Section 6.10: Simple Examples for the Lapace Transform

This section contains examples for solving ordinary differential equations and boundary value problems. The first example illustrates the use of Laplace transforms of ordinary differential equations. The second example shows how the step function should be included in the solution when the driving term is a Dirac delta function (i.e., impulse function). The third example discusses boundary value problems.

Example 6.10.1: Solve the ordinary differential equation

$$\frac{df(t)}{dt} + 2f(t) = 1 \qquad \text{with } f(0)=0$$

Take the Laplace transform of both sides to get

$$\left[sF(s) - f(t=0)\right] + 2F(s) = \frac{1}{s}$$

where F(s) is the Laplace transform of f(t). Solving for F(s) gives

$$F(s) = \frac{1}{s(s+2)}$$

Expanding in partial fractions

$$F(s) = \frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{s+2}$$

requires the values a=1/2 and b=-1/2. The inverse transform of

$$F(s) = \frac{1/2}{s} - \frac{1/2}{s+2}$$

provides the function f(t)

$$f(t) = \frac{1}{2}e^{0t} - \frac{1}{2}e^{-2t} = \frac{1}{2}\left(1 - e^{-2t}\right)$$

Notice how the technique easily handles boundary conditions and simple driving terms.

Example 6.10.2: A Second Order Differential Equation with an Impulse Driving Term

The Laplace transforms are particularly useful for finding the impulse response of a system. Once the impulse response (i.e., the Green function) is known, a solution to the differential equation with general driving term can be constructed. The present example illustrates how limiting procedures are important for physical interpretation. The correct use of the step function is also illustrated. Sometimes problems arise because there are three or four limits near t=0. These limits occur in

Definition of the Laplace Transform:	$\int_0^\infty dt\, f(t)\, e^{-st}$
Laplace transforms of derivatives:	$L\left[f''(t)\right] = s^2F(s) - sf(0^{\leftarrow}) - f'(0^{\leftarrow})$
Boundary conditions:	f(0)=0
Dirac delta functions:	$\delta(0)$

For example, suppose that an object situated at the origin is at rest at t=0 so that $x(0) = 0 = x'(0)$. If the object is hit with an impulse force at t=0 then what is the initial speed of the object? The initial condition requires the initial speed to be zero, but the impulse requires it to be nonzero. The answer is to arrange the limits so that the impulse hits immediately after t=0.

Solve the following ordinary differential equation using Laplace transforms.

$$\text{ODE:} \quad x''(t) + x(t) = \delta(t)$$
$$\text{ICs:} \quad x(0) = 0 = x'(0)$$

Not paying attentions to limits $t \to 0$ produces the transform

$$s^2 X(s) + X(s) = 1$$

Using partial fractions, the Laplace transform is

$$X(s) = \frac{i}{2}\frac{1}{s+i} - \frac{i}{2}\frac{1}{s-i}$$

which has the inverse transform

$$x(t) = \frac{i}{2}e^{-it} - \frac{i}{2}e^{it}$$

Notice that the first boundary condition x(0) is satisfied. However, x(t) seems to satisfy a different ODE $x''(t) + x(t) = 0$ and a different second boundary condition $x'(0) = 1$. In other words, the delta function in the original ODE can be eliminated so long as the IC is changed. The new boundary condition requires that the object be moving at t=0. It turns out that the solution is correct for t>0 but requires the step function be included.

Let's work the problem again, but displace the impulse to the infinitesimal time $t = \varepsilon > 0$. In this way, the boundary condition $x'(0) = 0$ is correct and, when the impulse hits at $t = \varepsilon$, we expect $x'(\varepsilon) \cong 1$.

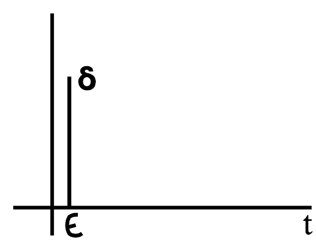

Figure 6.10.1: The impulse is infinitesimally displaced from t=0.

$$\text{ODE:} \quad x''(t) + x(t) = \delta(t - \varepsilon)$$
$$\text{ICs:} \quad x(0) = 0 = x'(0)$$

Again, using the initial conditions, the Laplace transform is

$$X(s) = e^{-s\varepsilon}\frac{i}{2}\left\{\frac{1}{s+i} - \frac{1}{s-i}\right\}$$

where theorem #8 in Topic 6.9.1 is used. The inverse transform is found by again using the same theorem to provide the new solution

$$x(t) = \frac{i}{2}\left[e^{-i(t-\varepsilon)} - e^{i(t-\varepsilon)}\right]\theta(t-\varepsilon)$$

At the end of the problem the small displacement in time is taken as zero $t = \varepsilon \to 0$.

