Section 2.5: D'alembert's Solution for $u(x,0) \neq 0$ and $u_t(x,0) = 0$

D'alembert's solution to the initial value problem in Equations 2.4.15 to 2.4.16

PDE: $u_{tt} - v^2 u_{xx} = 0$ (2.4.15)
IC1: $u(x,0) = f(x)$ (2.4.16)
IC2: $u_t(x,0) = g(x) = 0$ (2.4.17)

is given by Equation 2.4.25

$$u(x,t) = \frac{1}{2} f(\xi) + \frac{1}{2} f(\eta) + \frac{1}{2v} \int_{\xi}^{\eta} dz\, g(z) \qquad (2.4.25)$$

where $\xi = x - vt$ and $\eta = x + vt$. Substituting g(z)=0 gives

$$u(x,t) = \frac{1}{2} f(\xi) + \frac{1}{2} f(\eta) = \frac{1}{2} f(x - vt) + \frac{1}{2} f(x + vt)$$

which shows that the energy initially on the x-axis divides into two portions with one propagating along the ξ line and the other along the η line. For example, energy initially at x=0 follows the two characteristic curves defined by $\xi = 0$ and $\eta = 0$.

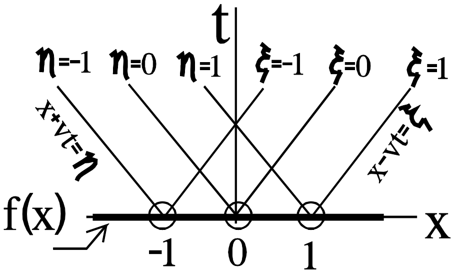

Figure 2.5.1: Energy initially on the x-axis divides into two portions which propagate in opposite directions.

As an important note, so far the function u has been described as "energy", but it might be a displacement (for a vibrating string, for example), or some other quantity. The notion of "energy flow" is only being used to develope an intuitive feeling for the characteristic curves.

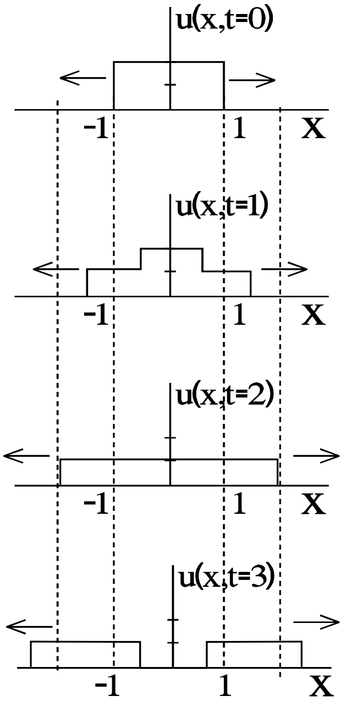

Figure 2.5.2: The square wave splits into two parts that move in opposite directions.

The sequence of graphs in Figure 2.5.2 shows how a wave (initially stationary) divides and flows along the x-axis. The speed of each half-wave is roughly ½ meters per second. The figure can be compared with Figure 2.5.1. Look at the top graph and note the edge at x=1. Half of the energy at x=1 moves to the right following the line $x = vt + 1$ while the other half moves to the left according to $x = -vt + 1$. Therefore the t=0 right-hand edge follows the two characteristic curves defined by $\xi = 2 = \eta$.

Now that the energy is understood to divide into two pieces, there is another question. How does the energy flow from two directions to reach the point x,t and thereby generate a solution $u(x,t)$? How should the coordinates ξ,η be interpreted in such a case.

As shown in Figure 2.5.2, the only energy u(x,t) that reaches the point x at time t is that which started on the x-axis at points ξ and η and flowed along the characteristics

$$\xi = x - vt \qquad \text{and} \qquad \eta = x + vt$$

respectively. The energy initially at point η divided into two halves with one part flowing along $\eta = x + vt$. Similarly, half of the energy at point ξ flows along the characteristic $\xi = x - vt$ to reach the point x,t. The amount of energy flowing along $\eta = x + vt$ is

$$\frac{1}{2}f(\eta) = \frac{1}{2}f(x+vt)$$

and the amount flowing along the $\xi = x - vt$ is

$$\frac{1}{2}f(\xi) = \frac{1}{2}f(x-vt)$$

Therefore the total energy reaching point x,t is

$$u(x,t) = \frac{1}{2}f(x-vt) + \frac{1}{2}f(x+vt)$$

as given by D'alembert's solution when g(x)=0.

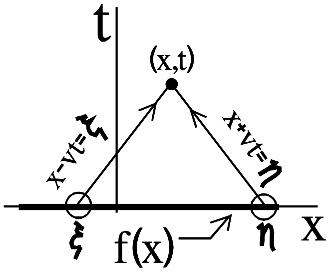

Figure 2.5.3: The energy reaching point (x,t) comes from two separate locations on the x-axis. Note that $\xi<\eta$.

