Chapter 6

6.1

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \rho(C) = 3. \quad \text{controllable}$$

$$O = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \quad \rho(O) = 1 \quad \text{not observable.}$$

6.2

$$[B \; AB] = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right],$$ It has full row rank, thus controllable

$$O = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & -2 & 4 \end{bmatrix}, \quad \rho(O) = 3 \quad \text{observable}$$

6.3

$$[AB \; A^2B \cdots A^nB] = A[B \; AB \cdots A^{n-1}B]$$

$$\rho([AB \; A^2B \cdots A^nB]) = \rho([B \; AB \cdots A^{n-1}B])$$

if and only if A is nonsingular.

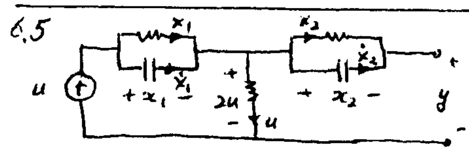
6.4 $\{A, B\}$ controllable $\Leftrightarrow$

$$\text{rank} \begin{bmatrix} A_{11} - sI & A_{12} & B_1 \\ A_{21} & A_{22} - sI & 0 \end{bmatrix} = n$$ for every s (In Theorem 6.1, it is stated for every eigenvalue of A. However, if s is not an eigenvalue, then $(A - sI)$ has rank n. Thus the statement holds for every s.)

$\Leftrightarrow [A_{21} \; A_{22} - sI]$ has full row rank

$\Leftrightarrow \{A_{22}, A_{21}\}$ controllable.

6.5



$$\dot{x}_1 = u - x_1, \quad \dot{x}_2 = -x_2$$

$$y = -x_2 + 2u$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [0 \; -1] x + 2u$$

$$C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \rho(C) = 1 \quad \text{not controllable}$$

$$O = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad \rho(O) = 1 \quad \text{not observable.}$$

6.6 For the state equation in Problem 6.1, we have $\mu = 3$. If the observability index is defined as the least integer such that

$$\rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu-1} \end{bmatrix}\right) = \rho\left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\nu} \end{bmatrix}\right)$$

then $\nu = 1$. (Note that the controllability and observability indices are defined in the text for controllable and observable state equations.)

For the state equation in Problem 6.2, we have $\mu_1 = 2$, $\mu_2 = 1$, $\mu = \max\{\mu_1, \mu_2\} = 2$ and $\nu = 3$.

6.7 $\mu_i = 1$ for all i and $\mu = 1$

6.8

$$\dot{x} = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad y = [1 \; 1] x$$

$$C = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}.$$ We select $P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

Then $P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and

$$PAP^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$$

$$\bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \bar{C} = CP^{-1} = [2 \; 1]$$

Thus $\bar{x} = Px$ will transform the equation to

$$\dot{\bar{x}} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [2 \; 1] \bar{x}$$

and the equation can be reduced to

$$\dot{\bar{x}}_1 = 3\bar{x}_1 + u$$

$$y = 2\bar{x}_1$$

This reduced equation is observable.

6.9 The state equation in Problem 6.5 is already in the form of (6.40), thus it can be reduced to

$$\dot{x}_1 = -x_1 + u$$
$$y = 0 \cdot x_1 + 2u$$

It is not observable, thus it can be further reduced to

$$y = 2u.$$

There is no state variable in the equation

6.10 From Corollary 6.8 or Fig. 6.9, we see that x_3 is not controllable. We rearrange the equation as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_3 \end{bmatrix} = \left[\begin{array}{cccc:c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 1 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \lambda_1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \hdashline 0 \end{bmatrix} u$$

$$y = [\,0 \;\; 1 \;\; 0 \;\; 1 \;\vdots\; 1\,]\,\bar{x}$$

Thus the equation can be reduced as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [\,0 \;\; 1 \;\; 0 \;\; 1\,]\,\tilde{x}$$

Using Corollary 6.8, we conclude that the reduced equation is controllable. Using Corollary 6.08 or Fig. 6.9, we see that x_1 and x_4 are not observable. We rearrange the equation as

