3.5 Nash Equilibria in Extensive Form

Def. 3.12 Nash equilibrium (seen before)

\[ J_1^* = J_1(u_1^*, u_2^*, ..., u_N^*) \leq J_1(u_1, u_2^*, u_3^*, ..., u_N^*) \]
\[ J_2^* = J_2(u_1^*, u_2^*, ..., u_N^*) \leq J_2(u_1^*, u_2, u_3^*, ..., u_N^*) \]
\[ J_N^* = J_N(u_1^*, u_2^*, ..., u_{N-1}^*, u_N) \leq J_N(u_1^*, u_2^*, ..., u_N, u_N) \]

3.5.1 Single-Act Games: Pure Strategy Nash Equilibria

Static informational game: each player has a single informational set (they are completely equivalent to Nash games in normal form).

Dynamic informational game: one of the players has some information about actions of other players, hence the player has several information sets

(Ex)

P2 knows that P1 played either L or P1 played M or R. P2 can not distinguish between M and R strategies of P1.
We can try to analyze this game in a recursive manner, as before.

In the case \( u_1 = L \Rightarrow u_2 = L \Rightarrow J_1^N = 0, \ J_2^N = -1 \)

In the right-hand part (zone) of the game we have a static uninformative game for which the Nash equilibrium can be found by using two matrices

\[
A = \begin{pmatrix}
3 & 0 \\
2 & -1 \\
L & R
\end{pmatrix} \quad B = \begin{pmatrix}
2 & 3 \\
1 & 0 \\
\text{no dotted lines}
\end{pmatrix}
\]

which implies the unique Nash equilibrium \( J_1^H = -1, \ J_2^H = 0 \) obtained for \( u_1 = R \) and \( u_2 = R \).

It follows from this analysis that \( u_1 \) should play \( R \) which will assure for him \( J_1^H = -1 \). In such a case the best strategy for \( P_2 \) is also \( R \) which produces \( J_2^H = 0 \).

However, this game is much richer on strategies to be played.

The normal form of this game can be obtained as follows:

- The number of pure strategies for \( P_1 \) is three: \( L, M, R \).
- The number of pure strategies for \( P_2 \) is four, that is: \( LL, LR, RL, RR \).
The corresponding normal game is

\[
A = \begin{bmatrix}
0 & 2 & 1 & 2 \\
3 & 0 & 1 & 3 \\
2 & -1 & -1 & 2
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-1 & 1 & -1 & 1 \\
2 & 3 & 3 & 2 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

which reveals that this game in fact has two Nash equilibria, \((0, 1)\) and \((-1, 0)\), none better than the other.

The second Nash equilibrium could have been also obtained from the game’s extensive form under assumption that P2 has only one information set, that is from

\[
\text{(I-game)}
\]

Its normal form is (now P2 has only two pure strategies L and R)

\[
A = \begin{bmatrix}
0 & 2 \\
3 & 0 \\
2 & -1
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-1 \\
2 & 3 \\
1 & 0
\end{bmatrix}
\]

which implies the unique Nash equilibrium \((0, 1) = (\frac{1}{2}, \frac{1}{2})\).

Note that game I is informationally uninformative.
From this example we can draw some conclusions:

- Existence of multiple Nash equilibria.
- Recursive procedure may find only one of the Nash equilibria.
- Informationally inferior game (with single information set) produces also a Nash equilibrium for the original informationally superior game (with two information sets).

**Proposition 3.7** Let (I) be an N-person single-act game that is informationally inferior to another single-act N-person game (II). Then any Nash equilibrium solution of (I) also constitutes a Nash equilibrium of (II).

Proof: not difficult

Hence, games with dynamic information in general admit multiple Nash equilibria (informational nonuniqueness).

To cope with the problem of informational nonuniqueness we impose more structure on the game through feedback, nested, ladder-nested, etc. structures. Even more, we characterize several types of Nash equilibria such as robust equilibrium, perfect equilibrium, sequential equilibrium, strategic equilibrium, and so on.

\[
A = B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{cases} 0,0 \text{ and } (1,1) \text{ are Nash equilibria}, \\ (0,1) \text{ is not a Nash equilibria if elements are} \end{cases}
\]
\[
A_e = \begin{bmatrix}
0 + \varepsilon_1 & 1 + \varepsilon_2 \\
1 + e_{21} & 1 + e_{22}
\end{bmatrix}, \quad \begin{bmatrix}
0 + \mu_1 & 1 + \mu_2 \\
1 + \mu_{12} & 1 + \mu_{22}
\end{bmatrix}
\]

We will show that there are infinitesimally small quantities \( \varepsilon \) and \( \mu \) that destroy Nash equilibrium (1,1), hence it is not robust equilibrium.

Take
\[
A_e = \begin{bmatrix}
0.1 & 0.98 \\
1.1 & 1.1
\end{bmatrix}, \quad B_u = \begin{bmatrix}
0.2 & 0.99 \\
0.99 & 1.01
\end{bmatrix}
\]

Hence, the Nash equilibrium now is only \( e = 1, u = 1 \Rightarrow V_H = (0.1, 0.2) \). Of course, we prefer to select Nash equilibria that are robust, especially when the data in \( A \) and \( B \) matrices are obtained empirically.

Note that in the bimatrix game at the top of page 9 the equilibrium \((0, -1)\) is not robust (if you change \( b_{13} \) to \(-0.99\)). Also, by changing \( b_{32} \) to \(-0.01\) we see that the Nash equilibrium \((-1, 0)\) is not robust.

Comment: As suggested by (Ana Lucia or Andrea), after the class, we can set up a normal form for the game that corresponds to the left information set, page 7.

\[
A = \begin{bmatrix}
0 & -2 \\
-1 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
-1 & 1 \\
1 & 0
\end{bmatrix}
\]

which reveals that \((0, 1)\) is the Nash eq

Hence, in this example, both Nash equilibria could have been detected from game's extensive form.