A time-invariant realization

$$\dot{x} = \begin{bmatrix} 3\lambda & -3\lambda^2 & \lambda^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad 2]\, x$$

4.26 $g(t,\tau) = \sin t\, e^{-t} \cdot e^{\tau} \cos \tau$

A time-varying realization

$$\dot{x} = 0 \cdot x + e^{t} \cos t\; u(t)$$

$$y = \sin t\, e^{-t} x$$

Because $g(t,\tau)$ cannot be expressed as $g(t-\tau)$, it cannot be realized as a linear time-invariant equation.

Chapter 5

5.1 The transfer function from u to y is

$$\hat{g}(s) = \frac{s \cdot \frac{1}{s}}{s + \frac{1}{s}} = \frac{s}{s^2+1}$$

If $u(t) = \sin t$, then

$$\hat{y}(s) = \hat{g}(s)\,\hat{u}(s) = \frac{s}{s^2+1} \cdot \frac{1}{s^2+1} = \frac{s}{(s^2+1)^2}$$

and $y(t) = 0.5\, t \sin t$

which is not bounded. Thus the network is not BIBO stable.

5.2 $$\hat{g}(s) = \int_0^{\infty} g(t) e^{-st}\, dt$$

Let $s = \sigma + j\omega$. If $\sigma \geq 0$, then

$$|e^{-st}| = |e^{-\sigma t}||e^{-j\omega t}| = e^{-\sigma t} \leq 1$$

for all t. If the system is BIBO stable, then $\int_0^{\infty} |g(t)| < \infty$. Thus, we have, for $\operatorname{Re} s \geq 0$,

$$|\hat{g}(s)| \leq \int_0^{\infty} |g(t)||e^{-st}|\, dt \leq \int_0^{\infty} |g(t)|\, dt < \infty$$

5.3 $$\int_0^{\infty} |g(t)|\, dt = \int_0^{\infty} \frac{1}{1+t}\, dt = \ln(1+t)\Big|_0^{\infty} = \infty$$

Thus the system is not BIBO stable.

For $g(t) = t e^{-t}$, we have

$$\hat{g}(s) = \mathcal{L}[g(t)] = \frac{1}{(s+1)^2}$$

all its poles have negative real parts, thus the system is BIBO stable.

5.4 $\hat{g}(s) = \dfrac{e^{-2s}}{s+1}$ irrational function of s.

$$g(t) = \mathcal{L}^{-1}[\hat{g}(s)] = \begin{cases} e^{-(t-2)} & \text{for } t \geq 2 \\ 0 & \text{for } t < 2 \end{cases}$$

$$\int_0^{\infty} |g(t)|\, dt = \int_2^{\infty} e^{-(t-2)}\, dt = \int_0^{\infty} e^{-\tau}\, d\tau$$

$= -e^{-\tau}\Big|_{\tau=0}^{\infty} = -[0-1] = 1$

Thus the system is BIBO stable.

5.5 

If $r(t) = \delta(t)$, then

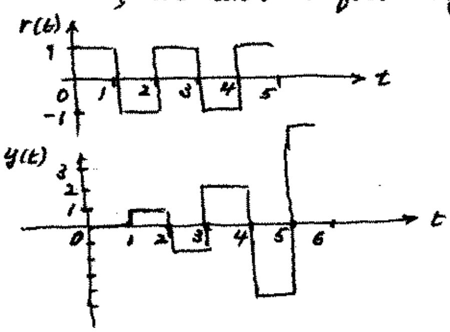
$$g_f(t) = y(t) = a\delta(t-1) - a^2\delta(t-2) + a^3\delta(t-3) - a^4\delta(t-4) + \cdots$$

$$\int_0^\infty |g_f(t)|\,dt = |a| + |a|^2 + |a|^3 + \cdots$$

$$= |a|\sum_{i=0}^{\infty}|a|^i = \begin{cases} \frac{|a|}{1-|a|} & \text{if } |a| < 1 \\ \infty & \text{if } |a| \geq 1 \end{cases}$$

Thus the feedback system is BIBO stable if and only if $|a| < 1$.

For $a = 1$, we have the following pair



The bounded input excites an unbounded output.