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_5 \\ \dot{x}_1 \\ \dot{x}_4 \end{bmatrix} = \left[\begin{array}{cc:cc} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \hdashline 1 & 0 & \lambda_1 & 0 \\ 0 & 1 & 0 & \lambda_2 \end{array}\right] \begin{bmatrix} x_2 \\ x_5 \\ x_1 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \hdashline 0 \\ 0 \end{bmatrix} u$$

$$y = [\,1 \;\; 1 \;\vdots\; 0 \;\; 0\,]\,\hat{x}$$

This is in the form of (6.44) and can be reduced to

$$\begin{bmatrix} \dot{x}_2 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [\,1 \;\; 1\,] \begin{bmatrix} x_2 \\ x_5 \end{bmatrix}$$

This is controllable and observable.

6.11 Select an arbitrary Q_2 such that $[Q_1\ Q_2]$ is nonsingular. Define

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} := [Q_1\ Q_2]^{-1}$$

Then

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} [Q_1\ Q_2] = \begin{bmatrix} P_1Q_1 & P_1Q_2 \\ P_2Q_1 & P_2Q_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I \end{bmatrix}$$

and $P_2Q_1 = 0$. Because Q_1 consists of all linearly independent columns of $[B\ AB\ \cdots\ A^{n-1}B] = 0$, we have

$$P_2B = 0 \quad \text{and} \quad P_2AQ_1 = 0$$

Consider the transformation $\bar{x} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} x$. Then

$$\bar{A} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} A\, [Q_1\ Q_2] = \begin{bmatrix} P_1AQ_1 & P_1AQ_2 \\ P_2AQ_1 & P_2AQ_2 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} B = \begin{bmatrix} P_1B \\ P_2B \end{bmatrix}$$

$$\bar{C} = C\,[Q_1\ Q_2] = [\,CQ_1\ \ CQ_2\,]$$

Because $P_2B = 0$ and $P_2AQ_1 = 0$, the equation is in the form of (6.40) and can be reduced to the controllable

$$\dot{\bar{x}}_1 = P_1AQ_1\bar{x}_1 + P_1Bu$$
$$y = CQ_1\bar{x}_1 + Du$$

6.12 Method 1: We may use elementary row operations to transform Q_1 into

$$PQ_1 = \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix}$$

The first n_1 rows of P yields P_1.

Method 2: Solve n_1 set of linear algebraic equations. The first row, p_1, of P_1 is the solution of

$$p_1 Q_1 = [1\ 0 \cdots 0] \quad (\text{first row of } I_{n_1})$$

The second row, p_2, of P_1, is the solution of $p_2 Q_1 = [0\ 1\ 0 \cdots 0]$ (second row of I_{n_1}) and so forth.

6.13 Consider

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Let $\rho(O) = n_2$ and let P_1 be $n_2 \times n$, consisting of n_2 linearly independent rows of O. Solve Q_1 from $P_1 Q_1 = I_{n_2}$, where Q_1 is $n \times n_2$. Then

$$\dot{\bar{x}}_1 = P_1 A Q_1 \bar{x}_1 + P_1 B u$$
$$y = C Q_1 \bar{x}_1 + Du$$

is zero-state equivalent to the original state equation.

6.14 Because the rows of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ and the rows of $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ are linearly independent, the equation is controllable. To be observable, the three columns of

$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ and the two columns of $\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

must be linearly independent. The three columns are not linearly independent; therefore, the equation is not observable.

6.15 To be controllable, the three rows of $\begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}$ must be linearly independent. This is not possible. To be observable, the three columns of

$$\begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{21} & c_{23} & c_{25} \\ c_{31} & c_{33} & c_{35} \end{bmatrix}$$

must be linearly independent. This can be easily achieved. For example, we may choose it as I_3.

