$\boxed{3.9}$ BALANCED REALIZATIONS

$\boxed{\text{Lemma } \boxed{3.20}}$ Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a state space realization of a transfer function $G(s)$. Suppose that there exists a symmetric matrix

$$P = P^T = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with P_1 nonsingular such that

$$AP + PA^T + BB^T = 0$$

Let the realization (A, B, C, D) be partitioned compatibly with P as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

then

$$\begin{bmatrix} A_{11} & B_1 \\ C_1 & D_1 \end{bmatrix}$$

is also a realization of $G(s)$. Even more, (A_{11}, B_1) is controllable if A_{11} is stable

Proof of Lemma 3.20

$$AP + PA^T + BB^T = \begin{bmatrix} A_{11}P_1 + P_1 A_{11}^T + B_1 B_1^T & P_1 A_{21} + B_1 B_2 \\ A_{21} P_1 + B_2 B_1 & B_2 B_2 \end{bmatrix} = 0$$

$(2,2) \Rightarrow B_2 = 0$

$(2,1) \Rightarrow A_{21} = 0$ (since P_1 is nonsingular)

$$\Rightarrow \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right] = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$$ since A_{22} is uncontrollable part

From the Lyapunov controllability Theorem we have that A_{11} stable $\Rightarrow$ (A_{11}, B_1) is controllable.

$\boxed{\text{Lemma 3.21}}$ Let $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a realization of $G(s)$.
Assume that there exists

$$Q = Q^T = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with Q_1 nonsingular such that

$$QA + QA^T + C^T C = 0$$

Let us partition the realization (A, B, C, D) compatibly with Q as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

then $\begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$ is also a realization of $G(s)$.

Moreover, (C_1, A_{11}) is observable if A_{11} is stable.

From Lemmas 3.20 and 3.21 we have the following algorithm for eliminating uncontrollable and unobservable modes:

1) Let $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a stable realization

2) Compute the controllability Grammian $P \geq 0$ from

$$AP + PA^T + BB^T = 0$$

3) Diagonalize P to get

$$P = [U_1 \; U_2] \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} [U_1 \; U_2]^T, \quad \Lambda_1 > 0, \; [U_1 \; U_2] \text{ unitary matrix}$$

4) then

$$G(s) = \begin{bmatrix} U_1^{T} A U_1 & U_1^T B \\ C U_1 & D \end{bmatrix}$$

is a controllable realization.

Similarly we can eliminate non-observable modes.

NOTE THAT WE MAY REMOVE MODES THAT ARE <u>BOTH</u> WEAKLY CONTROLLABLE AND WEAKLY OBSERVABLE

$\Rightarrow$ order reduction of the original system

(Example, p. 74)

$$G(s) = \frac{3s+18}{s^2+3s+18} \quad \Rightarrow \quad P = \begin{bmatrix} 0.5 & 0 \\ 0 & \alpha^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 1/\alpha^2 \end{bmatrix}$$

for α very small we have one weakly controllable mode.
If we remove it $\Rightarrow$ $G_r(s) = \frac{-1}{s+1}$ $\Rightarrow$ very different response from the original system.
The reason for this error is that the second mode is also <u>strongly observable</u>.

In MATLAB — balancing transformation

[ab, bb, cb, g, T] = balreal (a, b, c)   Also dbalreal for discrete-time systems

diagonal contr & obs. grammian

Goal to Balance Controllability and Observability Grammians

$$AP + PA^T + BB^T = 0 \Rightarrow P > 0 \quad (A \text{ asymptotically stable and } (A, B) \text{ controllable})$$

$$A^T Q + QA + C^T C = 0 \Rightarrow Q > 0 \quad (A \text{ asymptotically stable and } (A, C) \text{ observable})$$

In general $P \geq 0$, $Q \geq 0$.

By using a state transformation $\hat{x} = Tx$ we get

$$\left.\begin{array}{l} \hat{P} = TPT^T \\ \hat{Q} = T^{-T} Q T^{-1} \end{array}\right\} \Rightarrow \hat{P}\hat{Q} = TPQT^{-1}$$

Theorem 7.5 in Chen's book 1999. 183

$\Rightarrow$ The eigenvalues of the product of the Grammians are invariant under state transformation.