5.6 $\hat{g}(s) = \frac{s-2}{s+1}$

If $u(t) = 3$, then $y(t) \to \hat{g}(0)\cdot 3 = -6$

If $u(t) = \sin 2t$, then

$$y(t) \to |\hat{g}(j2)| \sin(2t + \measuredangle \hat{g}(j2))$$

$$= 1.26 \sin(2t + 1.25)$$

5.7

$$\hat{g}(s) = [-2\ \ 3]\begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= [-2\ \ 3]\begin{bmatrix} \frac{1}{s+1} & \frac{10}{s^2-1} \\ 0 & \frac{1}{s-1} \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{4}{s+1}$$

It is BIBO stable.

5.8 $g[k] = k(0.8)^k$, $k = 0, 1, 2, \cdots$

$$\hat{g}(z) = \mathcal{Z}[g[k]] = \frac{0.8z}{(z-0.8)^2}$$

Its poles lie inside the unit circle, thus the system is BIBO stable.

5.9 The matrix has eigenvalues -1 and 1, thus the equation is not asymptotically stable nor marginally stable.

5.10 The matrix has eigenvalues $-1, 0, 0$; thus the equation is not asymptotically stable. If the repeated eigenvalue 0 is a simple root of the minimal polynomial or, equivalently, has only Jordan blocks of order 1, then the equation is marginally stable. We compute the eigenvectors associated with $\lambda = 0$:

$$(A - \lambda I)v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

which yields two linearly independent eigenvectors $[0\ 1\ 0]'$ and $[1\ 0\ 1]'$. Thus the Jordan form of A is

$$\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the equation is marginally stable.

5.11 The matrix has eigenvalues $-1, 0, 0$; thus the equation is not asymptotically stable. We compute the eigenvectors associated with $\lambda = 0$:

$$(A - \lambda I)v = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

It has only one linearly independent

eigenvector. Thus its Jordan form is

$$\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

It has one Jordan block with order 2 associated with 0. Thus the equation is not marginally stable.

5.12 The matrix has eigenvalues 0.9, 1, 1; thus the discrete-time system is not asymptotically stable. Its Jordan form is

$$\hat{A} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus it is marginally stable.

5.13 The matrix has eigenvalues 0.9, 1 and 1 and its Jordan form, as in Prob. 5.11, is

$$\hat{A} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus the equation is not marginally stable, nor asymptotically stable.

5.14 $A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$. Select $N = I$.

$$\begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Equating the (i,j)th entry:

$(1,1)$: $-0.5m_{12} - 0.5m_{12} = -1 \Rightarrow m_{12} = 1$

$(2,2)$: $m_{12} - m_{22} + m_{12} - m_{22} = -1 \Rightarrow m_{22} = 1.5$

$(1,2)$: $-0.5m_{22} + m_{11} - m_{12} = 0 \Rightarrow m_{11} = 1.75$

$$M = \begin{bmatrix} 1.75 & 1 \\ 1 & 1.5 \end{bmatrix}$$

Leading principal minors $1.75 > 0$

$1.75 \times 1.5 - 1 \times 1 > 0$

M is positive definite. Thus all eigenvalues of A have negative real parts.

5.15

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} - \begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(1,1)$: $m_{11} - 0.25m_{22} = 1$

$(1,2)$: $-0.5m_{22} + 1.5m_{12} = 0$

$(2,2)$: $m_{22} - m_{11} + 2m_{12} - m_{22} = 1$

From these, we can obtain $m_{12} = 1.6$, $m_{22} = 4.8$, $m_{11} = 2.2$.

$$M = \begin{bmatrix} 2.2 & 1.6 \\ 1.6 & 4.8 \end{bmatrix} \quad \text{positive definite}$$

Thus all eigenvalues of A have magnitude less than 1. As a check, the eigenvalues of A are $-0.5 \pm j0.5$. Both have negative real parts and have magnitudes less than 1.

