Outline

Statistics of Random Variables

Some Random Variables

Discrete-Time Markov Chains

Birth-Death Systems

Continuous-Time Markov Chains

References

R. D. Yates and D. J. Goodman, "Probability and stochastic processes: a friendly introduction for electrical and computer engineers."

D. Bertsekas and R. Gallager, "Data networks."

Statistics of Random Variables (from [Yates and Goodman]):

Define X as some random variable (RV).

Discrete RV case

If X is a *discrete* RV, define $S_X = \{x_1, x_2, \ldots, x_N\}$ as the range of X. That is, the value of X belongs to S_X.

Probability Mass Function (PMF):

$$P_X(x) = P[X = x]$$

Properties of X with $P_X(x)$ and S_X:

a) $P_X(x) \geq 0 \ \forall \ x$

b) $\sum_{x \in S_X} P_X(x) = 1$

c) Given $B \subset S_X$, $P[B] = \sum_{x \in B} P_X(x)$

Continuous RV case

Define a and b as upper and lower bounds of X if X is a *continuous* RV.

Cumulative Distribution Function (CDF)

$$F_X(x) = P[X \leq x]$$

Probability Density Function (PDF):

$$f_X(x) = \frac{dF(x)}{dx}$$

Properties of X with PDF $f_X(x)$:

a) $f_X(x) \geq 0 \ \forall \ x$

b) $F_X(x) = \int_{-\infty}^{x} f_X(u)du$

c) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Expected Value:

The *mean* or first moment.

Continuous RV case…

$$E[X] = \mu_X = \int_a^b x \cdot f(x)dx$$

Discrete RV case…

$$E[X] = \mu_X = \sum_{k=1}^{N} x_k \cdot P_X(x_k)$$

Variance:

Second moment minus first moment-squared…

$$Var[X] = E\left[(X - \mu_X)^2\right] = E\left[X^2\right] - \mu_X^2$$

Continuous RV case…

$$E\left[X^2\right] = \int_a^b x^2 \cdot f_X(x)dx$$

Discrete RV case…

$$E\left[X^2\right] = \sum_{k=1}^{N} x_k^2 \cdot P_X(x_k)$$

Standard Deviation:

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

*n*th Moment:

Continuous RV case…

$$E\left[X^n\right] = \int_a^b x^n \cdot f_X(x)dx$$

Discrete RV case…

$$E\left[X^n\right] = \sum_{k=1}^{N} x_k^n \cdot P_X(x_k)$$

Random Variables (from [Yates and Goodman]):

Bernoulli:

$$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$

- A coin flipped. Is the result "heads" (1) or "tails" (0)?
- A frame is received. Is it error-free (1) or corrupted (0)?

$$E[X] = p \qquad Var[X] = p \cdot (1-p)$$

Geometric:

$$P_X(x) = \begin{cases} p \cdot (1-p)^{x-1} & x = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

$0 < p < 1$

- How many times must a coin be flipped before a result of "heads" is achieved?
- How many times must a pair of dice be rolled before a result of "7" is rolled?
- How many times must a frame be transmitted before it is received error-free at the destination?

$$E[X] = \frac{1}{p} \qquad Var[X] = \frac{1-p}{p^2}$$

Binomial:

$$P_X(x) = \begin{cases} \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} & x = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

- A coin is flipped *n* times. How many times is the result "heads"?
- A source originates *n* frames. How many frames are received correctly at the destination?

$$E[X] = n \cdot p \qquad Var[X] = n \cdot p \cdot (1-p)$$

Discrete Uniform:

$$P_X(x) = \begin{cases} \dfrac{1}{n-m+1} & x = m, m+1, m+2, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

- A fair die is rolled: What is the result?
- In a given time slot, a particular transceiver will transmit on one of *n* frequencies {1,2,…,*n*} with equal likelihood. Which frequency is used?

$$E[X] = \frac{m+n}{2} \qquad Var[X] = \frac{(n-m)\cdot(n-m+2)}{12}$$

Continuous Uniform:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x < b \\ 0 & \text{otherwise} \end{cases}$$

- Every day, the morning train arrives at a station some time in the range of 6AM to 7AM. How many minutes after 6AM will it arrive, tomorrow morning?
- A packet for a CBR application is scheduled to arrive at a queue some time in the interval bounded by t_1 and t_2. At what time does the packet actually arrive?

$$E[X] = \frac{a+b}{2} \qquad Var[X] = \frac{(b-a)^2}{12}$$

Exponential:

$$f_X(x) = \begin{cases} \lambda \cdot e^{-\lambda \cdot x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$\lambda > 0$

- How long will a telephone conversation between a customer and service representative last?
- How much time elapses between new arrivals to a queue?