Take for the columns of T^{-1} the eigenvectors of PQ then

$$PQ = T^{-1} \Lambda T \text{ with } \begin{cases} \lambda_i = \lambda(PQ) \\ \Lambda = \text{diag}(\lambda_1, \lambda_2, -, \lambda_n) \geq 0 \\ \text{note that } P \geq 0, Q \geq 0 \end{cases}$$

The eigenvectors can be chosen such that
(not unique in general)

THE CONTROLLABILITY AND OBSERVABILITY GRAMMIANS ARE EQUAL, that is

$$\begin{array}{l} \hat{P} = TPT^T = \Sigma \\ \hat{Q} = T^{-T} Q T^{-1} = \Sigma \end{array} \Rightarrow \boxed{\begin{array}{l} \text{BALANCED} \\ \text{REALIZATION} \\ \hat{P} = \hat{Q} = \Sigma \end{array}}$$

with

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, -, \sigma_n) \quad \text{and} \quad \Sigma^2 = \Lambda$$

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$ are known as Hankel singular values

Corollary 3.24

If a realization is not minimal then for any stable system $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ there exists a nonsingular T such that

$$G = \left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$$

has controllability and observability Grammians given by

$$P = \begin{bmatrix} \Sigma_1 & & \\ & \Sigma_2 & \\ & & 0 \\ & & & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} \Sigma_1 & & \\ & 0 & \\ & & \Sigma_3 \\ & & & 0 \end{bmatrix} \qquad \begin{array}{l} \Sigma_1 > 0 \\ \Sigma_2 > 0 \\ \Sigma_3 > 0 \end{array}$$

For a minimal realization (in which uncontrollable and unobservable modes are eliminated) we can get a balanced realization by using the following algorithm:

(1) Compute the controllability and observability Grammians $P > 0$ and $Q > 0$

(2) Find R such that $R^T R = P$

(3) Diagonalize RQR^T to get $RQR^T = U\Sigma^2 U^T$

(4) Let $T^{-1} = R^T U \Sigma^{-1/2}$ then

$$TPT^T = T^{-T} Q T^{-1} = \Sigma$$

with $G = \begin{bmatrix} TAT^{-1} & TB \\ CT^{-1} & D \end{bmatrix}$

being balanced.

Now

$$\Sigma = \left[\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \sigma_2 & & & 0 & \\ & & \sigma_r & & & \\ \hline & & & \sigma_{r+1} & & \\ & 0 & & & \ddots & \\ & & & & & \sigma_n \end{array}\right]$$

If $\sigma_r >> \sigma_{r+1}$ the remaining modes are weakly controllable and weakly observable. Thus, they can be neglected

Chapter 7 $\Leftarrow$ system order reduction $\Leftarrow$

We will do this chapter ...

Chapter 7
Model Reduction by Balanced Truncation

We have seen in Section 3.9 (Balancing Realization) that the system is balanced if the controllability and observability Grammians are equal, that is

$$A P + P A^T + B B^T = 0 \qquad P = Q \Rightarrow \text{balancing}$$
$$A^T Q + Q A + C^T C = 0$$

In general $P \neq Q$ and we need a state transformation to make them equal.

For a <u>minimal realization</u> (in which both uncontrollable and unobservable modes are removed) the balancing transformation algorithm is given on page 39 of the textbook

let
$$P = Q = \Sigma \qquad \text{then} \qquad \Sigma = \left(\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \sigma_2 & & & 0 & \\ & & \sigma_r & & & \\ \hline & & & \sigma_{r+1} & & \\ & 0 & & & \ddots & \\ & & & & & \sigma_n \end{array}\right)$$

$\sigma_1 \geq \sigma_2 \geq \ldots \sigma_r \geq \sigma_{r+1} \geq \ldots \geq \sigma_n \geq 0$
are known as <u>Hankel singular values</u>

If $\sigma_r > \sigma_{r+1}$ then

$$\Sigma_r = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_r \end{bmatrix} \quad \text{is an approximation for } \Sigma$$

and
$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} \overset{r\times r}{A_{11}} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] \approx \left[\begin{array}{c|c} \overset{r\times r}{A_{11}} & B_1 \\ \hline C_1 & D \end{array}\right] = G_r(s)$$