Example 2.5.1: Solve the following initial value problem

PDE: $u_{tt} = 9u_{xx}$
IC1: $u(x,0) = \sin x$
IC2: $u_t(x,0) = 0$

D'alembert's solution yields

$$u(x,t) = \frac{1}{2}f(\xi) + \frac{1}{2}f(\eta) = \frac{1}{2}\sin\xi + \frac{1}{2}\sin\eta = \frac{1}{2}\sin(x-vt) + \frac{1}{2}\sin(x+vt)$$

where v=3 in this case.

All of the solutions for g=0 work just as easily. The next topic considers the case of $u(x,0) = f(x) = 0$ but $0 \neq u_t(x,0) = g(x)$. The solution is simple, but requires some explanation regarding the limits on the integral.

Section 2.6: D'alembert's Solution for $u(x,0)=0$ and $u_t(x,0)\neq 0$

The previous sections have shown that the general solution to the following initial value problem (consisting of only initial conditions)

$$\text{PDE:}\quad u_{tt}-v^2u_{xx}=0 \qquad (2.4.15)$$

$$\text{IC1:}\quad u(x,0)=f(x) \qquad (2.4.16)$$

$$\text{IC2:}\quad u_t(x,0)=g(x) \qquad (2.4.17)$$

is given by D'alembert's solution

$$u(x,t)=\frac{1}{2}f(\xi)+\frac{1}{2}f(\eta)+\frac{1}{2v}\int_{\xi}^{\eta}dz\; g(z) \qquad (2.4.25)$$

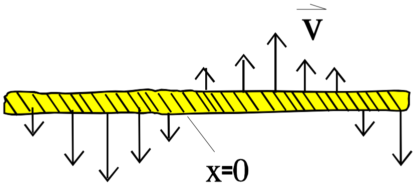

Figure 2.6.1: An infinitely long string initially coincident with the x-axis but with an initial speed that depends on position.

The initial conditions that $u(x,0)=f(x)=0$ and $u_t(x,0)=g(x)\neq 0$ is probably best exemplified by an infinitely long string that is initially stretched taught along the x-axis (no displacement) but each small section of string has an initial speed as shown in Figure 2.6.1. The initial displacement of the string from equilibrium is $u(x,0)=f(x)=0$. However, its initial speed is shown to be $u_t(x,0)=g(x)=\sin x$. Some parts of the string are shown to be initially moving upward while other parts are moving downward.

The best way to understand the integral term in D'alembert's solution (with f=0)

$$u(x,t)=\frac{1}{2v}\int_{\xi=x-vt}^{\eta=x+vt}dz\; g(z) \qquad (2.6.1)$$

is by looking at some examples.

Example 2.6.1: Sove the following initial value problem

$$\text{PDE:}\quad u_{tt}=v^2u_{xx}$$

$$\text{IC1:}\quad u(x,0)=f(x)=0$$

$$\text{IC2:}\quad u_t(x,0)=g(x)=1 \quad \forall x\in(-\infty,\infty)$$

Note that the entire string is moving along away from the x-axis as shown in Figure 2.6.2. The solution has the form

$$u(x,t)=\frac{1}{2v}\int_{\xi}^{\eta}dz\; g(z)$$

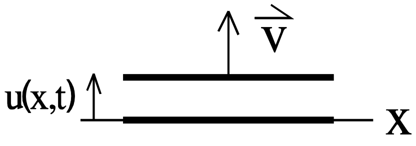

Figure 2.6.2: The whole string moves away from the x-axis for Example 2.6.1.

Substituting the initial condition into the integral gives

$$u(x,t)=\frac{1}{2v}\int_{\xi}^{\eta}dz\; g(z)=\frac{1}{2v}\int_{\xi}^{\eta}dz=\frac{\eta-\xi}{2v}=\frac{(x+vt)-(x-vt)}{2v}=t \qquad (2.6.2)$$

The Solution 2.6.2 agrees with Figure 2.6.2 since the displacement of the string u=t. The whole string moves away from the x-axis which is consistent with IC2. Notice, as exemplified here, that "u" is not necessarily energy density. In this example, u is displacement and u(x,0)=0 does not mean that the initial energy is zero.

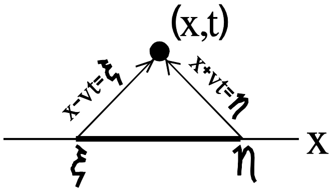

Figure 2.6.3: The cone of integration.

To continue with Example 2.6.1, look at the limits on the integral. The range of the integral covers only those points on the x-axis for which g(x) is non-zero and those points between ξ,η. Notice the order of ξ,η in Figure 2.6.3. The region between ξ,η, which defines the base of the cone of integration, is sometimes referred to a region of disturbance when g(x) is non-zero there. The following example indicates how the integration changes when g(x) is non-zero only within a limited region.

