$$\begin{bmatrix}1 & -0.5\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.5 & 0 & 0\\ 0 & -0.5 & -0.5\end{bmatrix}=\begin{bmatrix}0.5 & 0.25 & 0.25\\ 0 & -0.5 & -0.5\end{bmatrix}$$

$$N(s)=\left[\begin{array}{cc:c}1 & 0.5 & 2.5\\ 1 & 2.5 & 2.5\end{array}\right]$$

Thus a minimal realization is

$$\dot{x}=\left[\begin{array}{cc:c}-0.5 & -0.25 & -0.25\\ 1 & 0 & 0\\ \hdashline 0 & 0.5 & 0.5\end{array}\right]x+\left[\begin{array}{cc}1 & -0.5\\ 0 & 0\\ \hdashline 0 & 1\end{array}\right]u$$

$$y=\left[\begin{array}{cc:c}1 & 0.5 & 2.5\\ 1 & 2.5 & 2.5\end{array}\right]x$$

Chapter 8

8.1 $\dot{x}=\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}x+\begin{bmatrix}1\\ 2\end{bmatrix}u$

$$u=r-[k_1\ \ k_2]\,x$$

$$\dot{x}=\begin{bmatrix}2-k_1 & 1-k_2\\ -1-2k_1 & 1-2k_2\end{bmatrix}x+\begin{bmatrix}1\\ 2\end{bmatrix}r$$

$$\det\begin{bmatrix}s-2+k_1 & -1+k_2\\ 1+2k_1 & s-1+2k_2\end{bmatrix}=s^2+(k_1+2k_2-3)s+k_1-5k_2+3$$

$$\Delta_f(s)=(s+1)(s+2)=s^2+3s+2$$

$$\therefore\ k_1+2k_2-3=3 \Rightarrow k_1+2k_2=6$$

$$k_1-5k_2+3=2 \Rightarrow k_1-5k_2=-1$$

Solving these yields $k_2=1,\ k_1=4$.

8.2 $\Delta(s)=\det\begin{bmatrix}s-2 & -1\\ 1 & s-1\end{bmatrix}=(s-1)(s-2)+1$

$$=s^2-3s+3$$

$$\bar{k}=[3+3\ \ \ 2-3]=[6\ \ -1]$$

$$\bar{C}^{-1}=\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix},\quad \bar{C}=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

$$C=[b\ \ Ab]=\begin{bmatrix}1 & 4\\ 2 & 1\end{bmatrix},\quad C^{-1}=\frac{1}{-7}\begin{bmatrix}1 & -4\\ -2 & 1\end{bmatrix}$$

$$k=\bar{k}\bar{C}C^{-1}=[6\ \ -1]\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{-1}{7} & \frac{4}{7}\\ \frac{2}{7} & \frac{-1}{7}\end{bmatrix}$$

$$=[6\ \ 17]\begin{bmatrix}\frac{-1}{7} & \frac{4}{7}\\ \frac{2}{7} & \frac{-1}{7}\end{bmatrix}=\left[\frac{28}{7}\ \ \frac{7}{7}\right]=[4\ \ 1].$$

8.3 Select $F=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}$, $\bar{k}=[1\ \ 1]$

Solve $AT-TF=b\bar{k}$ or

$$\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}-\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$$

$$\begin{bmatrix}3t_{11}+t_{21} & 4t_{12}+t_{22}\\ -t_{11}+2t_{21} & -t_{12}+3t_{22}\end{bmatrix}=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$$

From the four equations, we can solve

$$T=\begin{bmatrix}0 & \frac{1}{13}\\ 1 & \frac{9}{13}\end{bmatrix},\quad T^{-1}=\begin{bmatrix}-9 & 1\\ 13 & 0\end{bmatrix}$$

$$k=\bar{k}T^{-1}=[1\ \ 1]\begin{bmatrix}-9 & 1\\ 13 & 0\end{bmatrix}=[4\ \ 1].$$

8.4 $\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$

We use (8.13) to compute feedback gain k. We compute

$$\Delta(s) = (s-1)^3 = s^3 - 3s^2 + 3s - 1$$

$$\Delta_f(s) = (s+2)(s+1+j)(s+1-j) = s^3 + 4s^2 + 6s + 4$$

$$\bar{k} = [4-(-3) \quad 6-3 \quad 4-(-1)] = [7 \quad 3 \quad 5]$$

$$\bar{C}^{-1} = \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -3 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

$$k = \bar{k}\bar{C}C^{-1} = [15 \quad 47 \quad -8].$$

8.5 If we place the eigenvalues of the state feedback system at $-2, -2, -3$, then the resulting system has transfer function

$$\hat{g}_f(s) = \frac{(s-1)(s+2)}{(s+2)^2(s+3)} = \frac{s-1}{(s+2)(s+3)}$$

The system is BIBO stable because $\hat{g}_f(s)$ has poles -2 and -3; it is asymptotically stable because the eigenvalues are $-2, -2$, and -3.

