**Math Background - Convex Sets**

**Convex Set**: If the line segment joining any two points of the set also belong to the set.

\[ x_1, x_2 \in S \]
\[ x = \lambda_1 x_1 + \lambda_2 x_2 \in S \]
\[ \lambda_1 + \lambda_2 = 1 \]
\[ \lambda_1, \lambda_2 \geq 0 \]
\[ \text{or } \left( \lambda x_1 + (1-\lambda)x_2 \in S' \right), 0 \leq \lambda \leq 1 \]

\[ x = \text{convex combination of } x_1 \text{ and } x_2. \]

This is a nonconvex set.

**Convex Hull**: \( H(S_4) \) is a minimal convex set that contains \( S_4 \).

For example, \( H(S_4) \) is an arbitrary set.

(Examples)

\( S_2 \)
It holds for the convex hull (and for the convex set)

\[ x \in \mathcal{H}(S) \iff x = \sum_{y \in S} \lambda_y y \quad , \quad \lambda_y \geq 0, \quad \sum_{y \in S} \lambda_y = 1 \]

**Polytope**: A convex hull of a finite number of points \( x_1, \ldots, x_k \) in \( \mathbb{E}^n \) is called a polytope.

(Ex)

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \]

\[ x_5 \]

**Simplex**: Let \( x_1, x_2, \ldots, x_k \) form a polytope in \( \mathbb{E}^n \).

If \( x_2 - x_1, x_3 - x_2, \ldots, x_k - x_{k-1} \) are \( k \) linearly independent vectors then \( \mathcal{H}(x_1, x_2, \ldots, x_k) \) defines a simplex.

**In plane (n=2)**

\[ x_1 \quad x_2 \quad x_3 \]

is a simplex

In \( \mathbb{E}^n \) space a simplex has no more than \( n+1 \) vertices.

**Hyperplane**: \( S = \{ x : p^T x = \lambda y \} \) defines a hyperplane.

\( p \) = normal to hyperplane, \( \lambda \) = scalar.

**Half space**

\[ S = \{ x : p^T x \leq \lambda y \} \quad , \quad x \in \mathbb{E}^n, \quad p \in \mathbb{E}^n \]
MINIMAX Theorem (Theorem 2.4, p.27)
In any matrix game $A$, the average security levels of the players in mixed strategies coincide, that is

$$
\bar{v}_m(A) = \min \max_{\mathbf{y}} \{ \mathbf{y}^T A \mathbf{z} \} = \max \min_{\mathbf{z}} \{ \mathbf{y}^T A \mathbf{z} \} = \bar{v}_m(A)
$$

where

$$\mathbf{y} = \{ y \in \mathbb{R}^m; y_i \geq 0, \sum_{i=1}^{m} y_i = 1 \}$$

$$\mathbf{z} = \{ z \in \mathbb{R}^n; z_i \geq 0, \sum_{j=1}^{n} z_j = 1 \}$$

**Proof**: We first need to prove the following lemma.

**Lemma 2.1** Let $A$ be an arbitrary $(m \times n)$-dimensional matrix. Then, either there exists

(i) a nonzero vector $\mathbf{y} \in \mathbb{R}^m, y \geq 0$, such that $A^T \mathbf{y} \leq \mathbf{0}$

(ii) a nonzero vector $\mathbf{z} \in \mathbb{R}^n, z \geq 0$, such that $A \mathbf{z} \leq \mathbf{0}$

Note that $\mathbf{0} \in \mathbb{R}^m$ denotes a zero vector.

**Proof of Lemma 2.1**

Let $H (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ denote the convex hull of $m + n$ vectors.

Let $\mathbf{e} = \{ \mathbf{e}_i \}$ unit vectors in $\mathbb{R}^n$.

Let $\mathbf{a} = \{ \mathbf{a}_i \}$ rows of $A^{m \times n}$.

Then $\mathbf{a}_i \in \mathbb{R}^n$ for all $i$. 

A vector $\mathbf{o}$ either belongs to $\mathcal{H}$ or does not belong to it.

