TEACHING
GENERALIZED PYTHAGORAS THEOREM

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ABSTRACT. Pythagoras theorem and its generalizations are given. The theorem is discussed in terms of matrices.

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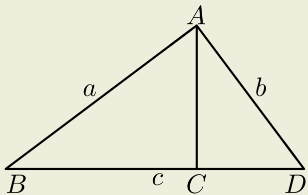
1. Introduction

The discussion here shows that starting from the simple Pythagoras theorem we can quickly proceed to the more sophisticated parts of mathematics. Just like analytic geometry, the theory of compound matrices [1] can also be used as a tool to study geometrical objects. Compound matrices investigate the properties of matrices even deeper than the theory of canonical matrices.

2. Pythagoras Theorem

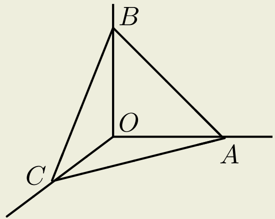
There are dozens of proofs for the Pythagoras theorem, the following is a simple one, with virtually no geometrical constructions.



Proof: Similar triangles BAC, CAD, and BAD have the length of their diagonals a, b, and c respectively, and the area of the triangle BAD is the sum of the areas of the triangles CAD and BAC. Since the area increases proportional to the square of the length, it follows that $a^2 + b^2 = c^2$, which is the Pythagoras theorem.

3. 3D Pythagoras Theorem

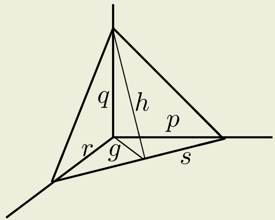
In the figure, take the areas of the right-angled triangles OAB, OBC, OCA, and the *diagonal triangle* ABC as a, b, c, and d, respectively.



3D Pythagoras Theorem: $a^2 + b^2 + c^2 = d^2$.

The proof is considered next.

4. Proof



From the figure we can write,
$h^2 = g^2 + q^2,\ r/g = s/p,$ or $g = pr/s,$ and $s^2 = p^2 + r^2.$
Thus, we have
$4d^2 = h^2 s^2 = (g^2 + q^2)s^2 = [(p^2 r^2/s^2) + q^2]s^2$
$= p^2 r^2 + q^2(p^2 + r^2) = p^2 q^2 + p^2 r^2 + q^2 r^2,$
or
$a^2 + b^2 + c^2 = d^2,$
which is the 3D Pythagoras theorem.

5. Some Facts

To discuss 4D Pythagoras theorem, clearly we cannot make use of pictures. So we switch to matrices. We claim that the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

represents a paralellogram in a three-dimensional space and, specifically, we have

$$\begin{bmatrix} p & 0 & r \\ 0 & q & r \end{bmatrix}$$

representing a parallelogram with twice the area of the diagonal triangle considered in the 3D Pythagoras theorem. The *components* of this parallelogram are given by the 2^{nd} compound of the above matrix (majors arranged in the lexical order):

$$\begin{bmatrix} pq & pr & -qr \end{bmatrix}.$$

The square of the length of this vector $4(a^2+b^2+c^2)$, gives $4d^2$, the square of the area of the parallelogram.

6. 4D Pythagoras Theorem

Condidering the 3^{rd} compound of the matrix

$$\begin{bmatrix} p & 0 & 0 & s \\ 0 & q & 0 & s \\ 0 & 0 & r & s \end{bmatrix}$$

we get the 4D pythagoras theorem $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 = g^2$. The statement of the nD Pythagoras theorem should be obvious.

We have mentioned compound matrices only in passing, but there is a deep theory behind them. For a statement of some theorems in this area by mathematical giants like Laplace, Cauchy, and Jacobi, see [2].

7. Conclusion

Anybody objecting to the notion of an nD Pythagoras theorem should be reminded of the following two formulas.
The number of k-cubes contained in an n-cube is

$$N(k) = \binom{n}{k} 2^{(n-k)}$$

and the hypervolume of an n-dimensional sphere of radius r is

$$V(n,r) = \frac{\pi^{n/2} r^n}{\Gamma((n+2)/2)}$$

where Γ is the gamma function.

It is imporatant to recognize that while physics investigates the physical universe, mathematics talks about the possible universe.

1. A. C. Aitken, *Determinants and Matrices*, Oliver and Boyd, London, 1951.
2. K. K. Nambiar and Shanti Sreevalsan, *Compound Matrices and Three Celebrated Theorems*, Mathematical and Computer Modelling **34** (2001), no. 3-4, 251–255. http://www.rci.rutgers.edu/~kannan/science/three_theorems_screen.pdf.

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