8.6 If we place the eigenvalues of the state feedback system at $1, -2$ and -3, then the resulting system has transfer function

$$\hat{g}_f(s) = \frac{(s-1)(s+2)}{(s-1)(s+2)(s+3)} = \frac{1}{s+3}$$

The system is BIBO stable. It is not asymptotically stable because the system has eigenvalue 1.

8.7 $\dot{x} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$

$$y = [2 \quad 0 \quad 0]x$$

We compute

$$(sI-A)^{-1} = \begin{bmatrix} s-1 & -1 & 2 \\ 0 & s-1 & -1 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} & \frac{-2s+3}{(s-1)^3} \\ 0 & \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$$

Thus the transfer function is

$$\hat{g}(s) = \frac{2}{s-1} + \frac{(-2s+3)\cdot 2}{(s-1)^3} = \frac{2s^2 - 8s + 8}{(s-1)^3}$$

Now we introduce

$$u = pr - kx$$

with $k = [15 \; 47 \; -8]$ as computed in Problem 8.4. Then the transfer function from r to y is

$$\hat{g}_f(s) = p \cdot \frac{2s^2 - 8s + 8}{s^3 + 4s^2 + 6s + 4}$$

In order to track any step reference input, we require

$$\hat{g}_f(0) = 1 \quad \text{or} \quad p \cdot \frac{8}{4} = 1 \Rightarrow p = \frac{4}{8} = 0.5$$

This completes the design.

8.8 $x[k+1] = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$

$$y[k] = [2 \quad 0 \quad 0]x[k]$$

$$\Delta(z) = \det(zI - A) = (z-1)^3 = z^3 - 3z^2 + 3z - 1$$

We use the procedure used in Problem 8.4:

$$\Delta_f(z) = z^3 = z^3 + 0 \cdot z^2 + 0 \cdot z + 0$$

$$\bar{k} = [0-(-3) \quad 0-3 \quad 0-(-1)] = [3 \quad -3 \quad 1]$$

The matrices $\bar{C}$ and C^{-1} are the same as those in Problem 8.4. Thus we have

$$k = \bar{k}\bar{C}C^{-1} = [1\ 5\ 2]$$

The state feedback system becomes

$$x[k+1] = (A - bk)x[k] + bu[k]$$

$$= \left(\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} [1\ 5\ 2] \right) x[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$$

$$= \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$$

Its zero-input response is

$$x[k] = A_f^k x[0] := \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix}^k x[0]$$

We compute

$$A_f^2 = \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix}^2 = \begin{bmatrix} 4 & 16 & 0 \\ -1 & -4 & 0 \\ 1 & 4 & 0 \end{bmatrix}$$

$$A_f^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we have, for any $x[0]$,

$$x[k] = 0 \quad \text{for } k \geq 3$$

8.9 Following Problem 8.7, we have

$$\hat{g}(z) = \frac{2z^2 - 8z + 8}{(z-1)^3}$$

Introduce $u = pr[k] - [1\ \ 5\ \ 2]x[k]$ yields the transfer function from r to y as

