Essence of the Root Locus Technique

In this chapter we study a method for finding locations of system poles. The method is presented for a very general set-up, namely for the case when the closed-loop system poles are functions of an unknown parameter. In most cases the parameter of interest is the system static gain $\boldsymbol{K}$ satisfying $-\infty < \boldsymbol{K} < +\infty$. However, any other unknown and variable system parameter affecting pole locations can be used instead of $\boldsymbol{K}$. The method is known as the root locus technique for solving polynomial equations with constant or variable parameters. It was originally presented in Ewans (1948, 1950).

The importance of the root locus method for control system theory lies in the fact that the location of the system poles determines the system stability and the system transient response. In some cases, the desired control system performance can be obtained by changing only the system static gain $\boldsymbol{K}$. It is known from that the choice of the system static gain determines the errors of the system steady state response in the sense that a bigger value for $\boldsymbol{K}$ implies smaller values for steady state errors (assuming that the system remains asymptotically stable). However, changing $\boldsymbol{K}$ causes the system transient response parameters also to change. If one is not able to achieve all the control system requirements by changing only the static gain $\boldsymbol{K}$ (the essence of the root locus method), one has to design a dynamic compensator (controller). The question of designing dynamic compensators by using the root locus method will be addressed in detail in Chapter 8.

The root locus technique allows adjustment of the system poles by changing the feedback system static gain. The closed-loop feedback system, in general, can be represented by a block diagram as given in Figure 7.1.

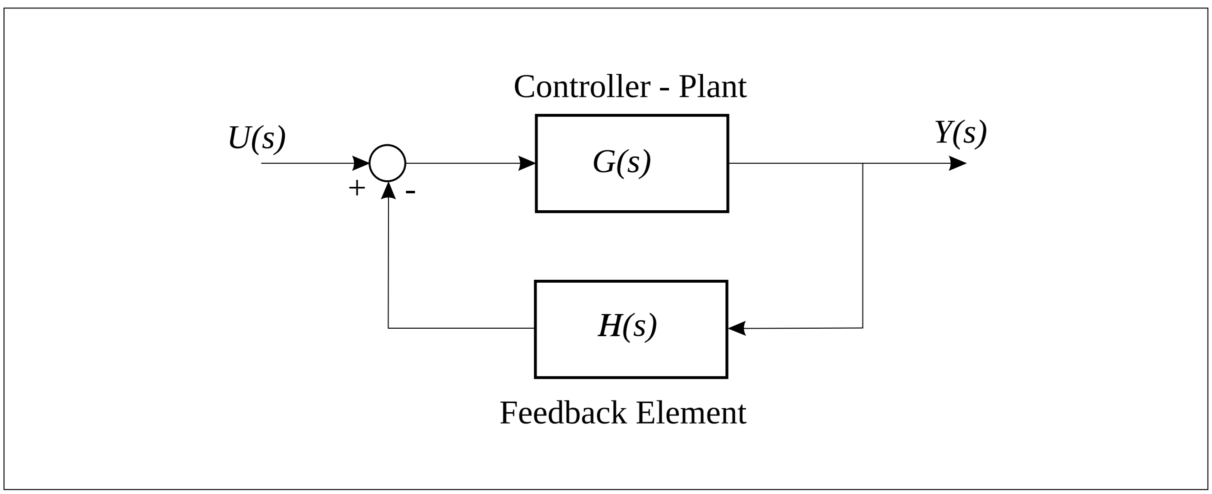

Figure 7.1: Block diagram of a feedback control system

The characteristic equation of the above closed-loop system is given by

$$1 + G(s)H(s) = 0$$

which can be written as

$$1 + KG_1(s)H_1(s) = 0$$

where K represents all static gains present in the loop. Usually the static gains come solely from the controller-plant element. That is, the plant $G_1(s)$ is controlled by changing the static gain K, which can be represented by a block diagram given in Figure 7.2.