5.16 Let $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. Then

$$A'M + MA = -\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix} = -\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}[a_1\ a_2\ a_3]$$

$$=: -\bar{N}'\bar{N} =: -N$$

It is clear that all eigenvalues of A have negative real parts and N is positive semidefinite. We compute

$$\begin{bmatrix} \bar{N} \\ \bar{N}A \\ \bar{N}A^2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ \lambda_1 a_1 & \lambda_2 a_2 & \lambda_3 a_3 \\ \lambda_1^2 a_1 & \lambda_2^2 a_2 & \lambda_3^2 a_3 \end{bmatrix} =: \Theta$$

$$\det \Theta = a_1a_2a_3 \det \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

$$= a_1a_2a_3 \det \begin{bmatrix} 1 & 0 & 0 \\ \lambda_1 & \lambda_2-\lambda_1 & \lambda_3-\lambda_1 \\ \lambda_1^2 & \lambda_2^2-\lambda_1^2 & \lambda_3^2-\lambda_1^2 \end{bmatrix}$$

$$= a_1a_2a_3(\lambda_2-\lambda_1)(\lambda_3-\lambda_1) \det \begin{bmatrix} 1 & 1 \\ \lambda_2+\lambda_1 & \lambda_3+\lambda_1 \end{bmatrix}$$

$$= a_1a_2a_3(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$$

$\det \Theta$ is nonzero if $a_i \neq 0$ and λ_i are distinct. Thus Θ has rank 3 and M is positive definite. (Corollary 5.5)

5.17 $M_1 = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ is not positive definite because $[1\ 0] M \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$. Its eigenvalues are 1 and 2. $M_2 = \begin{bmatrix} 2 & 1 \\ 1.9 & 1 \end{bmatrix}$ is not positive definite because

$$[0.5805\ \ -0.8142] M_2 \begin{bmatrix} 0.5805 \\ -0.8142 \end{bmatrix} = -0.0338.$$

Its leading principal minors are 2 and $(2 \times 1 - 1.9 \times 1) = 0.1$; both are positive. Therefore, the two assertions do not hold. Because

$$x' M_i x = \frac{1}{2}(x' M_i x + x' M_i' x) = x' \left(\frac{1}{2}(M_i + M_i')\right) x$$

we may check the positive definiteness of M_i by forming the symmetric matrix $\bar{M}_i = \frac{1}{2}(M_i + M_i')$ and then check $\bar{M}_i$. For example, we form

$$\bar{M}_1 = \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}\right) = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 3 \end{bmatrix}$$

It is not positive definite because its leading principal minors are 0 and -0.25. Similarly, we form

$$\bar{M}_2 = \begin{bmatrix} 2 & 1.45 \\ 1.45 & 1 \end{bmatrix}$$

It is not positive definite because its leading principal minors are 2 and $2 \times 1 - (1.45)^2 = -0.1025$

5.18 $A'M + MA + 2\mu M = -N$

$$(A' + \mu I')M + M(A + \mu I) = -N$$

If $N > 0$ and $M > 0$, then all eigenvalues of $(A + \mu I)$ have magnitudes less than 0, or $Re\ \lambda_i(A + \mu I) < 0$. Using Problem 3.19, we have

$$\lambda_i(A + \mu I) = \lambda_i(A) + \mu.$$ Thus

$$Re[\lambda_i(A)] + \mu < 0 \quad \text{or} \quad Re\ \lambda_i(A) < -\mu$$

5.19 $\rho^2 M - A'MA = \rho^2 N$

$$M - \left(\frac{1}{\rho}A'\right) M \left(\frac{1}{\rho}A\right) = N$$

If $N > 0$ and $M > 0$, then

$$\left|\lambda_i\left(\frac{1}{\rho}A\right)\right| < 1$$

Thus $|\lambda_i(A)| < \rho$

5.20 $g(t,\tau) = e^{-2|t| - |\tau|}$, for $t \geq \tau$

$$\int_{t_0}^{t} |g(t,\tau)| d\tau = e^{-2|t|} \int_{t_0}^{t} e^{-|\tau|} d\tau =: B$$

For $t_0 \leq t < 0$, we have

$$B = e^{2t} \int_{t_0}^{t} e^{\tau} d\tau = e^{2t}(e^{t} - e^{t_0}) < 1$$

For $t_0 \leq t$ with $t \geq 0$, we have

$$B = e^{-2t}\left[\int_{t_0}^{0} e^{\tau} d\tau + \int_{0}^{t} e^{-\tau} d\tau\right] \quad \text{with } t_0 < 0$$

$$= e^{-2t}\left[(1 - e^{t_0}) - (e^{-t} - 1)\right]$$

$$= e^{-2t}\left[-e^{-t} - e^{t_0} + 2\right] < \infty$$

Thus the system is BIBO stable.

