Chapter 4

(4.1) $\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x = Ax$

$\Delta(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2+1, \quad \lambda = \pm j$

Let $h(\lambda) = \beta_0 + \beta_1 \lambda \qquad f(\lambda) = e^{\lambda t}$

$\lambda = j: \quad e^{jt} = \beta_0 + j\beta_1$

$\lambda = -j: \quad e^{-jt} = \beta_0 - j\beta_1$

$\beta_1 = \dfrac{e^{jt}-e^{-jt}}{2j} = \sin t$

$\beta_2 = e^{jt} - j\beta_1 = \cos t$

$e^{At} = \cos t \, I + \sin t \, A = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

Thus

$x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$

(4.2) Find unit step response of

$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$

$y = [\,2 \quad 3\,]\, x$

Method 1: Laplace transform

$\hat{g}(s) = [\,2 \quad 3\,] \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= [\,2 \quad 3\,] \dfrac{1}{s^2+2s+2} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= \dfrac{5s}{s^2+2s+2}, \qquad \hat{u}(s) = \dfrac{1}{s}$

$\hat{y}(s) = \hat{g}(s)\,\hat{u}(s) = \dfrac{5s}{(s+1)^2+1} \cdot \dfrac{1}{s}$

$\therefore y(t) = 5e^{-t} \sin t$

Method 2: Using (4.7)

$\Delta(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+2 \end{bmatrix} = \lambda^2 + 2\lambda + 2$

$\lambda = -1 \pm j$

$f(\lambda) = e^{\lambda t}, \quad h(\lambda) = \beta_0 + \beta_1 \lambda$

$\lambda = -1-j: \quad e^{(-1-j)t} = \beta_0 + \beta_1(-1-j)$

$\lambda = -1+j: \quad e^{(-1+j)t} = \beta_0 + \beta_1(-1+j)$

$\Rightarrow \beta_0 = e^{-t} \sin t, \quad \beta_1 = e^{-t}(\sin t + \cos t)$

$e^{At} = \beta_0 I + \beta_1 \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$

$= \begin{bmatrix} e^{-t}(\sin t + \cos t) & e^{-t} \sin t \\ -2e^{-t} \sin t & e^{-t}(\cos t - \sin t) \end{bmatrix}$

$u(t) = 1 \quad$ for $t \geq 0$.

$y(t) = [\,2 \quad 3\,] \displaystyle\int_0^t e^{A(t-\tau)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 1 \cdot d\tau$

$= \displaystyle\int_0^t \left(5e^{-(t-\tau)} \cos(t-\tau) - 5e^{-(t-\tau)} \sin(t-\tau)\right) d\tau$

Consider

$5\displaystyle\int_0^t e^{-(t-\tau)} \cos(t-\tau)\, d\tau = -5 \int_0^t e^{-(t-\tau)} \frac{d}{d\tau} \sin(t-\tau)\, d\tau$

$= -5\left[e^{-(t-\tau)} \sin(t-\tau) \Big|_{\tau=0}^{t} - \displaystyle\int_0^t e^{-(t-\tau)} \sin(t-\tau)\, d\tau \right]$

Thus we have

$y(t) = -5\left[e^{-0} \cdot \sin 0 - e^{-t} \sin t \right] = 5e^{-t} \sin t.$

(4.3) Using e^{At} computed in Prob. 4.2. For $T=1$,

$A_d = e^{AT} = \begin{bmatrix} e^{-1}(\sin 1 + \cos 1) & e^{-1} \sin 1 \\ -2e^{-1} \sin 1 & e^{-1}(\cos 1 - \sin 1) \end{bmatrix}$

$= \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix}$

Because A is nonsingular (it has no zero eigenvalue), we may use (4.18) to compute

$b_d = A^{-1}(A_d - I)\, b = \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix}$

