For negative feedback system

$a=1: \quad y(t+1) = 1 - y(t)$

Because $y(t)=0$ for $0 \le t < 1$, we have

$$y(t) = 1 \quad \text{for } 1 \le t < 2$$
$$= 0 \quad \text{for } 2 \le t < 3$$
$$= 1 \quad \text{for } 3 \le t < 4$$
$$\vdots$$

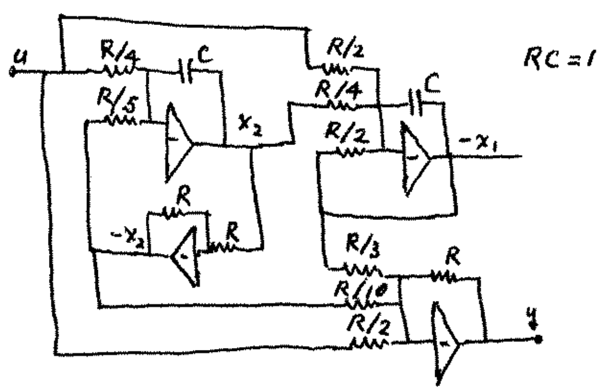
$a=0.5 \quad y(t+1) = 0.5(1-y(t))$. Thus

$$y(t) = 0 \quad \text{for } 0 \le t < 1$$
$$= 0.5 \quad \text{for } 1 \le t < 2$$
$$= 0.5(1-0.5) = 0.25 \quad \text{for } 2 \le t < 3$$
$$= 0.5(1-0.25) = 0.375 \quad \text{for } 3 \le t < 4$$
$$\vdots$$

2.14 $\dot{x}_1 = -2x_1 + 4x_2 + 2u$

$-\dot{x}_2 = -5x_2 + 4u$

$-y = -3x_1 - 10x_2 + 2u$



$RC = 1$

2.15 (a) Apply Newton's law in the tangential direction:

$$u\cos\theta - mg\sin\theta = ml\ddot{\theta}$$

Define $x_1 = \theta$, $x_2 = \dot{\theta}$. Then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{g}{l}\sin x_1 - \dfrac{u}{ml}\cos x_1 \end{cases}$$

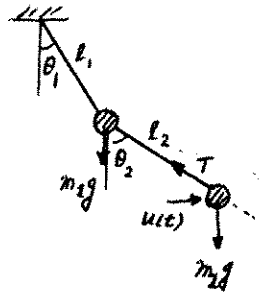
This is a nonlinear state equation.

If θ is small, then $\sin\theta \doteq \theta$, $\cos\theta \doteq 1$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/ml \end{bmatrix} u$$

This is a linearized equation.

(b)



$$T = m_2 g\cos\theta_2 + u\sin\theta_2$$
$$u\cos\theta_2 - m_2 g\sin\theta_2 = m_2 l_2 \ddot{\theta}_2$$
$$T\sin(\theta_2 - \theta_1) - m_1 g\sin\theta_1 = m_1 l_1 \ddot{\theta}_1$$

Define $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$, $x_4 = \dot{\theta}_2$

Then we have

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{-g}{l_1}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) + \frac{1}{m_1 l_1}\sin x_3 \sin(x_3 - x_1)\, u$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{-g}{l_2}\sin x_3 + \frac{\cos x_3}{m_2 l_2} u$$

This is a nonlinear equation. If $\theta_i \approx 0$, and $\theta_i\theta_j \approx 0$, then

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -g(m_1+m_2)/m_1 l_1 & 0 & m_2 g/m_1 l_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -g/l_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 l_2 \end{bmatrix} u$$

This is a linearized equation.

2.16 $m\ddot{h} = f_1 - f_2 = k_1\theta - k_2 u \qquad (1)$

$I\ddot{\theta} + b\dot{\theta} = (l_1 + l_2) f_2 - l_1 f_1 \qquad (2)$

Define $x_1 = h$, $x_2 = \dot{h}$, $x_3 = \theta$, $x_4 = \dot{\theta}$. Then

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{k_1}{m} x_3 - \dfrac{k_2}{m} u \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\dfrac{l_1 k_1}{I} x_3 - \dfrac{b}{I} x_4 + (l_1 + l_2) k_2 u \end{cases}$$

This state equation describes the airplane.