$$\hat{g}_f(z) = p \cdot \frac{2z^2 - 8z + 8}{z^3}$$

The condition to track any step reference sequence is

$$\hat{g}_f(1) = 1$$

(Theorem 5.D2). Thus we have

$$p \cdot \frac{2 - 8 + 8}{1} = 1 \Rightarrow p = \frac{1}{2}$$

and

$$\hat{g}_f(z) = \frac{0.5(2z^2 - 8z + 8)}{z^3}$$

Let $r[k] = a$, for $k \geq 0$, Then $\hat{r}(z) = \frac{az}{z-1}$

and

$$\hat{y}(z) = \frac{z^2 - 4z + 4}{z^3} \cdot \frac{az}{z-1}$$

which can be expanded as

$$\hat{y}(z) = \frac{az}{z-1} - a - \frac{4a}{z^2}.$$

Thus

$$y[k] = a - a\delta[k] - 4a\delta[k-2]$$

$k = 0 \quad y[0] = a - a = 0$

$k = 1 \quad y[1] = a$

$k = 2 \quad y[2] = a - 4a = -3a$

$k \geq 3 \quad y[k] = a = r[k]$

8.10 The equation is in Jordan form. It is clear that the Jordan block associated with eigenvalue 2 is controllable the two Jordan blocks associated with -1 are not (Corollary 6.8). Consider the subequation

$$\dot{x}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$P^{-1} = Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$\bar{x}_1 = Px_1$ will transform the equation into

$$\dot{\bar{x}}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \bar{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Thus the transformation

$$\bar{x} = \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ 0 & P \end{bmatrix} x$$

will transform the original equation into

$$\dot{\bar{x}} = \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right] \bar{x} + \left[\begin{array}{c} 0 \\ 1 \\ 1 \\ \hline 0 \end{array}\right] u$$

The 3-dimensional subequation is controllable and the three eigenvalues $\{2, 2, -1\}$ can be assigned to any values. The 1-dimensional subequation is not controllable; therefore, its eigenvalue -1 cannot be changed. Thus the answers to the first three questions are yes, yes and no. Because the uncontrollable eigenvalue -1 is stable the equation is stabilizable.

8.11

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u = Ax + bu \qquad (1)$$

$$y = [1 \quad 1]\, x = cx$$

Its transfer function is

$$\hat{g}(s) = [1 \quad 1] \begin{bmatrix} s-2 & -1 \\ 1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3s-4}{s^2-3s+3}$$

If $u = r - [4 \quad 1]\, x$, then

$$\hat{g}_f(s) = \frac{\hat{y}(s)}{\hat{r}(s)} = \frac{3s-4}{(s+1)(s+2)} = \frac{3s-4}{s^2+3s+2}$$

Two-dimensional state estimator:

$$F = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \quad l = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ (Use modal form with eigenvalues $-2 \pm j2$)

$$\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

or

$$\left[\begin{array}{cc|cc} 4 & -2 & -1 & 0 \\ 2 & 4 & 0 & -1 \\ \hline 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & 3 \end{array}\right] \begin{bmatrix} t_{11} \\ t_{21} \\ t_{12} \\ t_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Its solution can be computed as

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} 0.231 & 0.1986 \\ -0.1372 & -0.0866 \end{bmatrix}$$

We compute

$$T^{-1} = \begin{bmatrix} -12 & -27.5 \\ 19 & 32 \end{bmatrix}, \quad Tb = \begin{bmatrix} 0.6282 \\ -0.3105 \end{bmatrix}$$

Thus the estimator is

$$\dot{z} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} z + \begin{bmatrix} 0.6282 \\ -0.3105 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y \qquad (2)$$

$$\hat{x} = T^{-1} z = \begin{bmatrix} -12 & -27.5 \\ 19 & 32 \end{bmatrix} z$$

One-dimensional state estimator

$F = -3, \; l = 1,$

$T = [t_1 \quad t_2]$

$$[t_1 \quad t_2] \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} - (-3)[t_1 \quad t_2] = 1 \times [1 \quad 1] = [1 \quad 1]$$

$$[5t_1 - t_2 \quad t_1 + t_2] = [1 \quad 1] \Rightarrow t_1 = \frac{5}{21}, \; t_2 = \frac{4}{21}$$

or $T = \left[\frac{5}{21} \quad \frac{4}{21}\right]$

Thus the estimator is

$$\dot{z} = -3z + \frac{13}{21} u + y \qquad (3)$$

$$\hat{x} = \begin{bmatrix} 1 & 1 \\ \frac{5}{21} & \frac{4}{21} \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -4 & 21 \\ 5 & -21 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

8.12 The transfer function from r to y was computed in Problem 8.11 as

$$\hat{g}_f(s) = \frac{3s-4}{s^2+3s+2}$$

Now we apply

$$u = r - [4 \quad 1]\, \hat{x}$$

to the two-dimensional estimator in (2) of Problem 8.11 to yield

$$u = r - [4 \quad 1] \begin{bmatrix} -12 & -27.5 \\ 19 & 32 \end{bmatrix} z$$

$$= r + [29 \quad 78]\, z$$

Substituting this into (1) and (2) of Problem 8.11 yields

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 29 & 78 \\ 58 & 156 \end{bmatrix} z + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

$$\dot{z} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} z + \begin{bmatrix} 18.2166 & 48.9964 \\ -9.0036 & -24.2166 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0.6282 \\ -0.3105 \end{bmatrix} r$$