Thus the discretized equation with $T=1$ is

$$x[k+1] = \begin{bmatrix} 0.5083 & 0.3096 \\ -0.6191 & -0.1108 \end{bmatrix} x[k] + \begin{bmatrix} 1.0471 \\ -0.1821 \end{bmatrix} u[k]$$

$$y[k] = [2 \quad 3]\, x[k]$$

For $T=\pi$, we have

$$x[k+1] = \begin{bmatrix} -0.0432 & 0 \\ 0 & -0.0432 \end{bmatrix} x[k] + \begin{bmatrix} 1.5648 \\ -1.0432 \end{bmatrix} u[k]$$

$$y[k] = [2 \quad 3]\, x[k]$$

4.4

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad -1 \quad 0]\, x$$

Companion form

$$Q = [b \;\; Ab \;\; A^2b] = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & -4 \\ 1 & -2 & 0 \end{bmatrix}$$

$$P = Q^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0.5 & 0.5 & -0.5 \\ 0.25 & 0 & -0.25 \end{bmatrix}$$

$$\dot{\bar{x}} = Q^{-1}AQ\bar{x} + Q^{-1}bu$$

$$= \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -6 \\ 0 & 1 & -4 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = cQ\bar{x} = [1 \quad -4 \quad 8]\, \bar{x}$$

Modal form

Eigenvalues $-1+j,\ -1-j,\ -2$

Eigenvectors

$$v_1 = \begin{bmatrix} 0 \\ 0.5774j \\ -0.5774-0.5774j \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -0.5774j \\ -0.5774+0.5774j \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix}$$

$$Q = [\mathrm{Re}(v_1) \;\; \mathrm{Im}(v_1) \;\; v_3] = \begin{bmatrix} 0 & 0 & 0.7071 \\ 0 & 0.5774 & 0 \\ -0.5774 & -0.5774 & -0.7071 \end{bmatrix}$$

$$P = Q^{-1} = \begin{bmatrix} -1.7319 & -1.7319 & -1.7319 \\ 0 & 1.7319 & 0 \\ 1.4142 & 0 & 0 \end{bmatrix}$$

$$\dot{\bar{x}} = Q^{-1}AQ\bar{x} + Q^{-1}bu$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} -3.4638 \\ 0 \\ 1.4142 \end{bmatrix} u$$

$$y = cQ\bar{x} = [0 \quad -0.5774 \quad 0.7071]\, \bar{x}$$

4.5

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad -1 \quad 0]\, x$$

For unit step input ($u(t)=1$, for $t \geq 0$), we use MATLAB to obtain

$|y|_{max} = 0.55$, $|x_1|_{max} = 0.5$

$|x_2|_{max} = 1.05$, $|x_3|_{max} = 0.52$.

Let $\bar{x}_1 = x_1$, $\bar{x}_2 = 0.5x_2$, $\bar{x}_3 = x_3$ or

$$\bar{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = Px$$

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dot{\bar{x}} = PAP^{-1}\bar{x} + Pbu$, $y = cP^{-1}\bar{x}$ or

$$\dot{\bar{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & -4 & -2 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad -2 \quad 0]\, \bar{x}.$$

For this equation, we have

$|y|_{max} = 0.55$, $|\bar{x}_1|_{max} = 0.5$

$|\bar{x}_2|_{max} = 0.525$, $|\bar{x}_3|_{max} = 0.52$

The step input must have magnitude less than $(10/0.55) = 18.2$ to avoid saturation.

4.6 Direct verification

4.7 Direct verification

4.8 A necessary condition for two state equations to be equivalent is that they have the same set of eigenvalues. The first equation has eigenvalues 2, 2 and 1. The second equation has eigenvalue 2, 2 and -1. Thus, they are not equivalent.
Using the fact that the inverse of a triangular matrix is again triangular, we can readily verify that

$$[1\ -1\ 0]\begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}^{-1}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$=[1\ -1]\begin{bmatrix} s-2 & -1 \\ 0 & s-2 \end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}$$

$$=[1\ -1]\begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-2)^2} \\ 0 & \frac{1}{s-2} \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{(s-2)^2}$$

The second equation also has transfer function $1/(s-2)^2$. Thus, they are zero-state equivalent.

4.9 Define $Z=[Z_1\ Z_2 \cdots Z_r]$, where Z_i is $q\times q$ and Z is $q\times rq$, by

$$Z = C(sI-A)^{-1}$$

or

$$Z(sI-A)=C \qquad sZ=ZA+C$$

Using the forms of A and C, we have

$$sZ_1 = -\alpha_1 Z_1 - \alpha_2 Z_2 - \cdots - \alpha_r Z_r + I_q$$

$$sZ_2 = Z_1$$

$$\vdots$$

$$sZ_{r-1} = Z_{r-2}$$

$$sZ_r = Z_{r-1}$$

From these, we have

$$Z_2 = \frac{1}{s}Z_1,\quad Z_3=\frac{1}{s}Z_2=\frac{1}{s^2}Z_1,\ \cdots,\ Z_r=\frac{1}{s^{r-1}}Z_1$$