6.16 Consider

$$\dot{\bar{x}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + j\beta_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 - j\beta_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 + j\beta_2 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 - j\beta_2 \end{bmatrix} \bar{x} + \begin{bmatrix} b_1 \\ r_1 + j\eta_1 \\ r_1 - j\eta_1 \\ r_2 + j\eta_2 \\ r_2 - j\eta_2 \end{bmatrix} u$$

$$y = [c_1\ \ p_1 + jq_1\ \ p_1 - jq_1\ \ p_2 + jq_2\ \ p_2 - jq_2]\bar{x}$$

It is controllable $\Leftrightarrow b_1 \neq 0$; $r_i \neq$ or $\eta_i \neq 0$, $i = 1, 2$;
observable $\Leftrightarrow c_1 \neq 0$; $p_i \neq 0$ or $q_i \neq 0$, $i = 1, 2$.
(Corollaries 6.8 and 6.08)

The transformation $\bar{x} = Px$ with

$$P = \begin{bmatrix} 1 & & & & \\ & 1 & 1 & & \\ & j & -j & & \\ & & & 1 & 1 \\ & & & j & -j \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 1 & & & & \\ & 0.5 & -j0.5 & & \\ & 0.5 & j0.5 & & \\ & & & 0.5 & -j0.5 \\ & & & 0.5 & j0.5 \end{bmatrix}$$

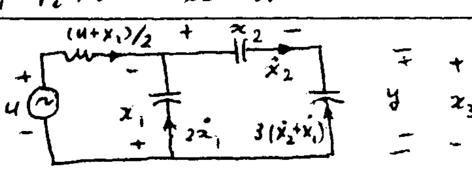
transforms the equation into

$$\dot{x} = \begin{bmatrix} \lambda_1 & & & & \\ & \alpha_1 & \beta_1 & & \\ & -\beta_1 & \alpha_1 & & \\ & & & \alpha_2 & \beta_2 \\ & & & -\beta_2 & \alpha_2 \end{bmatrix} x + \begin{bmatrix} b_1 \\ 2r_1 \\ -2\eta_1 \\ 2r_2 \\ -2\eta_2 \end{bmatrix} u$$

$$y = [c_1 \mid p_1\ \ q_1 \mid p_2\ \ q_2]\, x$$

Thus it is controllable $\Leftrightarrow b_1 \neq 0$; $b_{i1} = 2r_i \neq 0$ or $b_{i2} = -2\eta_i \neq 0$. It is observable $\Leftrightarrow c_1 \neq 0$; $c_{i1} = p_i \neq 0$ or $c_{i2} = q_i \neq 0$.

6.17



$$y = -x_2 - x_1$$
$$\dot{x}_2 = -3\dot{x}_2 - 3\dot{x}_1 \quad \Rightarrow \quad \dot{x}_2 = \frac{-3}{4}\dot{x}_1$$
$$0.5(u + x_1) + 2\dot{x}_1 = \dot{x}_2 = \frac{-3}{4}\dot{x}_1$$

$$\dot{x}_1 = -\frac{2}{11}x_1 - \frac{2}{11}u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{-2}{11} & 0\\ \frac{3}{22} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{-2}{11}\\ \frac{3}{22}\end{bmatrix}u$$

$$y = [-1 \quad -1]\,x$$

This two-dimensional equation describes the network.

$$C = \begin{bmatrix}\frac{-2}{11} & \frac{-2}{11}\times\frac{-2}{11}\\ \frac{3}{22} & \frac{-3}{22}\times\frac{-2}{11}\end{bmatrix},\ \rho(C)=1 \quad \text{not controllable}$$

$$O = \begin{bmatrix}-1 & -1\\ \frac{1}{22} & 0\end{bmatrix},\ \rho(C)=2 \quad \text{observable}$$

Now we introduce the voltage across the 3F capacitor as the third state variable x_3. Then we have $y = x_3$ and $x_3 = -x_1 - x_2$. Thus

$$\dot{x}_3 = -\dot{x}_2 - \dot{x}_1 = \frac{1}{22}x_1 + \frac{1}{22}u$$

and

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}\frac{-2}{11} & 0 & 0\\ \frac{3}{22} & 0 & 0\\ \frac{1}{22} & 0 & 0\end{bmatrix}x + \begin{bmatrix}\frac{-2}{11}\\ \frac{3}{22}\\ \frac{1}{22}\end{bmatrix}u$$

$$y = [0 \quad 0 \quad 1]\,x$$

This 3-dimensional equation describes the network. This equation is not controllable and not observable.