Of course, the exponential RV applies to the above questions only if the related processes conform to the "*memoryless*" system model.

$$E[X] = \frac{1}{\lambda} \qquad Var[X] = \frac{1}{\lambda^2}$$

Erlang:

$$f_X(x) = \begin{cases} \dfrac{\lambda^n \cdot x^{n-1} \cdot e^{-\lambda \cdot x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- How long will someone wait "on hold" for the next available service representative (given *n* other people are ahead in the queue and only 1 representative)?
- How much time elapses for *n* new packets to arrive at a queue?

As for the case of the exponential RV, the Erlang RV applies to the above questions only if the related processes conform to the "*memoryless*" system model.

$$E[X] = \frac{n}{\lambda} \qquad Var[X] = \frac{n}{\lambda^2}$$

Poisson:

$$P_X(x) = \begin{cases} \dfrac{(\lambda \cdot t)^x \cdot e^{-\lambda \cdot t}}{x!} & x = 0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

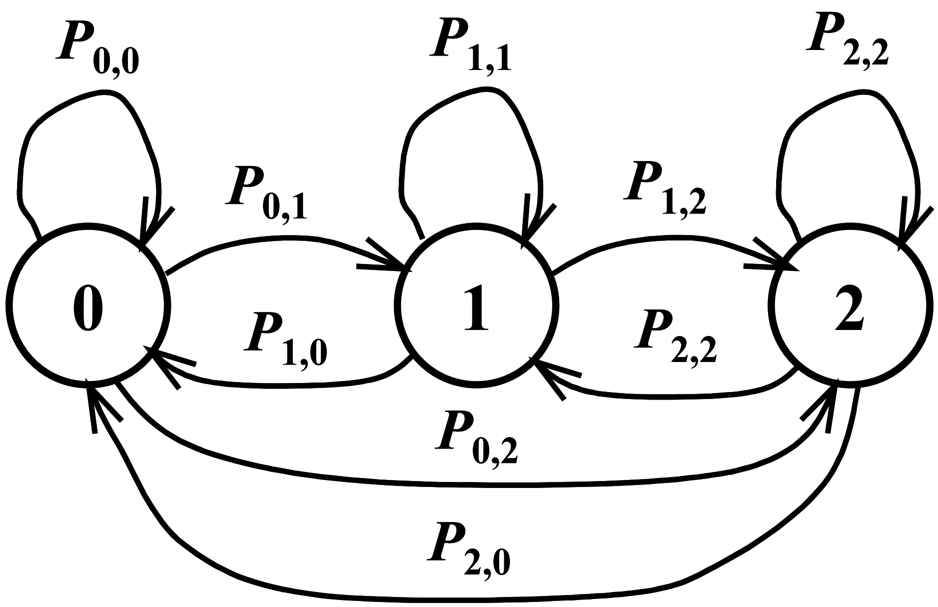
- Over the course of *t* seconds, how many customers call a service representative?
- Over the course of *t* seconds, how many new arrivals are received at a queue?

Of course, the Poisson RV is not the only possible arrival model.

$$E[X] = \lambda \cdot t \qquad Var[X] = \lambda \cdot t$$

Discrete Markov Chains (from [Bertsekas and Gallager])

Example: 3-state chain



$$P = \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} \\ P_{1,0} & P_{1,1} & P_{1,2} \\ P_{2,0} & P_{2,1} & P_{2,2} \end{bmatrix}$$

$P_{j,k} \equiv$ Probability of single-step transition from state j to state k

$$P_{j,k} \geq 0, \qquad \sum_{k=0}^{K-1} P_{j,k} = 1, \qquad j = 0,1,\ldots,J-1$$

$P^n_{j,k} \equiv$ Probability of n-step transition from state j to state k

Two states j and k are said to ***communicate*** if for some n and n', $P^{n}_{j,k} > 0$ and $P^{n'}_{k,j} > 0$.