**Case** $\mathbf{o} \in \mathcal{H}$ → A set of scalars $y_i$ and $\eta_j$ exists such that

$$
\mathbf{o} = \sum_{i=1}^{m} y_i \mathbf{a}_i + \sum_{j=1}^{n} \eta_j \mathbf{e}_j \quad \text{with} \quad \sum_{i=1}^{m} y_i + \sum_{j=1}^{n} \eta_j = 1
$$

with $y_i \geq 0$, $\eta_j \geq 0$

or in the scalar form (for every component of $\mathbf{o}$)

$$
\mathbf{o} = \sum_{i=1}^{m} y_i \mathbf{a}_y + \eta_j
$$

which implies

$$
\sum_{i=1}^{m} y_i \mathbf{a}_y = \mathbf{o} - \eta_j \leq 0 \quad \text{since } \eta_j \geq 0
$$

whence

$$
\sum_{j=1}^{n} \eta_j \leq \mathbf{a}^T \mathbf{y} \leq 1
$$

**Case** $\mathbf{o} \notin \mathcal{H}$

Here exists a hyperplane through the $\mathbf{o}$ such that

$$
\mathbf{z}^T \mathbf{x} = 0 \quad \text{and} \quad \mathbf{z}^T \mathbf{x} \geq 0 \quad \text{for } \mathbf{x} \in \mathcal{H}
$$
\[ z^T x > 0 \quad \text{for} \quad x \in H \]

let \( x = e_i \) \( \Rightarrow z^T e_i = z_i > 0 \)

let \( x = a_i \) \( \Rightarrow z^T a_i > 0 \quad \Rightarrow \quad A z > 0 \quad \text{q.e.d.} \)

**Proof of Theorem 2.4**

\[
\mu_m(A) = \min_y \max_z \{ y^T A z \} = \max_z \min_y \{ y^T A z \} = \mu_m(A)
\]

We have established before that \( \mu_m(A) \leq \mu_m(A) \)

From Lemma 2.1 it exists \( y^*_0 > 0 \) such that \( A y^*_0 \leq 0 \)

Also \( y^*_0^T A \leq 0 \)

since \( z > 0 \) \( \Rightarrow \) \( y^*_0^T A z \leq 0 \)

also \( \max_z \{ y^*_0^T A z \} \leq 0 \)

Since \( y^*_0 > 0 \) we have

\[
\mu_m(A) = \min_y \max_z \{ y^T A z \} \leq 0 \quad (1)
\]

Similarly, the second alternative of Lemma 2.1 implies that it exists \( z^0 \) vector such that \( A z^0 \geq 0 \)

or \( y^T A z^0 \geq 0 \) since \( y > 0 \)

or \( \min_y \{ y^T A z^0 \} \geq 0 \)

or \( \mu_m(A) = \max_z \min_y \{ y^T A z \} \geq 0 \quad (2) \)
(1) and (2) were obtained for an arbitrary matrix $A$. If we replace this matrix by a matrix obtained by shifting all entries by a constant $c$, that is, $A_j - c$, then the game lower and upper values in mixed strategies are

\[ \tilde{V}_m(A) - c \quad \text{and} \quad \tilde{Y}_m(A) = c \]

Combining this observation with (1) and (2), we have

\[ \tilde{V}_m(A) \leq c \quad \text{(3)} \]

\[ \tilde{Y}_m(A) \geq c \quad \text{(4)} \]

At least one of these inequalities must hold for an arbitrary constant $c$. In addition, we know that

\[ \tilde{Y}_m(A) \leq \tilde{V}_m(A) \]

which implies

\[ \tilde{Y}_m(A) = \tilde{V}_m(A) \]

Corollary 2.3 (of Theorem 2.4)

(-) The game has a saddle point in mixed strategies.

(-) The equilibrium strategies are the mixed security strategies.

(-) $\tilde{V}_m(A) = \tilde{Y}_m(A) = \tilde{V}_m(A)$
Comments on the proof of Minimax Theorem


"This theorem, the most important of game theory..."

Bazan and Olsder, pp. 75, "the original proof of the minimax theorem given by Von Neumann is nonelementary and rather complicated. ... The proof given here seems to be the simplest one available in the literature.