6.18 The equation is

$$\dot{x} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & -1\end{bmatrix}x + \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}u$$

$$y = [0 \quad 1 \quad 0]$$

$$C = \begin{bmatrix}1 & 1 & 1\\ 1 & 0 & -1\\ 0 & 1 & -1\end{bmatrix},\ \rho(C)=3 \quad \text{controllable}$$

$$O = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & -1\\ 0 & 1 & -1\end{bmatrix},\ \rho(O)=2 \quad \text{not observable}$$

The RC loop is in series with the current source, therefore the response due to x_1 will not affect the rest of the network. Thus the network is not observable.

6.19 Consider

$$\dot{x} = \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}x + \begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y = [2 \quad 3]\,x$$

Its eigenvalues are $-1 \pm j$. The necessary and sufficient condition for its discretized equation to be controllable is

$$T \neq \frac{2\pi}{|1-(-1)|}m = \frac{2\pi}{2}m = m\pi, \quad m = 1, 2, \cdots$$

For $T = 1$, the discretized equation was computed in Problem 4.3 as

$$x[k+1] = \begin{bmatrix}0.5083 & 0.3096\\ -0.6191 & -0.1108\end{bmatrix}x[k] + \begin{bmatrix}1.0471\\ -0.1821\end{bmatrix}u[k]$$

$$y[k] = [2 \quad 3]\,x[k]$$

As predicted by Theorem 6.9, it is controllable. Similarly, it is observable.

For $T = \pi$, we have, as computed in Prob. 4.3,

$$x[k+1] = \begin{bmatrix}-0.0432 & 0\\ 0 & -0.0432\end{bmatrix}x[k] + \begin{bmatrix}1.5648\\ -1.0432\end{bmatrix}u[k]$$

$$y[k] = [2 \quad 3]\,x[k]$$

It can be readily verified to be uncontrollable and unobservable and is consistent with Theorem 6.9.

6.20 $\dot{x} = \begin{bmatrix}0 & 1\\ 0 & t\end{bmatrix}x + \begin{bmatrix}0\\ 1\end{bmatrix}u \qquad y = [0 \quad 1]\,x$

$$M_0 = B = \begin{bmatrix}0\\ 1\end{bmatrix},\quad M_1(t) = -A(t)M_0(t) + \frac{d}{dt}M_0(t) = \begin{bmatrix}-1\\ -t\end{bmatrix}$$

$\text{rank}\begin{bmatrix}0 & -1\\ 1 & -t\end{bmatrix} = 2$ at every t. Thus the equation is controllable at every t (Theorem 6.12)

$$N_0(t) = [0 \quad 1],\quad N_1(t) = [0,\ t]$$

$$\text{rank}\begin{bmatrix}N_0(t)\\ N_1(t)\end{bmatrix} = \text{rank}\begin{bmatrix}0 & 1\\ 0 & t\end{bmatrix} = 1$$

Because Theorem 6.012 is a sufficient

condition, we cannot say anything about the observability of the equation.

The state transition matrix of the equation was computed in Problem 4.16 as

$$\Phi(t,t_0)=\begin{bmatrix}1 & -e^{0.5t^2}\int_{t_0}^{t}e^{0.5\tau^2}d\tau \\ 0 & e^{0.5(t^2-t_0^2)}\end{bmatrix}$$

We compute $C\Phi(\tau,t_0)=[0 \quad e^{0.5(\tau^2-t_0^2)}]$

and

$$W_0(t_0,t_1)=\int_{t_0}^{t_1}\begin{bmatrix}0 & 0 \\ 0 & e^{(\tau^2-t_0^2)}\end{bmatrix}d\tau$$

It is singular at every t_0. Thus the equation is not observable at every t.