and

$$(s^r+\alpha_1 s^{r-1}+\cdots+\alpha_r)Z_1 = s^{r-1}I_q$$

Thus we have

$$Z_1=\frac{s^{r-1}}{d(s)}I_q,\quad Z_2=\frac{s^{r-2}}{d(s)}I_q,\ \cdots,\ Z_r=\frac{1}{d(s)}I_q$$

where $d(s)=s^r+\alpha_1 s^{r-1}+\cdots+\alpha_r$

The transfer matrix is

$$G(s)=C(sI-A)^{-1}B = ZB = [Z_1 \cdots Z_r]\begin{bmatrix}N_1\\ \vdots \\ N_r\end{bmatrix}$$

$$=\frac{1}{d(s)}[N_1 s^{r-1}+N_2 s^{r-2}+\cdots+N_{r-1}s+N_r]$$

This completes the verification

4.10 Direct substitution

4.11

$$G(s)=\begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{s-2}{s+1} & \frac{s}{s+2} \end{bmatrix}$$

$$=\begin{bmatrix}0&0\\1&1\end{bmatrix}+\begin{bmatrix} \frac{2}{s+1} & \frac{2s-3}{(s+1)(s+2)} \\ \frac{-3}{s+1} & \frac{-2}{s+2} \end{bmatrix}$$

$$=\begin{bmatrix}0&0\\1&1\end{bmatrix}+\frac{1}{s^2+3s+2}\begin{bmatrix}2s+4 & 2s-3\\ -3s-6 & -2s-2\end{bmatrix}$$

Using (4.34), we have

$$\dot{x}=\left[\begin{array}{cc|cc} -3 & 0 & -2 & 0 \\ 0 & -3 & 0 & -2 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]x+\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]u$$

$$y=\left[\begin{array}{cc|cc} 2 & 2 & 4 & -3 \\ -3 & -2 & -6 & -2 \end{array}\right]x+\begin{bmatrix}0&0\\1&1\end{bmatrix}u$$

This is a 4-dimensional realization

4.12 Consider the first column:

$$\begin{bmatrix}\frac{2}{s+1}\\ \frac{s-2}{s+1}\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}+\frac{1}{s+1}\begin{bmatrix}2\\-3\end{bmatrix}$$

$$\dot{x}_1 = -x_1 + u_1$$

$$y_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1$$

The second column:

$$\begin{bmatrix} \dfrac{2s-3}{(s+1)(s+2)} \\ \dfrac{s}{s+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{1}{s^2+3s+2} \begin{bmatrix} 2s-3 \\ -2s-2 \end{bmatrix}$$

$$\dot{x}_2 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2$$

$$y_2 = \begin{bmatrix} 2 & -3 \\ -2 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2$$

Combining these yields

$$\dot{x} = \left[\begin{array}{c:cc} -1 & 0 & 0 \\ \hdashline 0 & -3 & -2 \\ 0 & 1 & 0 \end{array}\right] x + \left[\begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 1 \\ 0 & 0 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \left[\begin{array}{c:cc} 2 & 2 & -3 \\ -3 & -2 & -2 \end{array}\right] x + \left[\begin{array}{c:c} 0 & 0 \\ 1 & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This realization has dimension 3, one less than the one in Problem 4.11.

4.13 First row:

$$\left[\frac{2}{s+1} \quad \frac{2s-3}{(s+1)(s+2)} \right] = \frac{1}{s^2+3s+2} [2s+4 \quad 2s-3]$$

$$\dot{x}_1 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 2 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y_1 = [1 \quad 0] x_1 + [0 \quad 0] u$$

Second row:

$$\left[\frac{s-2}{s+1} \quad \frac{s}{s+2} \right] = [1 \quad 1] + \left[\frac{-3}{s+1} \quad \frac{-2}{s+2} \right]$$

$$= [1 \quad 1] + \frac{1}{s^2+3s+2} [-3s-6 \quad -2s-2]$$

$$\dot{x}_2 = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 & -2 \\ -6 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y_2 = [1 \quad 0] x_2 + [1 \quad 1] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left[\begin{array}{cc:cc} -3 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -3 & 1 \\ 0 & 0 & -2 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left[\begin{array}{cc} 2 & 2 \\ 4 & -3 \\ \hdashline -3 & -2 \\ -6 & -2 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left[\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left[\begin{array}{cc} 0 & 0 \\ \hdashline 1 & 1 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This realization has dimension 4, one more than the one in Problem 4.12 and the same as the one in Problem 4.11.

