P_1[n_1] \cdot P_2[n_2] \dots P_K[n_K] \text{ — property (2)}$$

$$\text{with } P_i[n_i] = \rho_i^{n_i} \cdot (1 - \rho_i), \quad \rho_i = \frac{\lambda_i}{\mu_i}$$

Normalizing,

$$P[n_1, n_2, \dots, n_K] = \frac{\rho_1^{n_1} \cdot \rho_2^{n_2} \dots \rho_K^{n_K}}{G} \quad (3.39)$$

$$G = \sum_{(n_1, n_2, \dots, n_K) \in S} \rho_1^{n_1} \cdot \rho_2^{n_2} \dots \rho_K^{n_K} \quad (3.40)$$

Truncated Chains ($M/M/\infty$)

$$P[n_1, n_2, \dots, n_k] = \frac{\rho_1^{n_1} \cdot \rho_2^{n_2} \cdots \rho_K^{n_K}}{G}$$

$$G = \sum_{(n_1, n_2, \dots, n_K) \in S} \frac{\rho_1^{n_1} \cdot \rho_2^{n_2} \cdots \rho_K^{n_K}}{n_1! \cdot n_2! \cdots n_K!}$$

$$\text{where } \rho_i = \frac{\lambda_i}{\mu_i}.$$

Note: $p_{0,i} = 1 - \rho_i$ is omitted from (3.39) and (3.40) as it would otherwise cancel exactly. Similarly, explicit recognition of $p_{0,i} = e^{-\rho_i}$ is omitted in the analysis of the $M/M/\infty$ case.