They can be combined as

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & 1 & 29 & 78 \\ -1 & 1 & 58 & 156 \\ \hline 1 & 1 & 16.2166 & 50.9964 \\ 0 & 0 & -11.0036 & -26.2166 \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0.6282 \\ -0.3105 \end{bmatrix} r$$

$$y = [1 \;\; 1 \; \vdots \; 0 \;\; 0] \begin{bmatrix} x \\ z \end{bmatrix} + 0 \cdot r$$

Its transfer function can be computed, using ss2tf in MATLAB, as

$$\hat{g}_f(s) = \frac{3s^3 + 7.9964s^2 + 7.9730s - 31.9576}{s^4 + 7s^3 + 21.999s^2 + 31.9986s + 16.002}$$

If we round the numbers to the nearest integers, then we have

$$\hat{g}_f(s) = \frac{3s^3 + 8s^2 + 8s - 32}{s^4 + 7s^3 + 22s^2 + 32s + 16} = \frac{(3s-4)(s^2+4s+8)}{(s+1)(s+2)(s^2+4s+8)} = \frac{3s-4}{s^2+3s+2}$$

Next we apply $u = r - [4 \;\; 1]\hat{x}$ to the one-dimensional estimator (3) in Prob. 8.11:

$$u = r - [4 \;\; 1] \begin{bmatrix} -4 & 21 \\ 5 & -21 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = r + 11y - 63z$$

Substituting this into (1) and (3) yields

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} (r + 11y - 63z)$$

$$= \begin{bmatrix} 13 & 12 \\ 21 & 23 \end{bmatrix} x - \begin{bmatrix} 63 \\ 126 \end{bmatrix} z + \begin{bmatrix} 1 \\ 2 \end{bmatrix} r$$

$$\dot{z} = -3z + \frac{13}{21}(r + 11y - 63z) + y$$

$$= \frac{-882}{21} z + \frac{164}{21} [1 \;\; 1] x + \frac{13}{21} r$$

They can be combined as

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 13 & 12 & -63 \\ 21 & 23 & -126 \\ \frac{164}{21} & \frac{164}{21} & \frac{-882}{21} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ \frac{13}{21} \end{bmatrix} r$$

$$y = [1 \;\; 1 \;\; 0] \begin{bmatrix} x \\ z \end{bmatrix}$$

Its transfer function is, using ss2tf in MATLAB,

$$\hat{g}_f(s) = \frac{3s^2 + 5s - 12}{s^3 + 6s^2 + 11s + 6} = \frac{(3s-4)(s+3)}{(s^2+3s+2)(s+3)} = \frac{3s-4}{s^2+3s+2}$$

Thus the use of state estimators will not affect $\hat{g}_f(s)$.

8.13 $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix}$

Desired poles: $-4 \pm j3$, $-5 \pm j4$

Select

$$F = \left[\begin{array}{cc|cc} -4 & 3 & 0 & 0 \\ -3 & -4 & 0 & 0 \\ \hline 0 & 0 & -5 & 4 \\ 0 & 0 & -4 & -5 \end{array}\right] \quad \bar{K}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In MATLAB, typing

```
a=[0 1 0 0;0 0 1 0;-3 1 2 3;2 1 0 0];
b=[0 0;0 0;1 2;0 2];
kb=[1 0 1 0;0 0 0 0];
f=[-4 3 0 0;-3 -4 0 0;0 0 -5 4;0 0 -4 -5];
t=lyap(a,-f,-b*kb)
k=kb*inv(t)
```

yields

$$K = \begin{bmatrix} 62.5 & 147 & 20 & 515.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The same result will be obtained if we use the function "place" or "acker" in MATLB. If kb is replaced by

kb = [1 0 0 0; 0 0 1 0] $\left(\bar{K}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right)$

then

$$K = \begin{bmatrix} -606.2 & -168 & -14.2 & -2 \\ 371.1 & 119.2 & 14.9 & 2.2 \end{bmatrix}.$$