4.14

$$G(s) = \left[\frac{-(12s+6)}{3s+34} \quad \frac{22s+23}{3s+34} \right]$$

$$= \left[-4 \quad \frac{22}{3}\right] + \left[\frac{130}{3s+34} \quad \frac{-679/3}{3s+34} \right]$$

$$= \left[-4 \quad \frac{22}{3}\right] + \frac{1}{s+34/3} \left[\frac{130}{3} \quad \frac{-679}{9} \right]$$

$$\dot{x} = \frac{-34}{3} x + \left[\frac{130}{3} \quad \frac{-679}{9} \right] u$$

$$y = x + \left[-4 \quad \frac{22}{3}\right] u$$

4.15 Using the formula in Problem 3.26, we write

$$\hat{g}(s) = c(sI-A)^{-1} b = \frac{1}{\Delta(s)} \left[cR_0 b s^{n-1} + cR_1 b s^{n-2} + \cdots + cR_{n-1} b \right]$$

The numerator of $\hat{g}(s)$ has degree $m \Leftrightarrow$

$cR_{n-m-1} b \neq 0$, $cR_i b = 0$ for $i = 0, 1, \cdots, n-m-2$

Using the formula in Problem 3.26, we have

$$cR_0 b = cb = 0$$

$$cR_1 b = cAb + \alpha_1 cb = 0 \Rightarrow cAb = 0$$

$$\vdots$$

$$cR_{n-m-2} b = 0 \Rightarrow cA^{n-m-2} b = 0$$

$$cR_{n-m-1} b \neq 0 \Rightarrow cA^{n-m-1} b \neq 0$$

4.16

$$\dot{x}_2 = t x_2 \rightarrow x_2(t) = x_2(0) e^{0.5t^2}$$

$$\dot{x}_1 = x_2(t) \rightarrow x_1(t) = \left(\int_0^t e^{0.5\tau^2} d\tau \right) x_2(0) + x_1(0)$$

Let $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $x(t) = \begin{bmatrix} \int_0^t e^{0.5\tau^2} d\tau \\ e^{0.5t^2} \end{bmatrix}$

Thus a fundamental matrix is

$$X(t) = \begin{bmatrix} 1 & \int_0^t e^{0.5\tau^2} d\tau \\ 0 & e^{0.5t^2} \end{bmatrix}$$

$$X^{-1}(t)=\frac{1}{e^{0.5t^2}}\begin{bmatrix} e^{0.5t^2} & -\int_0^t e^{0.5\tau^2}d\tau \\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & -e^{-0.5t^2}\int_0^t e^{0.5\tau^2}d\tau\\ 0 & e^{-0.5t^2}\end{bmatrix}$$

The state transition matrix is

$$\Phi(t,t_0)=X(t)X^{-1}(t_0)=\begin{bmatrix}1 & -e^{0.5t^2}\int_{t_0}^t e^{0.5\tau^2}d\tau\\ 0 & e^{0.5(t^2-t_0^2)}\end{bmatrix}$$

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$$\dot{x}_2(t)=-x_2(t) \longrightarrow x_2(t)=e^{-t}x_2(0)$$

$$\dot{x}_1(t)=-x_1(t)+e^{2t}x_2(t)=-x_1(t)+e^{t}x_2(0)$$

$$x_1(t)=e^{-t}x_1(0)+\int_0^t e^{-(t-\tau)}e^{\tau}x_2(0)\,d\tau$$

$$=e^{-t}x_1(0)+x_2(0)e^{-t}\int_0^t e^{2\tau}d\tau$$

$$=e^{-t}x_1(0)+\frac{1}{2}x_2(0)e^{-t}(e^{2t}-1)$$

$$x(0)=\begin{bmatrix}1\\0\end{bmatrix} \rightarrow x(t)=\begin{bmatrix}e^{-t}\\0\end{bmatrix}$$

$$x(0)=\begin{bmatrix}1\\2\end{bmatrix} \rightarrow x(t)=\begin{bmatrix}e^{t}\\2e^{-t}\end{bmatrix}$$

A fundamental matrix is

$$X(t)=\begin{bmatrix}e^{-t} & e^{t}\\ 0 & 2e^{-t}\end{bmatrix}$$

$$X^{-1}(t)=\frac{1}{2e^{-2t}}\begin{bmatrix}2e^{-t} & -e^{t}\\ 0 & e^{-t}\end{bmatrix}$$