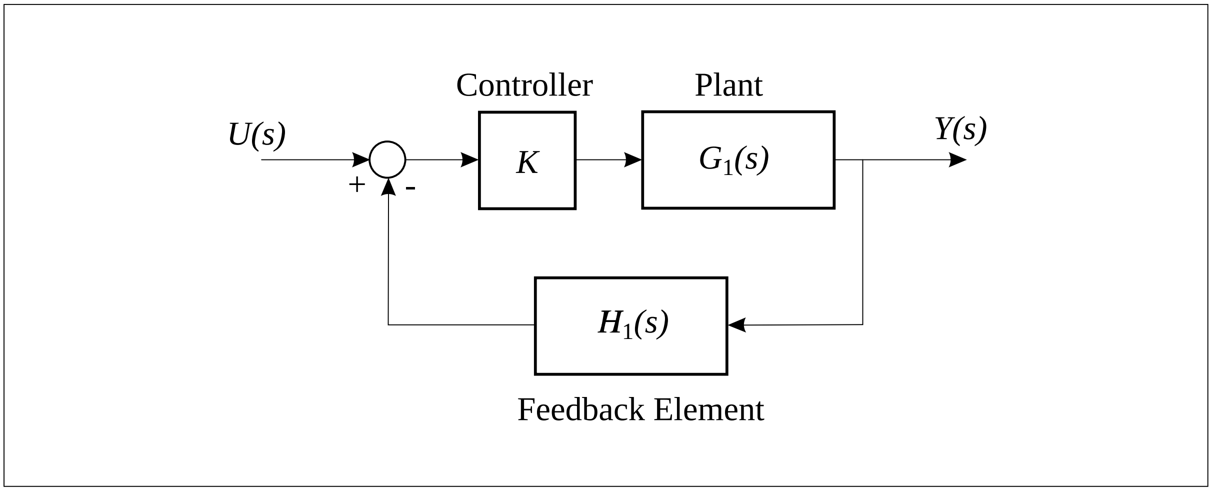

Figure 7.2: Feedback control system with a static controller in the direct path

The closed-loop transfer function of this system is given by

$$M(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{G(s)}{1 + KG_1(s)H_1(s)}$$

$$= \frac{G(s)}{1 + \dfrac{K(s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}}$$

$$= \frac{G(s)(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)}{(s^n + a_{n-1}s^{n-1} + \cdots + a_0) + K(s^m + b_{m-1}s^{m-1} + \cdots + b_0)}$$

$$n \geq m$$

The corresponding characteristic equation is

$$\left(s^{n} + a_{n-1}s^{n-1} + \cdots + a_{0}\right)$$
$$+K\left(s^{m} + b_{m-1}s^{m-1} + \cdots + b_{0}\right) = 0$$

The question to be answered by the root locus technique is: what can be achieved by changing the static gain K, theoretically, from $-\infty$ to $+\infty$? Can we find the location of system poles for all values of K? The positive answer to this question led to the development of the root locus technique. It was discovered by W. Ewans in 1948 and was mathematically formulated in 1950 in his famous paper (Ewans, 1950).

The main idea behind the root locus technique is hidden in equation

$$G_1(s)H_1(s) = -\frac{1}{K}$$

which is an algebraic equation involving complex numbers. It actually represents two equations (for real and imaginary parts, or for magnitudes and phase angles). In this book, *we will consider the root locus technique for* $0 \leq K < \infty$. This will simplify derivations and make the root locus plots clearer. The complementary root locus for negative values of K can be similarly derived. This equation produces the following equations for the magnitudes

$$\mid G_1(s)H_1(s) \mid = \frac{1}{K}$$

and for the phase angles

$$\measuredangle G_1(s)H_1(s) = (2l+1)\pi, \qquad l = 0, \pm 1, \pm 2, \ldots$$

If we factor $G_1(s)H_1(s)$ as

$$G_1(s)H_1(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

then, by using elementary algebra with complex numbers, we get

$$\mid G_1(s)H_1(s) \mid = \frac{\prod_{i=1}^{m} \mid s - z_i \mid}{\prod_{i=1}^{n} \mid s - p_i \mid} = \frac{1}{K}$$

and

$$\measuredangle G_1(s)H_1(s) = \sum_{i=1}^{m} \measuredangle(s - z_i) - \sum_{i=1}^{n} \measuredangle(s - p_i) = (2l+1)\pi$$

The last two equations are crucial for the development of the root locus technique.