Example 2.6.2: Solve the following initial value problem

PDE: $u_{tt} = v^2 u_{xx}$ with v=1

IC1: $u(x,0)=f(x)=0$

IC2: $u_t(x,0) = g(x) = \begin{cases} 1 & x \in [-1,1] \\ 0 & \text{elsewhere} \end{cases}$

The general solution is

$$u(x,t) = \frac{1}{2} f(\xi) + \frac{1}{2} f(\eta) + \frac{1}{2v}\int_{\xi}^{\eta} dz\, g(z) = \frac{1}{2}\int_{\xi=x-t}^{\eta=x+t} dz\, g(z)$$

where IC1 is used and v=1 is inserted into the limits of the integral. The second initial condition divides the x-axis into three regions. The question then arises as to how the integral should be evaluated. That is, what is the relation among the points ξ,η,-1,1,x ? The procedure to find u(x,t) is most easily understood using the "cone of integration" and dividing the x-t plane into 6 regions. The procedure will become clear as the example progresses. The end points of the region of the disturbance each produce two characteristic curves.

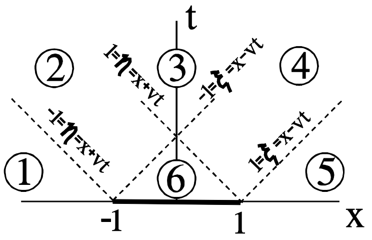

Figure 2.6.4: The three regions for the function g(x) produces six regions in the x-t plane by using the characteristic curves.

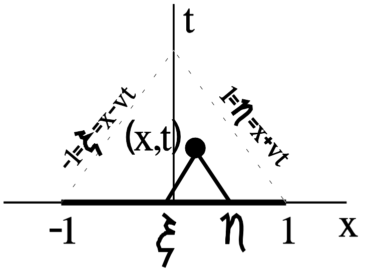

Figure 2.6.5: The cone at (x,t) in region 6 includes only points where g(x) is nonzero.

Examine each region separately.

Region 6: First consider region 6. Find the solution at all points (x,t) in region #6 as shown in Figure 2.6.4. A zoom view of region #6 appears in Figure 2.6.5. Draw a cone using the point (x,t) as the vertex. The sides of the cone are given by the characteristic equations.

$$\xi = x - t \qquad \eta = x + t$$

The points in the range (ξ,η) have a disturbance that contribute to the solution u(x,t). The integral is over the range (ξ,η) because $-1 \le \xi \le \eta \le 1$ and the function g(x) is therefore non-zero over (ξ,η). The integral becomes

$$u(x,t) = \frac{1}{2}\int_{\xi=x-t}^{\eta=x+t} dz\, g(z) = \frac{1}{2}\int_{\xi=x-t}^{\eta=x+t} dz = \frac{\eta-\xi}{2} = \frac{(x+t)-(x-t)}{2} = t$$

It's nice to say u(x,t)=t for region 6, but what values of x,t are encompassed by region 6? The values x,t can be determined by working with the characteristic equations which bound region 6. The points x,t to the right of the ξ line in Figure 2.6.5 are defined by

$$\xi > 1 \quad \text{which gives} \quad x - t > 1$$

The region to the left of the η line is defined by

$$\eta < 1 \quad \text{or equivalently} \quad x + t < 1$$

Combining the previous two conditions for region 6 gives a condition on the range of x

$$t - 1 \le x \le 1 - t$$

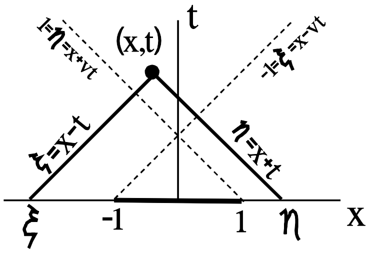

Figure 2.6.6: The point (x,t) is in region #3. Notice that the base of the cone (with vertex at x,t) contains points on the x-axis where g(x) is zero.

Region 3: Figure 2.6.6 shows a zoom view of region 3. The cone with vertex (x,t) has sides defined by the characteristics

$$\xi = x - t \quad \text{and} \quad \eta = x + t$$

The interval (ξ,η) covers more of the x-axis than just the interval [-1,1]. The solution to the initial value problem is

$$u(x,t) = \frac{1}{2} f(\xi) + \frac{1}{2} f(\eta) + \frac{1}{2v}\int_{\xi}^{\eta} dz\, g(z) = \frac{1}{2}\int_{\xi}^{\eta} dz\, g(z) = \frac{1}{2}\int_{-1}^{1} dz\, 1 = 1$$

since g=0 in [ξ,-1) and (1,η].

