Chapter 4 Review Exercises

4.1 Consider a solid rigid mass M rotating at angular speed $\dot{\theta}$ (in radians) about an origin fixed in space. Show the kinetic energy can be written as $T = I\dot{\theta}^2/2$ where $I = \int dm\, r^2$ and r is the distance to the mass dm from the origin. Start with $dT = (dm)\, v^2/2$.

4.2 Consider a system of N non-interacting point particles. Particle $\# i$ has mass m_i and vector $\vec{r}_n$ pointing from the origin to the particle. The center of mass can be written as $\vec{R} = \sum_{i=1}^{N} \vec{r}_i \frac{m_i}{M}$ where $M = \sum_{i=1}^{N} m_i$. Show the momentum of the system of N particles can be written as $\vec{P} = M\dot{\vec{R}}$. Further show the total externally applied force $F = \sum_{i=1}^{N} F_i$ accelerates the center of mass according to $\vec{F} = M\ddot{\vec{R}}$.

4.3 The torque is defined by $\vec{\tau} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i$ where Problem 4.2 defines the symbols. The angular momentum is defined by $\vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i$. Show $\vec{\tau} = \frac{d\vec{L}}{dt}$.

4.4 Define the center of mass as in Problem 4.2. Suppose the vector $\vec{r}_i$ represents the position of mass m_i in some arbitrary but fixed coordinate system while $\vec{r}_i\,'$ represents the position with respect to the origin of the center of mass system (i.e., place a coordinate system at point $\vec{R}$ defined in Problem 4.2). The vectors can be related by the relation $\vec{r}_i = R + \vec{r}_i'$. Show the total angular momentum can be written as

$$\vec{L} = \vec{R} \times \vec{P} + \sum_{i=1}^{N} \vec{r}_i' \times \vec{p}_i'$$

where $\vec{p}_i'$ is the momentum with respect to the center of mass coordinates.

4.5 Use the definitions in the previous problems to show the kinetic energy T of a solid body can be expressed as the sum of the kinetic energy of the center of mass and the motion about the center of mass.

$$T = \frac{1}{2} M\dot{\vec{R}}^2 + \frac{1}{2} I\dot{\theta}^2$$

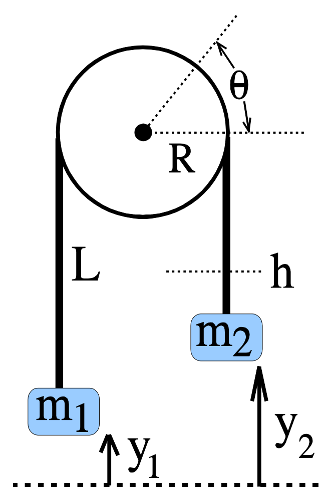

Figure P4.6: Pulley system

4.6 Assume the pulley has mass M and radius R and that it supports two masses as in Figure P4.6. Use the results of Problem 4.1.

a. Find moment of inertia I for the pulley with uniform mass distribution.

b. Write the total kinetic and potential energy for the system in terms of θ and $\dot{\theta}$

c. Use the Lagrangian to find the equation of motion and solve it.

4.7 Find the Equations of motion for the pulley system in Figure P4.6 for the case of a stretchable string with spring constant k. Assume the equilibrium length of the string is L (without masses attached), the string can be both compressed and stretched (obeys Hook's law), and the pulley is massless. Further assume that (without masses) $y_2 = h$ when $y_1 = 0$. Decouple and solve the equations of motion by using the new coordinates $y_+ = y_1 + y_2$ and $y_- = y_1 - y_2$.

4.8 Consider a cylinder of mass M, length L, and radius R constrained to roll down a plane as shown in Figure P4.8. Find the equation of motion and solve it.

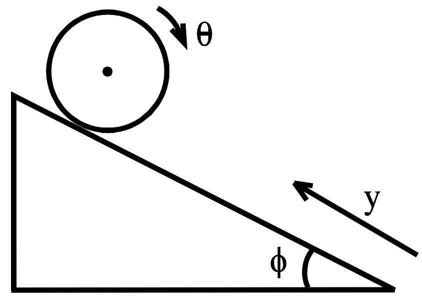

Figure P4.8: A cylinder rolling down the plane.

4.9 Consider a mass m connected to a spring with spring constant k. Assume the equilibrium position of the mass is at x=0.
a. Write the Hamiltonian for the system.
b. Use Hamilton's canonical equations to find an expression for $\dot{x}$ and $\dot{p}$.
c. Use the results of Part 4.9b to write an equation for position x alone and solve it.