Example 7.1: Given the open-loop transfer function

$$G(s)H(s) = \frac{K(s+1)}{s(s+2)(s+4)}$$

The locations of the open-loop poles and zeros are given in Figure 7.3.

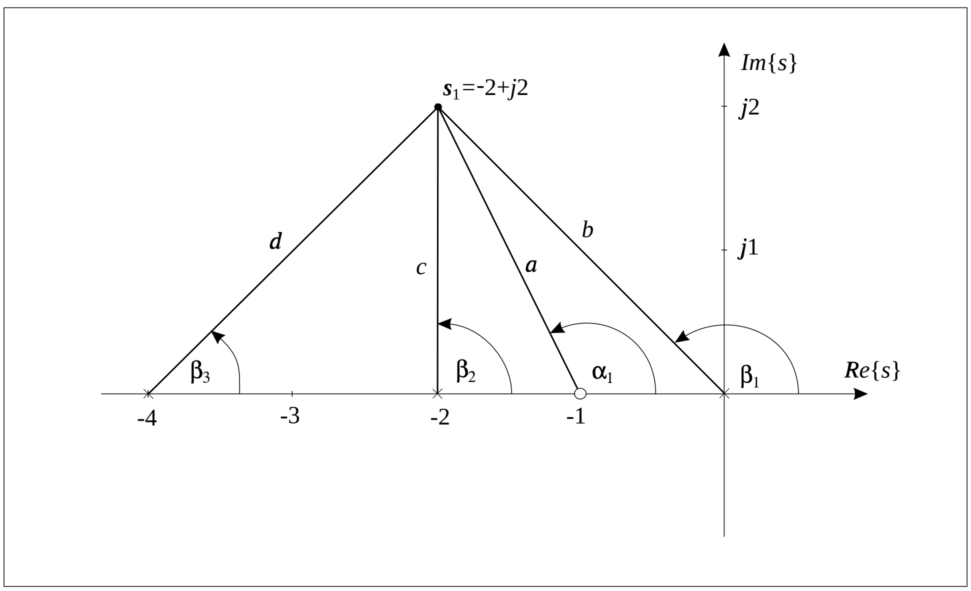

Figure 7.3: A point in the complex plain that does not lie on the root locus

Take any point s_1 in the complex plane. If that point belongs to the root locus, it must satisfy both equations (for the magnitude and for the phase). For example, for the point $s_1 = -2 + j2$ we have

$$\frac{| s_1 + 1 |}{| s_1 + 0 || s_1 + 2 || s_1 + 4 |} = \frac{1}{K} = \frac{a}{b \cdot c \cdot d}$$
$$= \frac{\sqrt{5}}{\sqrt{8} \cdot 2 \cdot \sqrt{8}} \Rightarrow K = \frac{16}{\sqrt{5}}$$

Thus, if the point s_1 belongs to the root locus, the static gain K at that point must be equal to $16/\sqrt{5}$.

It follows that for the point s_1 the following must be satisfied

$$\measuredangle G(s_1)H(s_1) = \measuredangle(s_1 + 1) - \measuredangle(s_1 + 0) - \measuredangle(s_1 + 2) - \measuredangle(s_1 + 4)$$
$$= (2l + 1)\pi$$

which leads to

$$\alpha_1 - \beta_1 - \beta_2 - \beta_3 = (2l+1)\pi, \qquad l = 0, \pm 1, \pm 2, \ldots$$

$$116.57^\circ - 135^\circ - 90^\circ - 45^\circ = -143.33^\circ \neq (2l+1)\pi$$

$$\text{for any} \quad l$$

We can conclude that the point $s_1 = -2 + j2$ cannot belong to the root locus since the phase equation is apparently not valid.

This example shows that only very selected points from the complex plane can belong to the root locus.

Minumum Phase Systems

It should be emphasized that from the root locus method it follows that the systems having unstable open-loop zeros become unstable for large values of the static gain K. Such kinds of systems are called *nonminimum phase systems*, in contrast to minimum phase systems whose definition is given below.

Definition 7.1 Systems having all open-loop zeros and poles in the closed left half of the complex plane excluding the origin are called *minimum phase systems*.