6.21

$$\dot{x}=\begin{bmatrix}0 & 0 \\ 0 & -1\end{bmatrix}x+\begin{bmatrix}1 \\ e^{-t}\end{bmatrix}u$$

$$y=[1 \quad e^{-t}]x$$

$$\Phi(t,\tau)=\begin{bmatrix}1 & 0 \\ 0 & e^{-(t-\tau)}\end{bmatrix}$$

$$\Phi(t,\tau)B(\tau)=\begin{bmatrix}1 & 0 \\ 0 & e^{-(t-\tau)}\end{bmatrix}\begin{bmatrix}1 \\ e^{-\tau}\end{bmatrix}=\begin{bmatrix}1 \\ e^{-t}\end{bmatrix}$$

$$W_c(t_0,t_1)=\int_{t_0}^{t_1}\begin{bmatrix}1 \\ e^{-t_1}\end{bmatrix}[1 \quad e^{-t_1}]\,d\tau$$

$$=\begin{bmatrix}t_1-t_0 & e^{-t_1}(t_1-t_0) \\ e^{-t_1}(t_1-t_0) & e^{-2t_1}(t_1-t_0)\end{bmatrix}$$

$\det W_c(t_0,t_1)=0$ for all t_0 and $t_1\geq t_0$. Thus the equation is not controllable at any t.

We use Theorem 6.012 to check observability

$$N_0(t)=[1 \quad e^{-t}]$$

$$N_1(t)=[1 \quad e^{-t}]\begin{bmatrix}0 & 0 \\ 0 & -1\end{bmatrix}+\frac{d}{dt}[1 \quad e^{-t}]$$

$$=[0 \quad -e^{-t}]+[0 \quad -e^{-t}]$$

$$=[0 \quad -2e^{-t}]$$

$$\text{rank}\begin{bmatrix}1 & e^{-t} \\ 0 & -2e^{-t}\end{bmatrix}=2$$ for all finite t. Thus the state equation is observable at every t.

We mention that in the time-invariant case, (A,B) is controllable if and only if (A',B') is observable. In the time varying case, it must be modified as $(A(t),B(t))$ is controllable at t_0 if and only if $(-A'(t),B'(t))$ is observable at t_0. See Problem 6.22.

6.22 Let $X(t)$ be a fundamental matrix of $\dot{x}=A(t)x$ or $\frac{d}{dt}X(t)=A(t)X(t)$.

Then

$$\frac{d}{dt}(X^{-1}(t)X(t))=\left(\frac{d}{dt}X^{-1}(t)\right)X(t)+X^{-1}(t)\frac{d}{dt}X(t)$$

$$=\frac{d}{dt}(I)=0$$ Thus

$$\frac{d}{dt}X^{-1}(t)=-X^{-1}(t)\left(\frac{d}{dt}X(t)\right)X^{-1}(t)$$

$$=-X^{-1}(t)A(t).$$

Let $X_1(t)$ be a fundamental matrix of $\dot{x}(t)=-A'(t)x(t)$ or $\frac{d}{dt}X_1(t)=-A'(t)X_1(t)$

Taking its transpose yields

$$\frac{d}{dt}X_1'(t)=-X_1'(t)A(t)$$

Thus we have $X_1'(t)=X^{-1}(t)$, $(X_1'(t))^{-1}=X(t)$

$$\Phi(t,\tau)=X(t)X^{-1}(\tau)$$

$$\Phi_1(t,\tau)=X_1(t)X_1^{-1}(\tau)$$

$$\Phi_1'(t,\tau)=(X_1')^{-1}(\tau)X_1'(t)=X(\tau)X^{-1}(t)$$

$$=\Phi(\tau,t)$$

Now $(A(t),B(t))$ is controllable at t_0 if and only if

$$W_c = \int_{t_0}^{t_1} \phi(t_1,\tau)B(\tau)B'(\tau)\phi'(t_1,\tau)\,d\tau$$

is nonsingular. Using

$$\phi(t_1,\tau) = \phi(t_1,t_0)\phi(t_0,\tau)$$

we write W_c as

$$W_c = \phi(t_1,t_0)\left(\int_{t_0}^{t_1} \phi(t_0,\tau)B(\tau)B'(\tau) \times \phi'(t_0,\tau)\,d\tau\right)\phi'(t_1,t_0)$$