Taking the Laplace transform of (1) and (2) and assuming $I \approx 0$ yield

$$m s^2 \hat{h}(s) = k_1 \hat{\theta}(s) - k_2 \hat{u}(s)$$

$$b s \hat{\theta}(s) = (l_1 + l_2) k_2 \hat{u}(s) - k_1 l_1 \hat{\theta}(s)$$

$$\hat{\theta}(s) = \frac{(l_1+l_2) k_2}{bs + k_1 l_1} \hat{u}(s)$$

$$m s^2 \hat{h}(s) = \left(\frac{(l_1+l_2) k_1 k_2}{bs + l_1 k_1} - k_2 \right) \hat{u}(s)$$

$$\hat{g}(s) = \frac{\hat{h}(s)}{\hat{u}(s)} = \frac{k_1 k_2 l_2 - k_2 b s}{m s^2 (bs + k_1 l_1)}$$

2.17 $m\ddot{y} = -k\dot{m} - mg$

$x_1 = y, \; x_2 = \dot{y}, \; x_3 = m, \; u = \dot{m}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{-k}{x_3} u - g \\ \dot{x}_3 = u \end{cases}$$

a nonlinear equation

2.18 Following Example 2.9, we have

$$y_1 = \frac{x_1}{R_1}, \quad A_1 dx_1 = (u - y_1) dt$$

Thus
$$\begin{cases} \dot{x}_1 = \dfrac{-1}{A_1 R_1} x_1 + \dfrac{1}{A_1} u \\ y_1 = \dfrac{1}{R_1} x_1 \end{cases}$$

$$\hat{g}_1(s) = \frac{\hat{y}_1(s)}{\hat{u}(s)} = \frac{1}{R_1}\left(s + \frac{1}{A_1 R_1}\right)^{-1} \frac{1}{A_1} = \frac{1}{A_1 R_1 s + 1}$$

Similarly

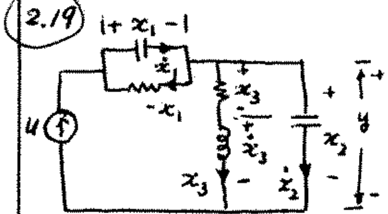
$$\hat{g}_2(s) = \frac{\hat{y}(s)}{\hat{y}_1(s)} = \frac{1}{A_2 R_2 s + 1}$$

The transfer function from u to y is

$$\hat{g}(s) = \hat{g}_1(s) \hat{g}_2(s) = \frac{1}{(A_1 R_1 s + 1)(A_2 R_2 s + 1)}.$$

The transfer function from u to y_1 in Fig. 2.13 depends on x_2 of the second tank; therefore, we must compute $\hat{g}(s) = \hat{y}(s)/\hat{u}(s)$ as a unit and do not have $\hat{g}(s) = \hat{g}_1(s)\hat{g}_2(s)$.

2.19



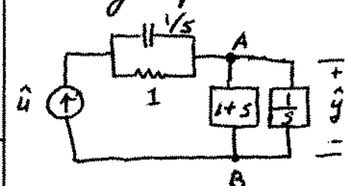
$$\dot{x}_1 = -x_1 + u$$

$$x_3 + \dot{x}_2 = u$$

$$x_3 + \dot{x}_3 = x_2 = y$$

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad y = [0 \;\; 1 \;\; 0] x$$

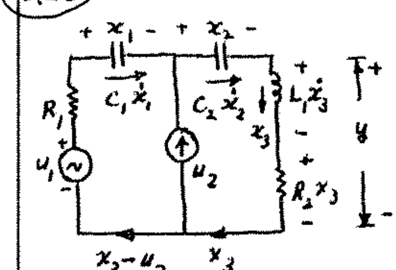
The transfer function can be computed from $c(sI - A)^{-1} b + d$ or, more easily, using impedances as



$$Z_{AB} = \frac{\frac{1}{s}(s+1)}{\frac{1}{s} + s + 1} = \frac{s+1}{s^2 + s + 1}$$

Thus we have $\hat{y}(s) = \dfrac{s+1}{s^2+s+1} \hat{u}(s)$.

2.20



The voltage across R_1 is $R_1(x_3 - u_2)$. We have

$$C_1 \dot{x}_1 = x_3 - u_2$$

$$C_2 \dot{x}_2 = x_3$$

From the outer loop, we have

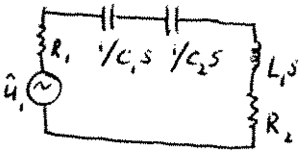
$L_1 \dot{x}_3 = -x_1 - x_2 - (x_3 - u_2) R_1 - R_2 x_3 + u_1$. Thus

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1/C_1 \\ 0 & 0 & 1/C_2 \\ \frac{-1}{L_1} & \frac{-1}{L_1} & \frac{-(R_1+R_2)}{L_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & -1/C_1 \\ 0 & 0 \\ \frac{1}{L_1} & \frac{R_1}{L_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = L_1 \dot{x}_3 + R_2 x_3 = -x_1 - x_2 - (x_3 - u_2) R_1 + u_1$$

$$= [-1 \;\; -1 \;\; -R_1] \underset{\sim}{x} + [1 \;\; R_1] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