4.10 Find the Hamiltonian for Problem 4.7 and then write expressions for $\dot{y}_1, \dot{y}_2, \dot{p}_1, \dot{p}_2$. You can start from the basic definition $H = \sum_i p_i \dot{q}_i - L$.

4.11 Find the Hamiltonian for Problem 4.8 and then use Hamilton's canonical relations.

4.12 Use the Poisson brackets to demonstrate the following relations

$$[A,A]=0 \qquad [A,B]=-[B,A] \qquad [A,c]=0$$
$$[A+B,C]=[A,C]+[B,C] \qquad [A,BC]=[A,B]C+B[A,C]$$

4.13 Use the Poisson brackets to show

$$\left[q_i, q_j\right] = 0 \quad \left[p_i, p_j\right] = 0 \qquad \left[q_i, p_j\right] = \delta_{ij}$$

where p_j is the momentum conjugate to q_j.

4.14 Suppose an electromagnetic field interacts with charged particle at $\vec{r}_i = \tilde{x}x_i + \tilde{y}y_i + \tilde{z}z_i$ through the vector potential $\vec{A}(\vec{r}_i)$ and electrostatic potential $\phi(\vec{r}_i)$, where $\tilde{x}, \tilde{y}, \tilde{z}$ represent unit vectors. The Lagrangian has the form

$$L = \sum_i \left\{ \frac{1}{2} m_i \dot{r}_i^2 - q_i \phi(\vec{r}_i) + \frac{q_i}{c} \vec{A}(\vec{r}_i) \cdot \dot{\vec{r}}_i \right\}$$

Find the canonical momentum p_{ix}. Explain why two terms appear in the result and what they physically mean.

4.15 Explain why a the following relation must hold for δx_i independent

$$\sum_{i=1}^{N} f(x_i)\delta x_i = 0 \quad \rightarrow f(x_i) = 0$$

This is similar to a step in the procedure to derive Lagranges Equation. Hint: Consider a matrix solution. Keep in mind that δx_1, for example, can have any number of values such as 0.1, 0.001 etc.

4.16 Assume periodic boundary conditions. Show how

$$0 = \delta I = \int_{t_1}^{t_2}\int_{\vec{r}_1}^{\vec{r}_2} dt\, d^3x \left[\frac{\partial \mathscr{L}}{\partial \eta}\delta\eta + \frac{\partial \mathscr{L}}{\partial \dot{\eta}}\frac{\partial}{\partial t}\delta\eta + \frac{\partial \mathscr{L}}{\partial(\partial_i \eta)}\partial_i \delta\eta \right]$$

leads to

$$\int_{t_1}^{t_2}\int_{\vec{r}_1}^{\vec{r}_2} dt\, d^3x \left[\frac{\partial \mathscr{L}}{\partial \eta} - \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \dot{\eta}} - \partial_i \frac{\partial \mathscr{L}}{\partial(\partial_i \eta)} \right]\delta\eta \quad = 0$$

Explain and show any necessary conditions of the limits of the spatial integral. Remark, according to the Einstein summation convention, repeated indices must be summed $i = 1,2,3$.

4.17 Suppose the Lagrange density has the form $\mathscr{L} = \frac{\rho}{2}\dot{\eta}^2 + \frac{\beta}{2}\left[(\partial_x \eta)^2 + (\partial_y \eta)^2\right]$ for 1-D motion, where ρ, β resemble the mass density and spring constant (Young's modulus) for the material, and $\eta = \eta(x, y, t)$. Find the equation of motion for η.

4.18 If $\mathscr{L} = \frac{\rho}{2}\dot{\eta}^2 + \frac{\beta}{2}(\nabla\eta)^2$ where $(\nabla\eta)^2 = \nabla\eta \cdot \nabla\eta$ and $\eta = \eta(x, y, z)$ then find the equation of motion for η.

4.19 Starting with $\mathscr{L} = i\hbar\psi * \dot{\psi} - \frac{\hbar^2}{2m}\nabla\psi * \cdot \nabla\psi - V(r)\,\psi * \psi$, show the alternate form of the Lagrange density by partial integration.

$$\mathscr{L} = i\hbar\psi^*\dot{\psi} + \frac{\hbar^2}{2m}\psi^*\,\nabla^2\psi - V(r)\,\psi^*\,\psi = \psi^*\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 - V \right)\psi$$

4.20 Show Hamiltonian

$$\mathscr{H} = \pi\dot{\psi} - \mathscr{L} = \psi * \left(-\frac{\hbar^2}{2m}\nabla^2 + V \right)\psi$$

based on the Lagrange density

$$\mathscr{L} = \psi^*\left(i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 - V \right)\psi$$