The state transition matrix is

$$\Phi(t,t_0)=X(t)X^{-1}(t_0)$$

$$=\begin{bmatrix}e^{-(t-t_0)} & \frac{1}{2}(e^{t}e^{t_0}-e^{-t}e^{t_0})\\ 0 & e^{-(t-t_0)}\end{bmatrix}$$

4.17 $\frac{\partial}{\partial t}\Phi(t_0,t)=X(t_0)\frac{\partial}{\partial t}X^{-1}(t)$

$$\frac{d}{dt}\left(X(t)X^{-1}(t)\right)=\dot{X}(t)X^{-1}(t)+X(t)\frac{d}{dt}X^{-1}(t)$$

$$=\frac{d}{dt}(I)=0$$

$$\therefore \frac{d}{dt}X^{-1}(t)=-X^{-1}(t)\dot{X}(t)X^{-1}(t)$$

$$=-X^{-1}(t)A(t)X(t)X^{-1}(t)$$

$$=-X^{-1}(t)A(t)$$

Thus we have

$$\frac{\partial}{\partial t}\Phi(t_0,t)=X(t_0)\left(-X^{-1}(t)\right)A(t)$$

$$=-\Phi(t_0,t)A(t)$$

4.18 $\frac{\partial}{\partial t}\Phi(t,t_0)=\begin{bmatrix}\frac{\partial}{\partial t}\varphi_{11}(t,t_0) & \frac{\partial}{\partial t}\varphi_{12}(t,t_0)\\ \frac{\partial}{\partial t}\varphi_{21}(t,t_0) & \frac{\partial}{\partial t}\varphi_{22}(t,t_0)\end{bmatrix}$

$$=\begin{bmatrix}a_{11}(t) & a_{12}(t)\\ a_{21}(t) & a_{22}(t)\end{bmatrix}\begin{bmatrix}\varphi_{11}(t,t_0) & \varphi_{12}(t,t_0)\\ \varphi_{21}(t,t_0) & \varphi_{22}(t,t_0)\end{bmatrix}$$

Substituting these $\frac{\partial}{\partial t}\varphi_{ij}(t,t_0)$ into

$$\frac{\partial}{\partial t}\det\Phi(t,t_0)=\frac{\partial}{\partial t}[\varphi_{11}\varphi_{22}-\varphi_{21}\varphi_{12}]$$

$$=\left(\frac{\partial}{\partial t}\varphi_{11}\right)\varphi_{22}+\varphi_{11}\left(\frac{\partial}{\partial t}\varphi_{22}\right)-\left(\frac{\partial}{\partial t}\varphi_{21}\right)\varphi_{12}-\varphi_{21}\left(\frac{\partial}{\partial t}\varphi_{12}\right)$$

and simple manipulation yield

$$\frac{\partial}{\partial t}\det\Phi(t,t_0)=(a_{11}(t)+a_{22}(t))\det\Phi(t,t_0)$$

Thus

$$\det\Phi(t,t_0)=\exp\left[\int_{t_0}^{t}(a_{11}(\tau)+a_{22}(\tau))\,d\tau\right]$$

4.19 $\Phi(t_0,t_0)=\begin{bmatrix}\Phi_{11}(t_0,t_0) & \Phi_{12}(t_0,t_0)\\ \Phi_{21}(t_0,t_0) & \Phi_{22}(t_0,t_0)\end{bmatrix}=I$

Thus $\Phi_{21}(t_0,t_0)=0$ and $\Phi_{22}(t_0,t_0)=I$.

$$\frac{\partial}{\partial t}\Phi_{21}(t,t_0)=0\cdot\Phi_{11}(t,t_0)+A_{22}(t)\Phi_{21}(t,t_0)$$

$$\frac{\partial}{\partial t}\Phi_{22}(t,t_0)=0\cdot\Phi_{12}(t,t_0)+A_{22}(t)\Phi_{22}(t,t_0)$$

