FINITE NASH GAMES

3.2 BIMATRIX GAMES

2-player finite NASH games (BIMATRIX GAMES) are non-zero sum games.

Two matrices $A^{m \times n}$ and $B^{m \times n}$ have information about costs of respectively $P_1$ and $P_2$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$$

$P_1$ chooses rows $i$ $\Rightarrow$ $(a_{i1}, b_{i1})$ is the game outcome

$P_2$ chooses columns $j$

Rational behavior: each player minimizes (max) losses/gains (winnings). Positive numbers in matrix $A$ ($B$) are losses for $P_1$ ($P_2$) and negative numbers in $A$ ($B$) are gains for $P_1$ ($P_2$)

Def. 3.1 NASH strategies

If the pair of inequalities

$$a_{i^*j^*} \leq a_{ij}, \quad b_{i^*j^*} \leq b_{ij^*}$$

is satisfied for all $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$

the pair $(a_{i^*j^*}, b_{i^*j^*})$ is the NASH equilibrium outcome of the bi-matrix game.

In general, as proven in Def. 3.1 NASH equilibria are ill-defined as demonstrated in the next examples.
(Example) \[ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \]

P1 examines the columns for every \(i\)-strategy of P2 and chooses the smallest element as indicated by dashed lines.

P2 examines the rows for every \(j\)-strategy of P1 and chooses the smallest element in the rows, as indicated by dashed lines.

In this game we have two Nash equilibria that satisfy Definition 3.1 (solid-line boxes)

\[ (i=1, j=1) \quad \text{and} \quad (i=2, j=2) \]

\[ (1,2) \text{ = game outcome = } (1,0) \]

Apparently, the players without any need for cooperation will choose \((i=2, j=2)\) since this strategy provides a better Nash equilibrium for both players.

**Definition 3.2** A pair of strategies \((i_j, j_1)\) is better than another pair \((i_2, j_2)\) if

- \(a_{i_1} < a_{i_2}\)
- \(b_{i_1} < b_{i_2}\)

with at least one strict inequality.

**Definition 3.2** A Nash equilibrium is admissible if there exists no better Nash equilibrium.

Note that ordering in the parts on numbers is not a complete operation since

\[ (1,2) < (3,4) \]

but \( (1,2) \not< (0,1) \)
(Example)
\[ A = \begin{bmatrix} 2 & 1 \\ +1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \]

\[ \Rightarrow \text{two Nash equilibria} \]
\[ (i=1, j=1) \Rightarrow (-2, -1) \]
\[ (i=2, j=2) \Rightarrow (-1, -2) \]

However, the result of the game is not any of the Nash equilibria since P1 choosing at the equilibria of A may choose \( i=1 \) and P2 may choose \( j=2 \) \( \Rightarrow (4,1) \) as a game outcome which is apparently worse for both players.

\[ (-1, -2) < (1, 1) \]
\[ (-2, -1) < (4,1) \]

This is a serious problem with Nash strategies since Nash equilibria are not recognized by the players \( \Rightarrow \) the outcome that may be worse for both players.

This leads to a conclusion that in the space of pure strategies, in the case of multiple Nash equilibria, the Nash equilibrium is not well-defined (unless we allow communications (cooperation), which is not the rule of the game).

(Example)
\[ A = \begin{bmatrix} 8 & 0 \\ 30 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 30 \\ 0 & 2 \end{bmatrix} \]

Here, \( i=1, j=1 \) unhappy the unique Nash equilibrium. The game is well-posed. The players play \( i=1, j=2 \). The outcome of the game is Nash equilibrium, equal to 8.
SECURED STRATEGIES AND MINIMAX SOLUTION

We use the same game as before.

The secured strategy for P1:
The row in A whose maximal element is
minimal (the row with the minimal maximal
element \( \min \max \{ a_{ij} \} \))

The secured strategy for P2:
The column in B with the minimal maximal
element ( \( \min \max \{ a_{ij} \} \))

\[
\begin{align*}
E(k) &= \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \\
A &= \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \\ B &= \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}
\end{align*}
\]

\( \mathbf{p}_d = 2 \)

\( \Box = \text{maximal row (column) element} \)

\( \Diamond = \text{the value of game (secure strategies played)} \)

Since both players use min max to find their
secured strategies, the corresponding solution
is known as min max solution.

Note that in this example the security
strategies coincide with Nash strategy
since \((-2, -1)\) is the unique Nash equilibrium.

Also

\[
A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \\ B = \begin{bmatrix} 2 & -1 \\ 4 & 0 \end{bmatrix}
\]

\( V_{\text{Nash}} = (-2, -1) \) for \( c = 1 \) and \( d = 2 \)

\( V_{\text{Nash}} = V_{\text{secure}} = 45 \)
In general, the min-max solution is "worse" than the Nash solution as demonstrated on the next example:

\[ A^*_N = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 2 & 2 \end{bmatrix}, \quad B^*_N = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \Rightarrow V_N = (-3, -1) \]
\[ c_N = 2, \quad d_N = 1 \]

\[ A^*_S = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ 2 & 2 \end{bmatrix}, \quad B^*_S = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \Rightarrow V_S = (1, -1) \]
\[ c_S = 4, \quad d_S = 4 \]

Mixed Strategies

(Ex)

\[ A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} \]

No Nash equilibrium here in pure strategies.

**Definition 3.6 Mixed Nash strategies**

A pair \( \{ y^* \in Y : y \geq 0, \sum_{i=1}^{m} y_i = 1 \} \),

\( z^* \in Z : z \geq 0, \sum_{j=1}^{n} z_j = 1 \)

is a Nash equilibrium solution to a bimatrix game with mixed strategies if

\[ y^* A z^* \leq y^* A z^* \]

\[ y^* B z^* \]

the pair \( (y^* A z^* , y^* B z^* ) \) is the game value of the Nash equilibrium.

**Theorem 3.1** Every bimatrix game has at least one Nash equilibrium in mixed strategies.

Computations of Nash equilibria in mixed strategies is pretty difficult.
3.3 $N$-Person Finite Nash Games in Normal Form

1. $N$-players $P_1, P_2, \ldots, P_N$
2. Each player has a finite number of strategies, $m_i$, with $m_i$ denoting a strategy of the player $i$.
3. For given $m_i$'s the cost functions of each player are
   $$ J^i = a^i_{m_1, m_2, \ldots, m_N}, \; i = 1, 2, \ldots, N $$
4. Rule of the game: each player minimizes his/her cost function independently (assuming that the other players are doing the same).

**Def. 3.7 Nash Equilibrium**

An $N$-tuple of strategies $(m_1^*, m_2^*, \ldots, m_N^*)$ is a Nash equilibrium if the following $N$ inequalities are satisfied:

$$J^1 = a_{m_1^*, m_2^*, \ldots, m_N^*}^1 \leq a_{m_1^*, m_2^*, \ldots, m_N^*}^1$$
$$J^2 = a_{m_1^*, m_2^*, \ldots, m_N^*}^2 \leq a_{m_1^*, m_2^*, \ldots, m_N^*}^2$$
$$J^N = a_{m_1^*, m_2^*, \ldots, m_{N-1}^*}^N \leq a_{m_1^*, m_2^*, \ldots, m_{N-1}^*}^N$$

**Theorem 3.2** Every $N$-person static finite game in normal form admits at least one Nash equilibrium in mixed strategies (no proof).

In general, Nash equilibria are not unique, difficult to find (calculate) and lead to ill-posedness of the corresponding game.