Because $\phi(t_1,t_0)$ is nonsingular, we conclude $(A(t),B(t))$ is controllable if and only if

$$\int_{t_0}^{t} \phi(t_0,\tau)B(\tau)B'(\tau)\phi'(t_0,\tau)\,d\tau \quad (*)$$

is nonsingular. Now $(-A'(t),B'(t))$ is observable if and only if

$$W_{10} = \int_{t_0}^{t} \phi_1'(\tau,t_0)B(\tau)B'(\tau)\phi_1(\tau,t_0)\,d\tau$$

is nonsingular. Using $\phi_1'(\tau,t_0) = \phi(t_0,\tau)$, we write W_{10} as

$$W_{10} = \int_{t_0}^{t} \phi(t_0,\tau)B(\tau)B'(\tau)\phi'(t_0,\tau)\,d\tau$$

which is identical to $(*)$. This establishes that $(A(t),B(t))$ is controllable if and only if $(-A'(t),B(t))$ is observable.

6.23 $(-A,B)$ is controllable if and only if

$$[B\ (-A)B\ (-A)^2B \cdots (-A)^{n-1}B]$$
$$= [B\ -AB\ A^2B\ -A^3B\ \cdots \pm A^{n-1}B]$$

has full row rank. Because

$$[B\ -AB\ A^2B\ -A^3B\ \cdots]$$
$$= \underset{(n\times np)}{[B\ AB\ A^2B\ A^3B\ \cdots]} \underset{(np\times np)}{\begin{bmatrix} I & 0 & 0 & 0 & \\ 0 & -I & 0 & 0 & \\ 0 & 0 & I & 0 & \\ 0 & 0 & 0 & -I & \\ & & & & \ddots \end{bmatrix}}$$

The $(np\times np)$ matrix is clearly nonsingular, thus $[B\ AB\ A^2B \cdots]$ and $[B\ -AB\ A^2B \cdots]$ have the same rank, and (A,B) is controllable if and only if $(-A,B)$ is controllable.

The assertion is not true in the time-varying case. For example, $(A(t),B(t))$ in Problem 6.21 is not controllable at any t. Consider $(-A(t),B(t))$ or

$$-A(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\quad B(t) = \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix}$$

we have

$$\phi(t,\tau) = \begin{bmatrix} 1 & 0 \\ 0 & e^{(t-\tau)} \end{bmatrix}$$

$$\phi(t,\tau)B(\tau) = \begin{bmatrix} 1 \\ e^{(t-\tau)}e^{-\tau} \end{bmatrix} = \begin{bmatrix} 1 \\ e^{t-2\tau} \end{bmatrix}$$

$$W_c(t_0,t_1) = \int_{t_0}^{t_1} \begin{bmatrix} 1 \\ e^{t_1-2\tau} \end{bmatrix} [1\ \ e^{t_1-2\tau}]\,d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 1 & e^{t_1-2\tau} \\ e^{t_1-2\tau} & e^{2(t_1-2\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} t_1-t_0 & \frac{1}{3}e^{t_1}(e^{-3t_0}-e^{-3t_1}) \\ \frac{1}{3}e^{t_1}(e^{-3t_0}-e^{-3t_1}) & \frac{1}{5}e^{2t_1}(e^{-5t_0}-e^{-5t_1}) \end{bmatrix}$$

for any t_0, we can find a t_1 so that $W_c(t_0,t_1)$ is nonsingular and $(-A(t),B(t))$ is controllable at any t although $(A(t),B(t))$ is not.