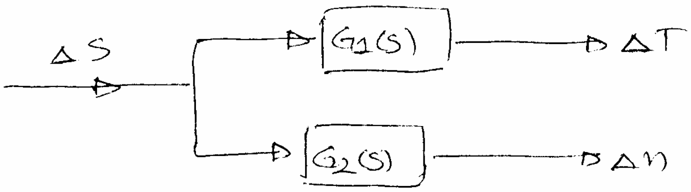
approximated by

Example: Fusion Reactor Control (Plummer, 1995 paper) 7

Linearized model

$$\frac{\Delta T}{\Delta S} = \frac{-3.81\cdot 10^{-12} s + 24.28\, 10^{-14}}{11.688 s^2 + 5.735 s + 1} \Rightarrow \frac{\Delta T}{\Delta S} = G_1(s)$$

$$\frac{\Delta n}{\Delta S} = \frac{s + 0.258}{s^2 + 0.4906 s + 0.085} \Rightarrow \frac{\Delta n}{\Delta S} = G_2(s)$$



two uncoupled systems with a common input

Let us apply the balancing order reduction technique to $G_2(s)$. We use MATLAB to solve this problem

Problem is solved in MATLAB 4.2. Upper numbers are corrected values obtained by MATLAB 5

```
>> num = [1  0.258]
>> den = [1  0.4906  0.085]
>> [A,B,C,D] = tf2ss(num,den)
```

$$\Rightarrow A = \begin{bmatrix} -0.49069 & -0.8855 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \;\; 0.258], \quad D = 0$$

% controllability and observability gramians

```
>> P = lyap(A, B*B')
>> K = lyap(A', C'*C)
```

$$\Rightarrow P = \begin{bmatrix} 1.0192 & 0 \\ 0 & 11.92 \end{bmatrix}, \quad K = \begin{bmatrix} 1.8126 & 0.3893 \\ 0.3893 & 0.088 \end{bmatrix}$$

note that the first mode is weakly controllable and strongly observable. The second mod is the other way around

($\sigma(P)$ and $\sigma(K)$ are controllability and observability margins)

```
>> [Ab, Bb, Cb, sigma, trans] = balreal(A,B,C)
```

$$\Rightarrow A_b = \begin{bmatrix} -0.3146 & -0.1736 \\ 0.1736 & -0.176 \end{bmatrix}, \quad B_b = \begin{bmatrix} -1.0317 \\ 0.2538 \end{bmatrix}, \quad C_b = [-1.0317 \;\; -0.2538]$$

$$T = \begin{bmatrix} -0.6215 & 1.4134 \\ -1.5899 & -6.4622 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1.6918 & 0 \\ 0 & 0.1830 \end{bmatrix}$$

nice separation of Hankel's singular

Due to the fact that

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 1.6918 & 0 \\ 0 & 0.1830 \end{bmatrix} \qquad \sigma_1 >> \sigma_2$$

we can approximate this system by

$$\left.\begin{array}{lll} >> A_r = A_b(1,1) & \Rightarrow & A_r = -0.3146 \\ >> B_r = B_b(1,1) & \Rightarrow & B_r = -1.0317 \\ >> C_r = C_b(1,1) & & C_r = -1.0317 \end{array}\right\} \Rightarrow \begin{array}{l} \dot{x}_r = -0.3146 x_r - 1.0317 u \\ y_r = -1.0317 \end{array}$$