The new solution satisfies x(0)=0 since the step function gives $\theta(t-\varepsilon)\big|_{t=0} = 0$. The second initial condition is also satisfied but requires (1) property # 4 in Topic 3.4.4 namely $f(x)\delta'(x-\varepsilon) = -f'(\varepsilon)\delta(x-\varepsilon)$ and (2) $\delta(t-\varepsilon) = d\theta(t-\varepsilon)/dt$. To see that the second BC is satisfied, calculate $x'(0)$ to get

$$x'(t)=\frac{i}{2}\left[-i\,e^{-i(t-\varepsilon)}-ie^{i(t-\varepsilon)}\right]\theta\left(t-\varepsilon\right)+\frac{i}{2}\left[e^{-i(t-\varepsilon)}-e^{i(t-\varepsilon)}\right]\delta\left(t-\varepsilon\right)$$

The initial speed found to be zero by setting t=0 and noting

$$\theta\left(0-\varepsilon\right)=0=\delta\left(0-\varepsilon\right)$$

Now, finally show that the ODE $x''(t)+x(t)=\delta(t-\varepsilon)$ is satisfied. Taking the derivatives and simplifying provides

$$x''(t)+x(t)=2\delta(t-\varepsilon)+\frac{i}{2}\left[e^{-i(t-\varepsilon)}-e^{i(t-\varepsilon)}\right]\delta'\left(t-\varepsilon\right)$$

The right hand side can be simplified by using the weak equality

$$x(t)\delta'(t-\varepsilon)=-x'(\varepsilon)\delta(t-\varepsilon)$$

to get

$$x''(t)+x(t)=2\delta(t-\varepsilon)-\frac{i}{2}\left[-i\,e^{-i(t-\varepsilon)}-ie^{i(t-\varepsilon)}\right]_{t=\varepsilon}\delta\left(t-\varepsilon\right)=2\delta(t-\varepsilon)-\delta(t-\varepsilon)=\delta(t-\varepsilon)$$

which shows the ODE is satisfied.

Notice that by setting $\varepsilon=0$, the original ordinary differential equation

$$x''(t)+x(t)=\delta(t)\quad\text{with }x(0)=0=x'(0)$$

is satisfied. Any apparent contradictions at t=0 can be resolved by taking the proper order for the limits.

Example 6.10.3: Solve a BVP

PDE: $u_t(x,t)=u_{xx}(x,t)$ with x,t>0

BCs: $u(x=0,t)=1$ $\qquad u(x\to\infty,t)=0$

IC: $u(x,0)=0$

Let's take the Laplace transform of the PDE with respect to the time t coordinate. The PDE becomes

$$sU(x,s)-u(x,t=0)=\frac{\partial^2}{\partial x^2}U(x,s)$$

where the second term is zero by the initial condition IC. The general solution to the resulting differential equation

$$\frac{\partial^2}{\partial x^2}U(x,s)=sU(x,s)$$

is given by

$$U(x,s)=c_1e^{\sqrt{s}\,x}+c_2e^{-\sqrt{s}\,x}$$

The second boundary condition of $u(x\to\infty,t)=0$ requires that $c_1=0$. To find c_2 in $U(x,s)=c_2e^{-\sqrt{s}\,x}$ where c_2 might depend on "s". The first boundary condition of $u(x=0,t)=1$ can be applied after applying the Laplace transform to it. The IC becomes

$$U(x=0,s)=1/s$$

Applying it to the solution gives

$$\frac{1}{s}=U(x=0,s)=c_2$$

The solution is now

$$U(x,s) = \frac{1}{s} e^{-\sqrt{s}\,x} \tag{6.10.1}$$

The tables show that the closest transform is

$$L_T\left[\operatorname{erfc}\frac{a}{2\sqrt{t}}\right] = \frac{1}{s} e^{-a\sqrt{s}} \tag{6.10.2}$$

Comparing the last two equations, we see that

$$-\sqrt{s}\,x = -\frac{a}{2}\sqrt{s} \quad \rightarrow \quad a = 2x$$

The solution is therefore

$$u(x,t) = \operatorname{erfc}\left(\frac{x}{\sqrt{t}}\right)$$