Problem 5.15 Assuming that the desired final state of a discrete system represented by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is $\mathbf{x}(3) = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$ find the control sequence that transfers the system from $\mathbf{x}(0)$ to $\mathbf{x}(3)$.

SOLUTION: Let us start with equation (5.18) for $n = 3$, i.e.

$$\mathbf{x}(3) - \mathbf{A}^3\mathbf{x}(0) = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

Since

$$\begin{aligned} \mathbf{A}^2 &= \begin{bmatrix} -1 & 3 & 1 \\ -4 & 7 & 4 \\ -1 & -1 & 1 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} -7 & 8 & 4 \\ -10 & 17 & 11 \\ 1 & -4 & 0 \end{bmatrix} \\ \mathbf{AB} &= \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{A}^2\mathbf{B} = \begin{bmatrix} 5 \\ 15 \\ 1 \end{bmatrix}, \quad \mathbf{A}^3\mathbf{x}(0) = \begin{bmatrix} 5 \\ 18 \\ -3 \end{bmatrix} \end{aligned}$$

the previous equation becomes

$$\begin{bmatrix} -5 \\ -19 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 5 \\ 1 & 5 & 15 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

The solution of this system gives the required control sequence as

$$\begin{bmatrix} u(2) \\ u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0.5455 \\ -2.2727 \\ -0.5455 \end{bmatrix}$$

Problem 2)

$$\mathcal{C} = [B \;\; AB] = \begin{bmatrix} b_1 & -b_1+b_2 \\ b_2 & -2b_2 \end{bmatrix}$$

$$\det \mathcal{C} = -2b_1b_2 - b_2(b_2-b_1) = -2b_1b_2 - b_2^2 + b_2b_1$$

$$\det \mathcal{C} = -b_2^2 - b_1b_2 = -b_2(b_2+b_1) \neq 0$$

$$\Rightarrow \boxed{b_2 \neq 0} \text{ and } \boxed{b_1+b_2 \neq 0} \Rightarrow \text{controllable}$$

$$\mathbb{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ -c_1 & c_1-2c_2 \end{bmatrix}$$

$$\det \mathbb{O} = c_1^2 - 2c_1c_2 + c_1c_2 = c_1^2 - c_1c_2 = c_1(c_1-c_2) \neq 0$$

$$\Rightarrow \boxed{c_1 \neq 0} \text{ and } \boxed{c_1-c_2 \neq 0} \Rightarrow \text{observable}$$

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

$$y = [c_1 \;\; c_2]\, x$$

(1) $y(t) = c_1x_1(t) + c_2x_2(t) \Rightarrow y(0) = c_1x_1(0) + c_2x_2(0)$

(2) $\dot{y}(t) = c_1\dot{x}_1(t) + c_2\dot{x}_2(t) = c_1(-x_1(t)+x_2(t)+b_1u(t)) + c_2(-2x_2(t)+b_2u(t))$

(1) $y(0) = \alpha_1+\alpha_2 = c_1x_1(0) + c_2x_2(0)$

(2) $\dot{y}(0) = -\alpha_1-2\alpha_2 = -c_1x_1(0) + (c_1-2c_2)x_2(0) + c_1b_1u(0) + c_2b_2u(0)$

(1)
(2)
$$\begin{bmatrix} c_1 & c_2 \\ -c_1 & c_1-2c_2 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \alpha_1+\alpha_2 \\ -\alpha_1-2\alpha_2-c_1b_1u(0)-c_2b_2u(0) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \underbrace{\begin{bmatrix} c_1 & c_2 \\ -c_1 & c_1-2c_2 \end{bmatrix}^{-1}}_{\text{invertible if the system is observable that is, } c_1\neq 0 \text{ and } c_1-c_2\neq 0} \begin{bmatrix} \alpha_1+\alpha_2 \\ -\alpha_1-2\alpha_2-c_1b_1u(0)-c_2b_2u(0) \end{bmatrix}$$