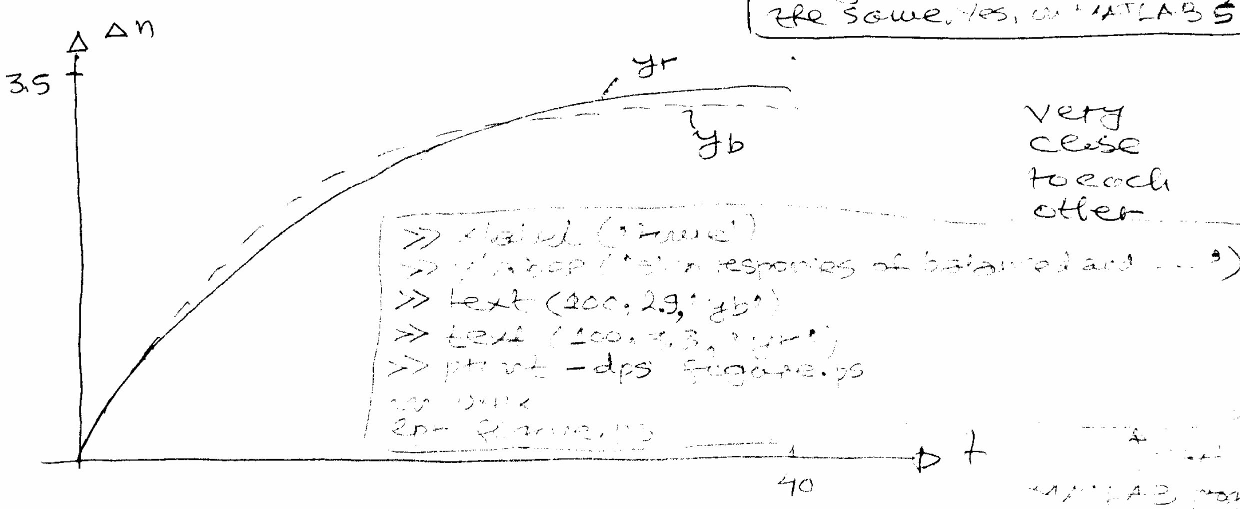
$u \approx \Delta S$

$y_r = \Delta n$

If we plot the step responses of the original second order system and the reduced first order system we get

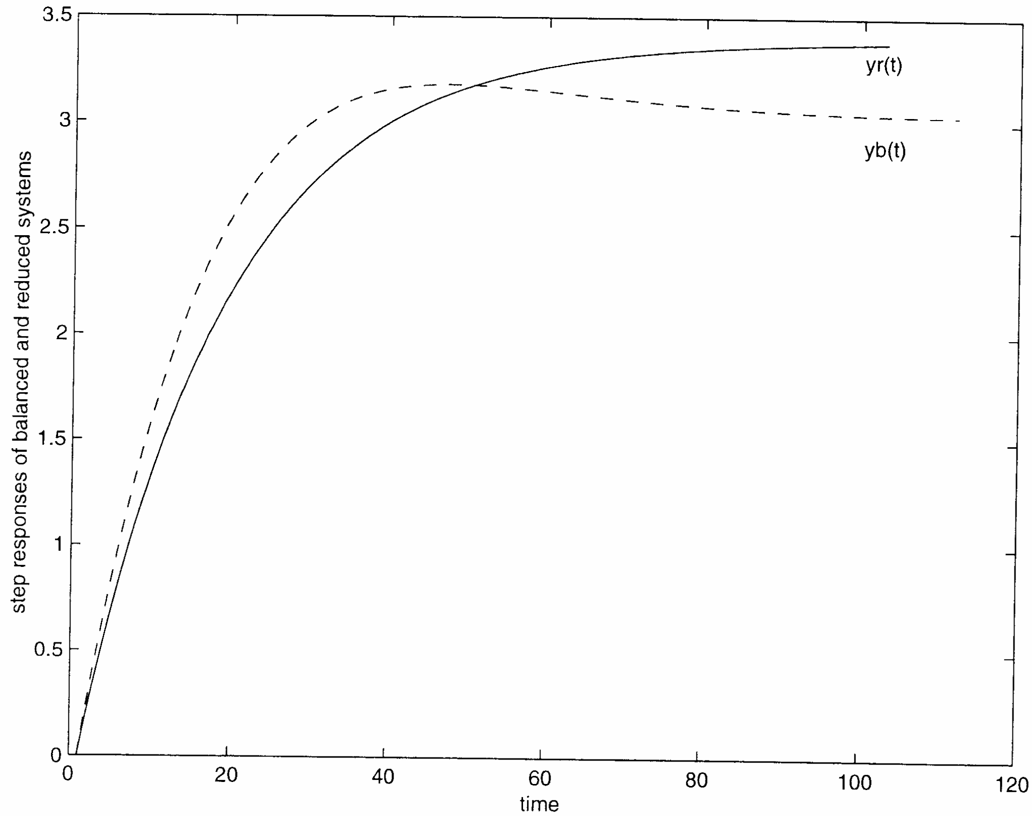
```
>> Db = 0
>> Dr = 0
>> yb = step(Ab, Bb, Cb, Db)
>> yr = step(Ar, Br, Cr, Dr)
>> plot(yb)
>> hold
>> plot(yr, '--')
```

Note that MATLAB's error
$\lambda(A) = -0.2453 \pm j0.1576$
$\lambda(A_b) = -0.2453 \pm j0.1592$
they should be exactly the same. Yes, MATLAB's