If all states communicate, the Markov chain is ***irreducible***.

A Markov chain is ***periodic*** if $\exists$ some integer $m \geq 1$ such that $P^{m}_{j,j} > 0$ and some integer $d > 1$ such that $P^{n}_{j,j} > 0$ only if n is a multiple of d.

A Markov chain is ***aperiodic*** if none of its states are periodic.

Defining the probability p_k as the probability that the system is in state k, the probability distribution $\{p_k \mid k = 0,1,\ldots\}$ is a stationary distribution for the Markov chain if:

$$p_k = \sum_{j=0}^{\infty} p_j \cdot P_{j,k} \qquad \text{(3A.1)}$$

Of particular interest for this class are the *irreducible* and *aperiodic* Markov chains with *stationary* distributions.

Multiplying (3A.1) by $\sum_{j=0}^{\infty} P_{k,j} = 1$ yields the ***global balance equations***:

$$p_k \cdot \sum_{j=0}^{\infty} P_{k,j} = \sum_{j=0}^{\infty} p_j \cdot P_{j,k}$$

"…at equilibrium, the frequency of transitions out of k equals the frequency of transitions into k."

Birth-Death Systems

Discrete Markov chain where $P_{j,k} = 0$ if $|j - k| = 0$. Assume further that $P_{j,j+1} > 0$ and $P_{j+1,j} > 0 \ \forall\ j$.

Steady state conditions yields ***detailed balance equations***:

$$p_k \cdot P_{k,j} = p_j \cdot P_{j,k}$$

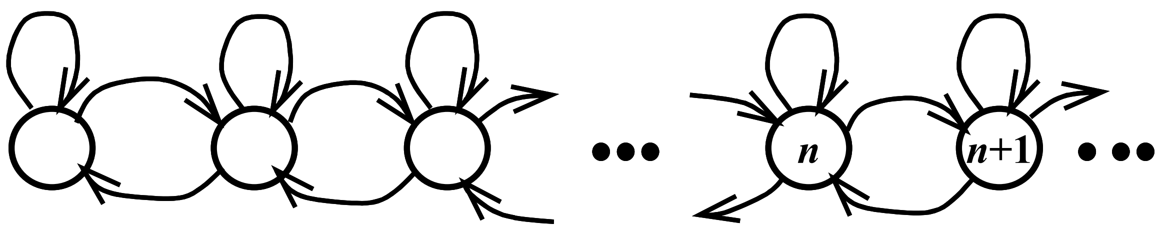


Figure 3A.1

Continuous-Time Markov Chains

Has the following properties upon entering state *j*:

1. Time spent in *j* is *exponentially* distributed with parameter ν_j
2. When process leaves state *j*, it will enter state *k* with probability $P_{j,k}$ where $\sum_k P_{j,k} = 1$

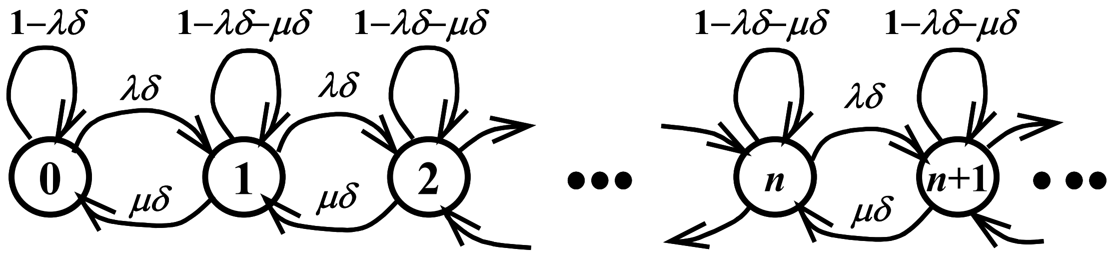


Figure 3.6

Under steady state conditions:

$$p_n \cdot \lambda \cdot \delta + o(\delta) = p_{n+1} \cdot \mu \cdot \delta + o(\delta)$$

$$\Rightarrow p_n \cdot \lambda = p_{n+1} \cdot \mu \qquad (3.20)$$