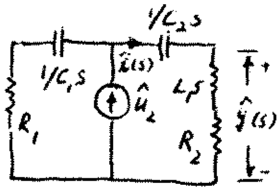
We use impedances to compute transfer functions. If $u_2 = 0$, the circuit reduces to



$$\hat{g}_1(s) = \frac{\hat{y}(s)}{\hat{u}_1(s)} = \frac{R_2 + L_1 s}{R_1 + R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} + L_1 s}$$

$$= \frac{s^2 + (R_2/L_1)s}{s^2 + \left(\frac{R_1+R_2}{L_1}\right)s + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{L_1}}$$

If $u_1 = 0$, then



$$\hat{i}(s) = \frac{\left(R_1 + \frac{1}{C_1 s}\right)\hat{u}_2(s)}{R_1 + R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s} + L_1 s}$$

$$\hat{g}_2(s) = \frac{\hat{y}(s)}{\hat{u}_2(s)} = \frac{\left(R_1 s + \frac{1}{C_1}\right)\left(s + \frac{R_2}{L_1}\right)}{s^2 + \left(\frac{R_1+R_2}{L_1}\right)s + \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\frac{1}{L_1}}$$

Therefore

$$\hat{y}(s) = \hat{g}_1(s)\hat{u}_1(s) + \hat{g}_2(s)\hat{u}_2(s)$$
$$= [\hat{g}_1(s) \;\; \hat{g}_2(s)]\begin{bmatrix}\hat{u}_1(s)\\ \hat{u}_2(s)\end{bmatrix}$$

Note that the denominator of $\hat{g}_i(s)$ is different from $\det(sI - A)$. The former has degree 2, the latter has degree 3.

2.21 Let I be the moment of inertia of the bar and mass about the hinge. Then

$$I\ddot{\theta} = l_2 u - k_1(\theta l_1)l_1 - k_2(l_2\theta - y)l_2$$
$$m_2\ddot{y} = k_2(l_2\theta - y) \qquad \text{for } \theta \text{ small}.$$

Define $x_1 = \theta$, $x_2 = \dot{\theta}$, $y = x_3$ and $x_4 = \dot{y}$. Then

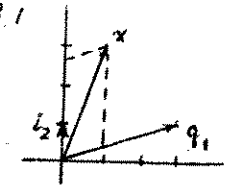
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1 l_1^2 + k_2 l_2^2)/I & 0 & k_2 l_2/I & 0 \\ 0 & 0 & 0 & 1 \\ k_2 l_2/m_2 & 0 & -k_2/m_2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{l_2}{I} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \;\; 0 \;\; 1 \;\; 0]x$$

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{k_2 l_2^2}{Im_2 s^4 + [m_2(k_1 l_1^2 + k_2 l_2^2) + Ik_2]s^2 + k_1 k_2 l_1^2}$$

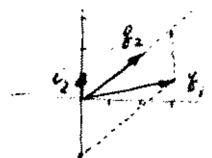
Chapter 3

3.1



Because $x = \frac{1}{3}q_1 + \left(2\frac{2}{3}\right)i_2 = [q_1 \;\; i_2]\begin{bmatrix}1/3\\ 8/3\end{bmatrix}$, the representation of x with respect to $\{q_1, i_2\}$ is $[\frac{1}{3} \;\; \frac{8}{3}]'$. Indeed, we have

$$x = \begin{bmatrix}3 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1/3\\ 8/3\end{bmatrix} = \begin{bmatrix}1\\ 3\end{bmatrix}.$$



Because $q_1 = -2i_2 + 1.5q_2 = [i_2 \;\; q_2]\begin{bmatrix}-2\\ 1.5\end{bmatrix}$, the representation of q_1 with respect to $\{i_2, q_2\}$ is $[-2 \;\; 1.5]'$. Indeed we have

$$q_1 = \begin{bmatrix}0 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}-2\\ 1.5\end{bmatrix} = \begin{bmatrix}3\\ 1\end{bmatrix}$$

3.2

$$\|x_1\|_1 = 2 + 3 + 1 = 6$$
$$\|x_1\|_2 = \sqrt{4+9+1} = \sqrt{14} = 3.74$$
$$\|x_1\|_\infty = 3$$
$$\|x_2\|_1 = 1 + 1 + 1 = 3$$
$$\|x_2\|_2 = \sqrt{1+1+1} = \sqrt{3} = 1.732$$
$$\|x_2\|_\infty = 1$$

3.3

$$q_1 = x_1/\|x_1\| = \frac{1}{3.74}\begin{bmatrix}2\\ -3\\ 1\end{bmatrix}$$

$$u_2 = x_2 - (q_1' x_2)q_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix} - \frac{0}{3.74}q_1 = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

$$q_2 = u_2/\|u_2\| = \frac{1}{1.732}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

Note that x_1 and x_2 are already orthogonal. Therefore q_1 and q_2 are their normalized vectors.

3.4 If $n > m$, AA' is symmetric and has rank m. If $m = n$, then $A'A = I_n$ and