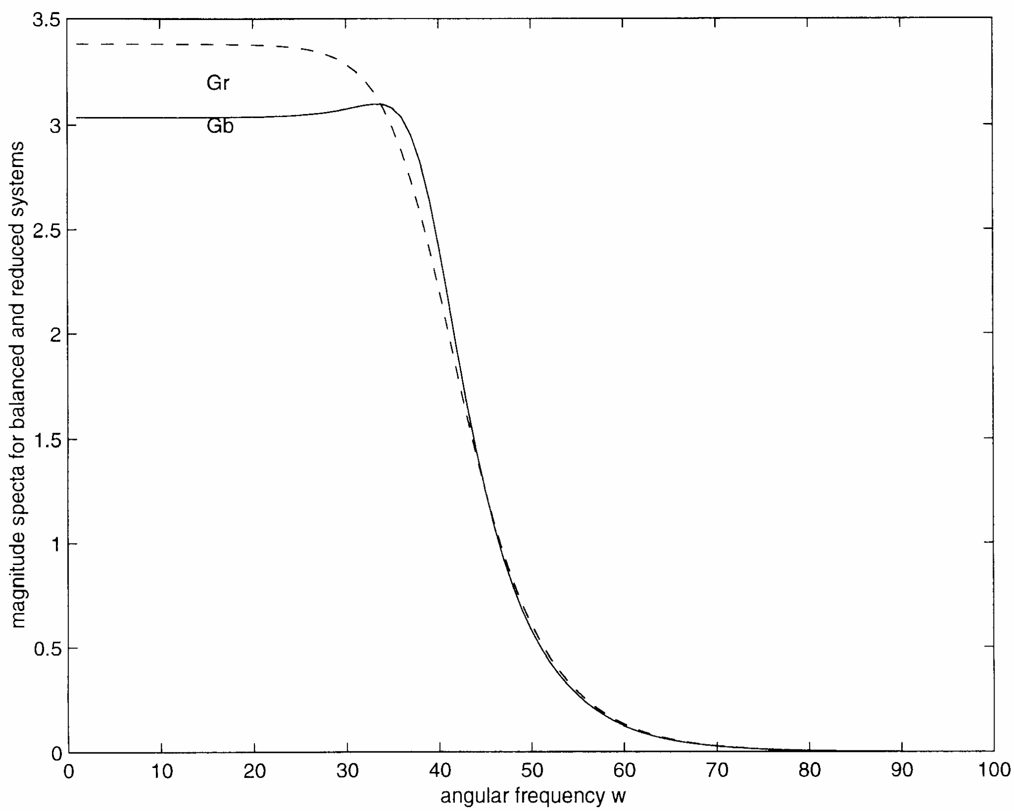


It is interesting that balancing realization works very well on the second order system. Imagine its importance in reducing high-order systems.

Note that balancing changes a lot of the original matrix A.



yr(t)
yb(t)
step responses of balanced and reduced systems
time



Gr
Gb
magnitude specta for balanced and reduced systems
angular frequency w

Some Important Results from Chapter 7

Let $G(s)$ be stable and balanced

$$G(s) = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}$$ A stable, (A,B) controllable, (A,C) observable

$$A_b\Sigma + \Sigma A_b^T + B_b B_b^T = 0$$
$$A_b^T\Sigma + \Sigma A_b + C_b^T C_b = 0$$

Partition $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$ and $G = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$

$\boxed{\text{Theorem 7.1}}$ If Σ_1 and Σ_2 have no diagonal entries in common then both subsystems (A_{ii}, B_i, C_i) are asymptotically stable

No proof. In the fusion reactor example $A_{11} = -0.3146$
$A_{22} = -0.176$

$\boxed{\text{Theorem 7.2}}$ Let $\Sigma_1 = \text{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2}, \dots, \sigma_r I_{s_r})$

and $\Sigma_2 = \text{diag}(\sigma_{r+1} I_{s_{r+1}}, \sigma_{r+2} I_{s_{r+2}}, \dots, \sigma_N I_{s_N})$

