Section 4.5: Note on Interplay Between Boundary Conditions and the Partial Differential Equation

This section focuses on the interplay between time-dependent boundary conditions and the partial differential equation. The previous section shows that the transformation that changes non-homogeneous boundary conditions into homogeneous ones can change the partial differential equation into a non-homogeneous one.

To make some general observations consider the following partial differential equation and single boundary condition.

$$\text{PDE:}\quad u_t(x,t) = k u_{xx}(x,t)$$

$$\text{BC:}\quad u(0,t) = f(t)$$

Defining the function

$$v(x,t) = u(x,t) - f(t)$$

gives the new boundary value problem

$$\text{PDE2:}\quad v_t(x,t) = k\, v_{xx}(x,t) + f_t(t)$$

$$\text{BC2:}\quad v(0,t) = 0$$

Now the partial differential equation has a forcing function.

What about a boundary value problem that has a boundary condition with a derivative. Consider

$$\text{PDE3:}\quad u_t(x,t) = k u_{xx}(x,t)$$

$$\text{BC3:}\quad u_x(x,t) = g(t)$$

We might define a new function

$$v(x,t) = u(x,t) - \int^{x} dx\, g(t) = u(x,t) - x\, g(t)$$

Then the boundary value problem becomes

$$\text{PDE4:}\quad v_t(x,t) - k v_{xx}(x,t) = -x\, g_t(t)$$

$$\text{BC4:}\quad v_x(0,t) = 0$$

The above examples make it clear that non-homogeneous boundary conditions can be transformed to homogeneous ones, but in general, the price to be paid is that a forcing function appears in the PDE. Incidently, it is also possible to transform a non-homogeneous PDE into a homogeneous one at the expense of making non-homogeneous boundary conditions.

The reader should recognize that the first step in solving boundary value problems is to transform nonhomogeneous BCs into homogeneous ones and only then attempt to solve the problem. This is because standard techniques are available for solving non-homogeneous PDEs.