Additional comments on the proof from the textbook

(3) $\bar{V}_m(A) \leq c$

(4) $\underline{V}_m(A) \geq 0$

one of them holds for an arbitrary constant $c$

We also know that

The lower game value in mixed strategies

$\underline{V}_m(A) = \bar{V}_m(A) + \kappa$, for some $\kappa \geq 0 \hspace{1cm} (5)$

Let us choose the constant $c$ as

$c = \bar{V}_m(A) + \frac{1}{2} \kappa$

then

(3) $\bar{V}_m(A) \leq \bar{V}_m + \frac{1}{2} \kappa = \bar{V}_m(A) - \kappa + \frac{1}{2} \kappa \Rightarrow 0 \leq -\frac{1}{2} \kappa \Rightarrow \kappa \leq 0$

(4) $\underline{V}_m(A) \geq \bar{V}_m(A) + \frac{1}{2} \kappa \Rightarrow 0 \geq \frac{1}{2} \kappa \Rightarrow \kappa \leq 0$

Hence, (3) and (4) contradict (5) unless $\kappa = 0$

which implies $\bar{V}_m(A) = \bar{V}_m(A)$. 
(23) Computation of Mixed Strategies

a) Graphical Approach (for 2x2, 2xn, nx2, games)

Example:

\[
\begin{bmatrix}
3 & 0 \\
1 & 1
\end{bmatrix}
\Rightarrow \begin{align*}
V &= 0, \quad V &= 1 \\
V_m &= V_m = V_m = ?
\end{align*}
\]

To find the mixed strategy of P1 we assume that P2 plays only pure strategies, that is

\[(z_1 = 1, z_2 = 0) \quad \text{or} \quad (z_1 = 0, z_2 = 1)\]

The problem is to find \((y_1, y_2) \quad \text{subject to} \quad y_1 \geq 0, y_2 \geq 0, y_1 + y_2 = 1\)

For \(z_1 = 1, z_2 = 0\) we have

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
3 & 0 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} = 3y_1 - y_2 = V(A)
\]

\[
V(A) = \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
3 & 0 \\
1 & 1
\end{bmatrix} ^{-1} \begin{bmatrix}
0 \\
1
\end{bmatrix} = \begin{bmatrix}
y_1^* \\
y_2^*
\end{bmatrix}
\]

For \((y_1, y_2) = (0, 1)\) we have \(V(A) = \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} = \begin{bmatrix}
3 & 0 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
0 \\
1
\end{bmatrix} = y_2^*
\]

\[
\begin{align*}
3y_1 - y_2 &= y_2^* \\
y_1 + y_2 &= 1
\end{align*}
\]

\[
\begin{align*}
y_1^* &= \frac{2}{3}, \quad y_2^* = \frac{3}{5} \quad \Rightarrow \quad V_m = 0.6
\end{align*}
\]
Similarly, for P2 we get

\[ \begin{align*}
\text{for } (y_1 &= 1, y_2 = 0) \\
&I = (1 \ 0) (3 \ 0) (z_1) = (3 \ 0) (z_1) = 3z_1 \\
&\text{for } (y_1 = 0, y_2 = 1) \\
&I = (0 \ 1) (3 \ 0) (z_1) = (-1 \ 1) (z_1) = -z_1
\end{align*} \]

\[ V_m = 0.6 \text{ is guaranteed for P2 if he plays his mixed security strategy}
\]
\[ (z_1^* = \frac{1}{6}, z_2^* = \frac{4}{6}) \]

For higher order problems \( n \geq 3 \) and \( m \geq 3 \), we use the linear programming

\[ \text{zero-sum game } \leftrightarrow \text{matrix game} \]

\[ \text{normal game} \leftrightarrow \text{linear programming problem} \]

Matrix games can be simplified by eliminating non-dominant rows and columns, if any.

\[ \text{strictly if i dominates row } k \text{ of any } a_{ij} \leq a_{kj}, j = 1, 2, \ldots, n \]

\[ \text{and if at least one of the strict inequalities holds.} \]

\[ \text{Column } j \text{ dominates column } c \text{ if } \]
\[ a_{ij} \geq a_{cj}, \quad c = 1, 2, \ldots, m \]

\[ \text{and if for at least one } i \text{ the strict inequality holds.} \]

\[ \text{strictly dominated rows and columns may be deleted.} \]