Problem 8.34
The system is represented in the linearized form by

\[ \Delta \dot{x}(t) = a_0 \Delta x(t) + b_0 \Delta f(t) \]

where

\[ a_0 = \frac{\partial}{\partial x} (x f e^{-f}) \big|_{x_n=1, f_n=0} = f_n e^{-f_n} = 0, \quad b_0 = \frac{\partial}{\partial f} (x f e^{-f}) \big|_{x_n=1, f_n=0} = x_n e^{-f_n} - f_n x_n e^{-f_n} = 1 \]

The linearized system and its initial condition are given by

\[ \Delta \dot{x}(t) = 0 \Delta x(t) + \Delta f(t) = \Delta f(t), \quad \Delta x(0) = x(0) - x_n(0) = 0.9 - 1 = -0.1 \]

Problem 8.35
For the second-order nonlinear system

\[ \ddot{x}(t) = -2 \dot{x}(t) \cos(f(t)) - (1 + f(t)) \dot{x}(t) + f^2(t) + 1 = \mathcal{F}(x(t), \dot{x}(t), f(t)) \]

the linearized equation is given by

\[ \Delta \ddot{x}(t) + a_1 \Delta \dot{x}(t) + a_0 \Delta x(t) = b_1 \Delta f(t) + b_0 \Delta f(t) \]

where

\[ a_1 = -\frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial \dot{x}(t)} \big|_{t_n} = 2 \cos(f(t)) \big|_{f(t)=f_n(t)=0} = 2 \]

\[ a_0 = -\frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial x(t)} \big|_{t_n} = (1 + f(t)) \big|_{f(t)=f_n(t)=0} = 1 \]

\[ b_1 = \frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial f(t)} = 0 \]

\[ b_0 = \frac{\partial \mathcal{F}(x(t), \dot{x}(t), f(t))}{\partial f(t)} \big|_{x_n(t)=1, f_n(t)=0} = 2 \dot{x}(t) \sin(f(t)) - x(t) \big|_{x_n(t)=1, f_n(t)=0} = -1 \]

The initial conditions for the linearized system are

\[ \Delta x(0) = x(0) - x_n(0) = 1.1 - 1 = 0.1, \quad \Delta \dot{x}(0) = \dot{x}(0) - \dot{x}_n(0) = 0.1 - 0 = 0.1 \]

Hence, we have the following linearized second-order system

\[ \Delta \ddot{x}(t) + 2 \Delta \dot{x}(t) + \Delta x(t) = -\Delta f(t), \quad \Delta x(0) = 0.1, \quad \Delta x_n(0) = 0.1 \]

The corresponding state space form is given by

\[ \begin{bmatrix} \Delta \dot{x}_1(t) \\ \Delta \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta f(t) \]

\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} = \Delta x_1(t) \]

The linearized system response due to \( \Delta f(t) = e^{-2t} \) can be obtained via the use of the Laplace transform

\[ (s^2 \Delta X(s) - s \Delta x(0) - \Delta \dot{x}(0)) + 2(s \Delta X(s) - \Delta x(0)) + \Delta X(s) = -\Delta F(s) \]

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which leads to
\[
\Delta X(s) = -\frac{1}{(s+1)^2(s+2)} + \frac{0.1s + 0.3}{(s+1)^2} = \frac{(s+2)(0.1s + 0.3) - 1}{(s+1)^2(s+2)} = \frac{1.1}{s+1} - \frac{0.8}{(s+1)^2} - \frac{1}{s+2}
\]

The inverse Laplace transform implies
\[
\Delta x(t) = \mathcal{L}^{-1}\{\Delta X(s)\} = (1.1e^{-t} - 0.8te^{-t} - e^{-2t})u(t)
\]

**Problem 8.36**
The nominal values for the state variable \(x(t)\) are obtained for the nominal system input \(f_n(t) = 1\) for the nominal system initial conditions, that is, by solving the following differential equation
\[
\ddot{x}_n(t) + 2\dot{x}_n(t) = 2, \quad x_n(0) = 0, \quad \dot{x}_n(0) = 1.1
\]