Next, determine the values of x,t that define region #3. For region 3,

$$\xi < -1 \qquad \text{and} \qquad \eta > 1$$

which means that

$$x - t < -1 \qquad \text{and} \qquad x + t > 1$$

so

$$1 - t < x < t - 1$$

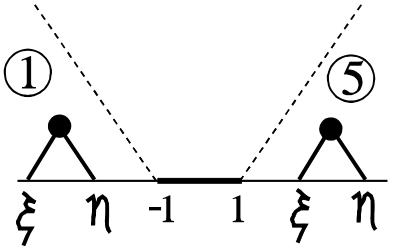

Figure 2.6.7: The cones for regions 1 and 5 do not intersect any of the disturbance at t=0.

Regions 1 and 5: In regions 1 and 5, a cone from the point (x,t) does not intersect any of the disturbance that exists in the interval [-1,1]. The intergral gives

$$u(x,t) = \frac{1}{2} f(\xi) + \frac{1}{2} f(\eta) + \frac{1}{2v}\int_{\xi}^{\eta} dz\, g(z) = \frac{1}{2}\int_{\xi}^{\eta} dz\, 0 = 0$$

Regions 1 and 5 are defined by

$$\eta < -1 \;\rightarrow\; x < -1 - t \qquad \text{and} \qquad \xi > 1 \;\rightarrow\; x > 1 + t$$

respectively.

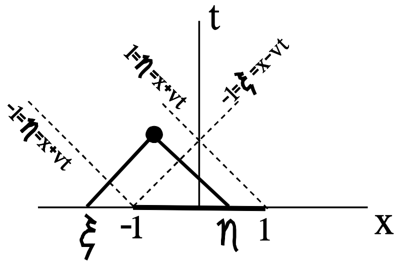

Figure 2.6.8: Region 2.

Region 2: The cone emanating from the point (x,t) intersects the disturbance in the range [-1,η] . The solution is therefore

$$u(x,t) = \frac{1}{2v}\int_{\xi}^{\eta} g(z)\;\; dz = \frac{1}{2}\int_{-1}^{\eta} dz\;\; 1 = \left.\frac{z}{2}\right|_{-1}^{\eta} = \frac{\eta + 1}{2} = \frac{x + t + 1}{2}$$

Region 2 is defined by

$$-1 < \eta < 1 \qquad \text{and} \qquad \xi < -1$$

which gives

$$-1 < x + t < 1 \quad \text{and} \qquad x - t < -1$$

Region 4: This region is very similar to region 2. The solution is

$$u(x,t) = \frac{1}{2v}\int_{\xi}^{\eta} g(z)\,dz = \frac{1}{2}\int_{\xi}^{1} dz\,1 = \frac{1-(x-t)}{2}$$

Region 4 is defined by

$$-1 < \xi < 1 \qquad \text{and} \qquad \eta > 1$$

which gives

$$-1 < x - t < +1 \quad \text{and} \quad x + t > 1$$

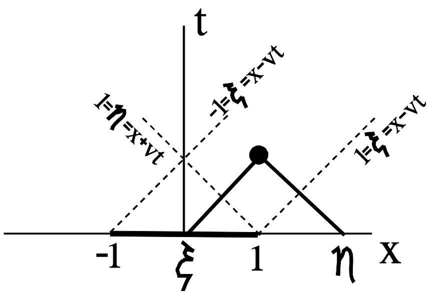

Figure 2.6.9: Region 4.

The full solution: The full solution is given by

$$u(x,t) = \begin{cases} 0 & \text{Region 1} & x < 1-t \\ (x+t+1)/2 & \text{Region 2} & -1-t < x < 1-t \;\; x < -1+t \\ 1 & \text{Region 3} & 1-t < x < t+1 \\ (1-x+t)/2 & \text{Region 4} & -1+t < x < 1+t \;\; x > 1-t \\ 0 & \text{Region 5} & 1+t < x \\ t & \text{Region 6} & x > 1+t \end{cases}$$

The initial value problems for which $u(x,0) \neq 0$ *and* $u_t(x,0) \neq 0$ are solved by combining the techniques given in this section and the last one. As a note, if $u_t(x,0) = g(x) \neq 0$ in two subintervals (a,b) and (c,d) then there are 14 regions to consider for the integral as indicated in Figure 2.6.10. On the other hand, based on this method of subdividing the x-t plane, there are only three regions if $u_t(x,0) = g(x) \neq 0$ at precisely one point.

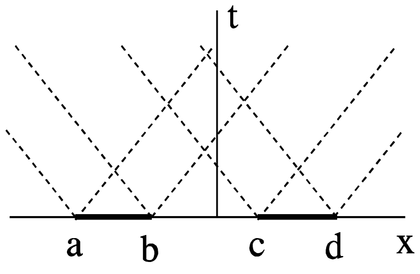

Figure 2.6.10: The 14 regions.