and $\sigma_1 > \sigma_2 > \dots > \sigma_r > \sigma_{r+1} > \dots > \sigma_N$

$s_1 + s_2 + \dots + s_N = n$

then **the reduced system**

$$G_r(s) = \begin{bmatrix} A_{11} & B_1 \\ C_1 & D \end{bmatrix}$$

is balanced and asymptotically stable. Furthermore

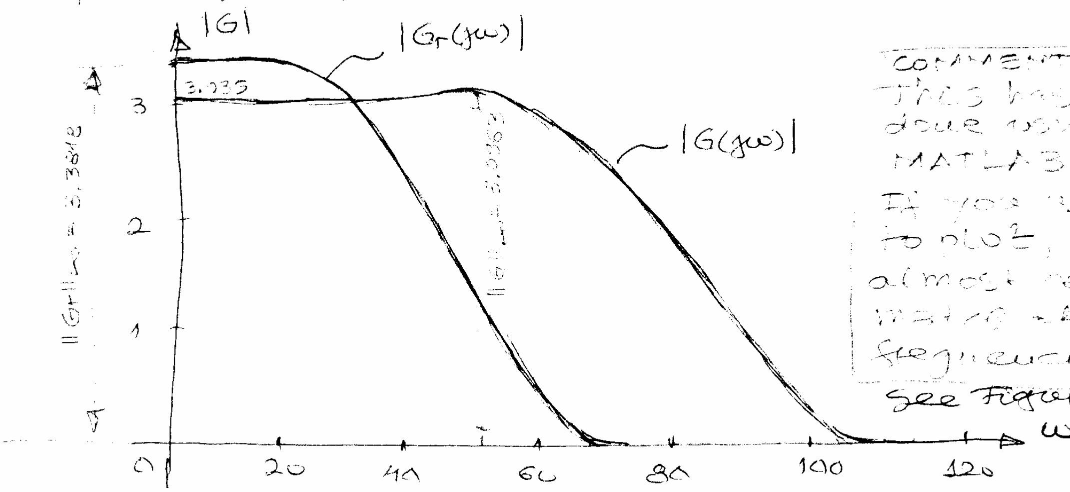
$$\|G(s) - G_r(s)\|_\infty \le 2(\sigma_{r+1} + \sigma_{r+2} + \dots + \sigma_N)$$

FR-Example $\|G(s) - G_r(s)\|_\infty \le 2 \cdot \sigma_2 = 2 \cdot 0.1830 = 0.366$

```
>> numr = Br * Cr          => 1.0633
>> denr = [1  0.3146]

>> [Re,Im] = nyquist(num,den)

>> [Rer,Imr] = nyquist(numr,denr)

>> G = Re + j*Im            => |G| = Ga
>> Ga = abs(G)
>> Gr = Rer + j*Imr         => |Gr| = Gra
>> Gra = abs(Gr)
>> hinG = max(Ga)           => ||G||∞ = 3.0963
>> hinGr = max(Gra)         => ||Gr||∞ = 3.3818

>> plot(Ga)
>> hold
>> plot(Gra,'--')
```

note that for the single input single output systems

$\|G\|_\infty = \max_\omega |G(j\omega)|$



COMMENT:
This has been done using MATLAB 4.2
If you use MATLAB 5 to plot, you get almost perfect match at high frequencies

See Figure on page 12

According to theorem 7.2 we have

$$\left\| \frac{s + 0.258}{s^2 + 0.4906s + 0.085} - \frac{Br \cdot Cr}{s + Ar} \right\|_\infty \le 2\sigma_2 = 2 \cdot 0.1830 = 0.366$$

In order to improve the approximation in frequency bands of interest we can use the frequency weighted balanced model reduction

$$\| W_o(s) [G(s) - G_r(s)] W_i(s) \|_\infty$$

W_o, $W_i \in RH_\infty$ (stable matrices) are known as the output and input weighting matrices. (Section 7.2)

Another interesting theorem related to the Hankel singular values and H_∞ norm is stated in Chapter 4.

$\boxed{\text{THEOREM 4.5}}$

$$\sigma_1 \le \| G(s) \|_\infty \le 2 \sum_{i=1}^{n} \sigma_i$$

Also

$$\| G(s) \|_\infty \le \int_0^\infty \| g(t) \|_2 dt \le 2 \sum_{i=1}^{n} \sigma_i$$

FR Example

$$\| G(s) \|_\infty \le 2 (1.6918 + 0.1830) = 3.7496$$

for the reduced system

$$A_{11} (\sigma_{1r}) + A_{11} (\sigma_{1r}) + B_1 \cdot B_1 = 0 \implies \sigma_{1r} = -\frac{B_1^2}{2 A_{11}} = \frac{(1.0317)^2}{2 \times (0.3146)} = 1.692$$

$$\implies \| G_r(s) \|_\infty \le 2 \cdot \sigma_{1r} = 2 \times 1.692 = 3.384$$

(pretty good upper bound the actual value is $\| G_r(s) \|_\infty = 3.3818$

Note that MATLAB has the ~~control~~ tool box for ROBUST CONTROL

```
>> [Gb, sig] = sysbal(G)
```

$\implies$ Gb = balanced, sig = Hankel Singular values

```
>> Gr = strunc(Gb, 2)
```

does truncation to the second order system