Applying the Laplace transform we have
\[
(s^2X_n(s) - sx_n(0) - \dot{x}_n(0)) + 2(sX_n(s) - x_n(0)) = \frac{2}{s}
\]

The complex domain solution is given by
\[
X_n(s) = \frac{1.1s + 2}{s^2(s+2)} = \frac{0.05}{s} + \frac{1}{s^2} - \frac{0.05}{s+2}
\]

so that in the time domain we have
\[
x_n(t) = 0.05 + t - 0.05e^{-2t}, \quad t > 0 \quad \Rightarrow \quad \dot{x}_n(t) = 1 + 0.1e^{-2t}
\]

For the given second-order nonlinear system
\[
\ddot{x}(t) = -2\dot{x}(t)f(t) - (1 - f(t))x(t) + f^2(t) + 1 = F(x(t), \dot{x}(t), f(t))
\]

the linearized equation is given by
\[
\Delta \ddot{x}(t) + a_1 \Delta \dot{x}(t) + a_0 \Delta x(t) = b_1 \Delta f(t) + b_0 \Delta f(t)
\]

where
\[
a_1 = -\left. \frac{\partial F(x(t), \dot{x}(t), f(t))}{\partial \dot{x}(t)} \right|_{x_n(t) = f_n(t) = 1} = 2
\]
\[
a_0 = -\left. \frac{\partial F(x(t), \dot{x}(t), f(t))}{\partial x(t)} \right|_{x_n(t) = f_n(t) = 1} = 0
\]
\[
b_1 = \left. \frac{\partial F(x(t), \dot{x}(t), f(t))}{\partial f(t)} \right|_{x_n(t) = f_n(t) = 1} = 0
\]
\[
b_0 = \left. \frac{\partial F(x(t), \dot{x}(t), f(t))}{\partial f(t)} \right|_{x_n(t) = f_n(t) = 1} = -2\dot{x}(t) + x(t) + 2f(t) = 0.05 + t - 0.25e^{-2t} = b_0(t)
\]

The linearized second-order system is given by
\[
\Delta \ddot{x}(t) + 2\Delta \dot{x}(t) = (0.05 + t - 0.25e^{-t}) \Delta f(t)
\]
with the initial conditions
\[ \Delta x(0) = x(0) - x_n(0) = 0, \quad \Delta \dot{x}(0) = \dot{x}(0) - \dot{x}_n(0) = 1 - 1.1 = -0.1 \]

Choosing the state space variables as \( x_1(t) = \Delta x(t) \) and \( x_2(t) = \Delta \dot{x}(t) \) with \( f(t) = \Delta f(t) \) and \( y(t) = x_1(t) \), we obtain the following state space form
\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} = 
\begin{bmatrix}
0 & 1 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} + 
\begin{bmatrix}
0 \\
b_0(t)
\end{bmatrix} f(t)
\]
\[
y(t) = [1 \ 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\]

Problem 8.37
Represent first the pendulum equation as
\[
\ddot{\theta}(t) = -\frac{mgL}{l} \sin(\theta(t)) + \frac{l}{I} f(t) = F(\theta(t), f(t))
\]
The linearized pendulum equation is given by
\[
\Delta \ddot{\theta}(t) + a_1 \Delta \dot{\theta}(t) + a_0 \Delta \theta(t) = b_1 \Delta \dot{f}(t) + b_0 \Delta f(t)
\]
where
\[
a_1 = -\frac{\partial F(\theta(t), f(t))}{\partial \theta(t)} = 0, \quad a_0 = \frac{\partial F(\theta(t), f(t))}{\partial \theta(t)} \bigg|_{\theta(t) = 0, f(t) = 0} = -\frac{mgL}{l} \cos(\theta(t)) \bigg|_{\theta(t) = 0, f(t) = 0} = -\frac{mgL}{l},
\]
\[
b_1 = \frac{\partial F(\theta(t), f(t))}{\partial f(t)} = 0, \quad b_0 = \frac{\partial F(\theta(t), f(t))}{\partial f(t)} \bigg|_{\theta(t) = 0, f(t) = 0} = \frac{l}{I}
\]
leading to
\[
\Delta \ddot{\theta}(t) - \frac{mgL}{l} \Delta \dot{\theta}(t) = \frac{l}{I} \Delta f(t)
\]
The initial conditions for the linearized system are given by
\[
\Delta \theta(0) = \theta(0) - \theta_n(0) = \theta_0 - 0 = \theta_0, \quad \Delta \dot{\theta}(0) = \dot{\theta}(0) - \dot{\theta}_n(0) = \omega_0 - 0 = \omega_0
\]