The equation

$$\frac{\partial}{\partial t}\Phi_{22}(t,t_0)=A_{22}(t)\Phi_{22}(t,t_0)$$

with $\Phi_{22}(t_0,t_0)=I$ yields the unique solution of $\Phi_{22}(t,t_0)$. The equation

$$\frac{\partial}{\partial t}\Phi_{21}(t,t_0)=A_{22}(t)\Phi_{21}(t,t_0)$$

with $\Phi_{21}(t_0,t_0)=0$ yields the unique solution $\Phi_{21}(t,t_0)\equiv 0$. With $\Phi_{21}\equiv 0$, then

$$\frac{\partial}{\partial t}\Phi_{11}(t,t_0)=A_{11}(t)\Phi_{11}(t,t_0)+A_{12}(t)\cdot 0$$

$$=A_{11}(t)\Phi_{11}(t,t_0)$$

4.20 $\dot{x} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} x$

$\dot{x}_1 = -\sin t \, x_1(t) \longrightarrow x_1(t) = e^{\cos t} x_1(0)$

$\dot{x}_2 = -\cos t \, x_2(t) \longrightarrow x_2(t) = e^{-\sin t} x_2(0)$

$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} e^{\cos t} \\ 0 \end{bmatrix}$

$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow x(t) = \begin{bmatrix} 0 \\ e^{-\sin t} \end{bmatrix}$

Fundamental matrix

$$X(t) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}$$

$$X^{-1}(t) = \begin{bmatrix} e^{-\cos t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}$$

State transition matrix

$$\Phi(t,t_0) = X(t)X^{-1}(t_0) = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{-\sin t + \sin t_0} \end{bmatrix}$$

4.21 $X(t) = e^{At} C e^{Bt}$, $X(0) = I \cdot C \cdot I = C$

$$\dot{X}(t) = A e^{At} C e^{Bt} + e^{At} C e^{Bt} B = A X(t) + X(t) B$$

4.22 $\dot{A}(t) = A_1 e^{A_1 t} A(0) e^{-A_1 t} + e^{A_1 t} A(0) e^{-A_1 t}(-A_1)$

$= A_1 A(t) - A(t) A_1$

$\det(\lambda I - A(t)) = \det(e^{A_1 t} \lambda I e^{-A_1 t} - A(t))$

$= \det[e^{A_1 t}(\lambda I - A(0)) e^{-A_1 t}]$

$= \det e^{A_1 t} \det e^{-A_1 t} \det(\lambda I - A(0))$

$= \det(\lambda I - A(0))$ (independent of t)

4.23 The equation is periodic with period $T = 2\pi$.

$$X(t) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}$$

$$X(t+2\pi) = \begin{bmatrix} e^{\cos(t+2\pi)} & 0 \\ 0 & e^{-\sin(t+2\pi)} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix} = X(t)$$

From (4.76), we have $\bar{A} = 0$ and

$$P(t) = e^{\bar{A}t} X^{-1}(t) = \begin{bmatrix} e^{-\cos t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}.$$

The transformation $\bar{x}(t) = P(t)x(t)$ will transform the equation into

$$\dot{\bar{x}} = 0 \cdot \bar{x} = 0$$

4.24 $\dot{x} = Ax + Bu$

$y = Cx$

Consider $\bar{x} = P(t)x = e^{-At}x$. Then

$\bar{A} = [P(t)A + \dot{P}(t)]P^{-1}(t)$

$= [e^{-At}A - e^{-At}A] e^{At} = 0$

$\bar{B} = P(t)B = e^{-At}B$

$\bar{C} = CP^{-1}(t) = Ce^{At}$

4.25 $g(t) = t^2 e^{\lambda t}$

$g(t,\tau) = g(t-\tau) = (t-\tau)^2 e^{\lambda(t-\tau)}$

$= (t^2 - 2t\tau + \tau^2) e^{\lambda t} e^{-\lambda\tau}$

$$= [e^{\lambda t} \;\; te^{\lambda t} \;\; t^2 e^{\lambda t}] \begin{bmatrix} \tau^2 e^{-\lambda\tau} \\ -2\tau e^{-\lambda\tau} \\ e^{-\lambda\tau} \end{bmatrix}$$

A time-varying realization:

$$\dot{x} = 0 \cdot x + \begin{bmatrix} t^2 e^{-\lambda t} \\ -2t e^{-\lambda t} \\ e^{-\lambda t} \end{bmatrix} u(t)$$

$$y = [e^{\lambda t} \;\; te^{\lambda t} \;\; t^2 e^{\lambda t}] x$$

$$\hat{g}(s) = \mathcal{L}[g(t)] = \frac{2}{(s-\lambda)^3}$$

$$= \frac{2}{s^3 - 3\lambda s^2 + 3\lambda^2 s - \lambda^3}$$