$g(t,\tau) = \sin t\,(e^{-(t-\tau)}) \cos \tau$

$$\int_{t_0}^{\infty} |g(t-\tau)| d\tau \leq \int_{t_0}^{t} e^{-(t-\tau)} d\tau = e^{-t} \int_{t_0}^{t} e^{\tau} d\tau$$

$$= e^{-t}[e^{t} - e^{t_0}] = 1 - e^{-(t - t_0)} \leq 1$$

for all t_0 and $t \geq t_0$. Thus the system is BIBO stable.

5.21 $\dot{x} = 2tx + u$, $y = e^{-t^2} x$

For this scalar equation, we have

$$\phi(t, t_0) = e^{\int_{t_0}^{t} 2\tau d\tau} = e^{(t^2 - t_0^2)}$$

Thus $g(t,\tau) = e^{-\tau^2}\phi(t,\tau)\cdot 1 = e^{-\tau^2+t^2-\tau^2}$

$= e^{-\tau^2}$

$\int_{t_0}^{t} |g(t,\tau)|\, d\tau = \int_{t_0}^{t} e^{-\tau^2} d\tau < \infty$ for all t_0 and $t \geq t_0$ because $e^{-\tau^2} < e^{-|\tau|}$ for $1 \leq \tau \leq \infty$.

Thus the equation is BIBO stable.

Because $|\phi(t,t_0)| = e^{t^2-t_0^2} \to \infty$ as $t \to \infty$, the equation is not marginally stable, nor asymptotically stable.

(5.22) $\bar{x} = e^{-t^2}x$ or $P(t) = e^{-t^2}$ and $P^{-1}(t) = e^{t^2}$.

Using (4.70), we have

$$\bar{A} = [P(t)A(t) + \dot{P}(t)]P^{-1}(t)$$
$$= [2te^{-t^2} - 2te^{-t^2}]e^{t^2} = 0$$

Thus the equivalent equation is

$$\dot{\bar{x}} = 0\cdot\bar{x} + e^{-t^2}u$$
$$y = e^{-t^2}e^{t^2}\cdot\bar{x} = \bar{x}$$

$\bar{g}(t,\tau) = C(t)\phi(t,\tau)B(\tau) = 1\times 1\times e^{-\tau^2} = e^{-\tau^2}$

The impulse response remains unchanged, therefore the equation is BIBO stable. The zero-input response is governed by the time-invariant equation $\dot{\bar{x}} = 0\cdot\bar{x}$ with eigenvalue 0. Thus the equation is marginally stable; it is not asymptotically stable. The transformation $P(t) = e^{-t^2}$ is not a Lyapunov transformation because $P^{-1}(t) = e^{t^2}$ is not bounded. Therefore marginal and asymptotic stabilities are not invariant under the transformation.

(5.23) $\dot{x} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix} x \quad$ for $t_0 \geq 0$

$\dot{x}_1(t) = -x(t) \rightarrow x(t) = e^{-t}x_1(0)$

$\dot{x}_2(t) = -e^{-3t}x_1(t) = -e^{-4t}x_1(0)$

$\rightarrow x_2(t) = 0.2\,[e^{-5t} - 1]\,x_1(0) + x_2(0)$

$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} e^{-t} \\ 0.2\,(e^{-5t}-1) \end{bmatrix}$

$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$X(t) = \begin{bmatrix} e^{-t} & 0 \\ 0.2\,(e^{-5t}-1) & 1 \end{bmatrix}$$

$$X^{-1}(t) = \begin{bmatrix} e^{t} & 0 \\ 0.2\,(e^{t} - e^{-4t}) & 1 \end{bmatrix}$$

$$\phi(t,t_0) = X(t)X^{-1}(t_0) = \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0.2\,(e^{-5t+t_0} - e^{-4t_0}) & 1 \end{bmatrix}$$

For $t_0 \geq 0$ and $t \geq t_0$, every entry of $\phi(t,t_0)$ is bounded, thus the equation is marginally stable. A necessary condition for $\|\phi(t,t_0)\| \to 0$ as $t \to \infty$ is that every entry approaches zero. This is not the case, thus the equation is not asymptotically stable.