Problem 8.38
The linearized system is given by
\[
\Delta \dot{x}(t) = 
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\Delta x(t)
\]
with
\[
a_{11} = \frac{\partial (x_1(t)x_2(t) - \sin(x_1(t)))}{\partial x_1(t)} \bigg|_{(x_{1a}, x_{2a}, x_{3a}) = (0, 1, 1)} = x_2(t) - \cos(x_1(t)) \big|_{(x_{1a}, x_{2a}, x_{3a}) = (0, 1, 1)} = 0
\]
\[
a_{12} = \frac{\partial (x_1(t)x_2(t) - \sin(x_1(t)))}{\partial x_2(t)} \bigg|_{(x_{1a}, x_{2a}, x_{3a}) = (0, 1, 1)} = x_1(t) \big|_{(x_{1a}, x_{2a}, x_{3a}) = (0, 1, 1)} = 0
\]
\[
a_{13} = \frac{\partial (x_1(t)x_2(t) - \sin(x_1(t)))}{\partial x_3(t)} \bigg|_{(x_{1a}, x_{2a}, x_{3a}) = (0, 1, 1)} = 0
\]
\[ a_{21} = \frac{\partial (1 - 3x_2(t)e^{-x_1(t)})}{\partial x_1(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = 3x_2(t)e^{-x_1(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = 3 \]

\[ a_{22} = \frac{\partial (1 - 3x_2(t)e^{-x_1(t)})}{\partial x_2(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = -3e^{-x_1(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = -3 \]

\[ a_{23} = \frac{\partial (1 - 3x_2(t)e^{-x_1(t)})}{\partial x_3(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = 0 \]

\[ a_{31} = \frac{\partial (x_1(t)x_2(t)x_3(t))}{\partial x_1(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = x_2(t)x_3(t) \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = 1 \]

\[ a_{32} = \frac{\partial (x_1(t)x_2(t)x_3(t))}{\partial x_2(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = x_1(t)x_3(t) \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = 0 \]

\[ a_{33} = \frac{\partial (x_1(t)x_2(t)x_3(t))}{\partial x_3(t)} \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = x_1(t)x_2(t) \bigg|_{(x_{1n}, x_{2n}, x_{3n}) = (0, 1, 1)} = 0 \]

Hence, the linearized system is given by

\[ \Delta \dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 3 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Delta x(t) \]

**Problem 3.39**

For the given nonlinear system

\[ \dot{x}_1(t) = f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)} \]

\[ \dot{x}_2(t) = x_1(t) \sin(f(t)) - \sin(x_2(t)) \]

\[ y(t) = \sin(x_1(t)) \]

the linearized system matrices are given by

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad C = [c_1 \ c_2], \quad D = 0 \]

where

\[ a_{11} = \frac{\partial (f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)})}{\partial x_1(t)} = \frac{f(t)}{x_1(t)}, \quad a_{12} = \frac{\partial (f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)})}{\partial x_2(t)} = e^{-f(t)} \]

\[ a_{21} = \frac{\partial (x_1(t) \sin(f(t)) - \sin(x_2(t)))}{\partial x_1(t)} = \sin(f(t)), \quad a_{22} = \frac{\partial (x_1(t) \sin(f(t)) - \sin(x_2(t)))}{\partial x_2(t)} = -\cos(x_2(t)) \]

\[ b_1 = \frac{\partial (f(t) \ln(x_1(t)) + x_2(t)e^{-f(t)})}{\partial f(t)} = \ln(x_1(t)) - x_2(t)e^{-f(t)} \]

\[ b_2 = \frac{\partial (x_1(t) \sin(f(t)) - \sin(x_2(t)))}{\partial f(t)} = x_1(t) \cos(f(t)) \]

\[ c_1 = \frac{\partial (\sin(x_1(t)))}{\partial x_1(t)} = \cos(x_1(t)), \quad c_2 = \frac{\partial (\sin(x_1(t)))}{\partial x_2(t)} = 0 \]
The state space linearized model, with the coefficients evaluated at the nominal points \( x_{1n}, x_{2n} \) and \( f_n \) is given by

\[
\begin{bmatrix}
\Delta \dot{x}_1(t) \\
\Delta \dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix}
\frac{f_1}{x_{1n}^2} & e^{-f_n} \\
\frac{1}{\sin(f_n)} & -\cos(x_{2n})
\end{bmatrix} \begin{bmatrix}
\Delta x_1(t) \\
\Delta x_2(t)
\end{bmatrix} + \begin{bmatrix}
\ln(x_{1n}) - x_{2n}e^{-f_n} \\
x_{1n} \cos(f_n)
\end{bmatrix} \Delta f(t)
\]

**Problem 8.40**

For the Volterra predator-prey mathematical model

\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) + x_1(t)x_2(t) \\
\dot{x}_2(t) &= x_2(t) - x_1(t)x_2(t)
\end{align*}
\]

the linearized system matrices are given by

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad C = [c_1 \ c_2], \quad D = 0
\]

where

\[
\begin{align*}
a_{11} &= \frac{\partial(-x_1(t) + x_1(t)x_2(t))}{\partial x_1(t)} = x_2(t) - 1, \\
a_{12} &= \frac{\partial(-x_1(t) + x_1(t)x_2(t))}{\partial x_2(t)} = x_1(t) \\
a_{21} &= \frac{\partial(x_2(t) - x_1(t)x_2(t))}{\partial x_1(t)} = x_2(t), \\
a_{22} &= \frac{\partial(x_2(t) - x_1(t)x_2(t))}{\partial x_2(t)} = 1 - x_1(t)
\end{align*}
\]

The state space linearized model, with the coefficients evaluated at the nominal points \( x_{1n} = 0 \) and \( x_{2n} = 0 \) is given by

\[
\begin{bmatrix}
\Delta \dot{x}_1(t) \\
\Delta \dot{x}_2(t)
\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix}
\Delta x_1(t) \\
\Delta x_2(t)
\end{bmatrix}
\]

**Problem 8.41**

Introducing the following change of the variables \( x_1(t) = \theta_i(t), \ x_2(t) = \dot{\theta}_i(t), \ x_3(t) = \theta_m(t), \ x_4(t) = \dot{\theta}_m(t) \), we obtain a system of four first-order differential equations

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{k}{J_i}x_1(t) - \frac{B_i}{J_i}x_2(t) + \frac{k}{J_i}kx_3(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= \frac{k}{J_m}x_1(t) - \frac{k}{J_m}x_3(t) - \frac{B_m}{J_m}x_4(t) + f(t)
\end{align*}
\]

which can be put in the state space matrix form as

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t) \\
\dot{x}_4(t)
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\frac{k}{J_i} & -\frac{B_i}{J_i} & \frac{k}{J_i} & 0 \\
0 & 0 & 0 & 1 \\
\frac{k}{J_m} & 0 & -\frac{k}{J_m} & -\frac{B_m}{J_m}
\end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ f(t) \end{bmatrix}
\]

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