In the case of several equilibria additional criteria have to be imposed in order to make a decision which one to choose as the solution to the game.

b) STACKELBERG STRATEGIES (sequential decision making)

Let P1 be the leader and P2 the follower.

The followers problem is very simple

$$\min_{u_2} J_2 (\tilde{u}_1, u_2)$$

with guess $\tilde{u}_1$.

The leader has a tougher problem. The leader must also solve the followers problem and find the corresponding reaction curve

(a) $$\min_{u_2} J_2 (u_1, u_2) \Rightarrow u_2 = u_2^*(u_1) = \frac{\partial J_2(u_1, u_2)}{\partial u_2}$$

(b) Now the leader has to solve Gisoym optimization problem

$$\min_{u_1} J_2 (u_1, u_2^*(u_1)) = \min_{u_1} F_1(u_1)$$

In general the leader guarantees that the strategy of the follower will be from the followers reaction curve and then solves the constrained optimization problem of the leader.
The reader's optimization problem is

\[ I(u_1, u_2, \lambda) = f_{\lambda}(u_1, u_2) + \lambda \frac{\partial J_2(u_1, u_2)}{\partial u_2} \]

which leads to

\[
\begin{align*}
(1) & \quad \frac{\partial I}{\partial u_1} = 0 \\
(2) & \quad \frac{\partial I}{\partial u_2} = 0 \\
(3) & \quad \frac{\partial I}{\partial \lambda} = 0
\end{align*}
\]

(1)-(3) can have the unique solution, no solution or many solutions.

The reader's optimization problem can be viewed as a constrained optimization problem, which can be solved by using techniques from nonlinear programming.

\[ \min J_i \text{ along the reaction curve } u^*_i(u_i) \]

Note that the Stackelberg solution is preferred by the reader (the reader can not be worse than Nash). In general the reader is doing better with the Stackelberg strategy then with the Nash strategy. The follower can be better or worse. If the follower is worse, he would have to play Nash, but the Stackelberg strategy is imposed on him by the rules of the game.
In "Hash figure" since $u^1(w_1)$ is (almost) tangent to the iso-cost of the player one, he cannot achieve any benefits by taking the role of the leader since $S_1 = N$. However, if $P_2$ is the leader then the iso-cost curve that is tangent to $u^1(w_2)$ is outside of his iso-cost Hash curve. Hence his losses are reduced. Even more, the losses of $P_1$ are also reduced since $J_{5_2} < J_{4_1}$.

In some cases both players prefer to be followers (these are so-called Stackelberg strategies). Hence, in such case none of the players is interested in announcing his strategy first— and they react.

3) COOPERATIVE GAMES

Pareto strategies

It can be seen from the "Hash figure" that the players can do much better than Hash (light grey shaded area). This can happen if they play Stackelberg or if they decide to cooperate. If they decide to cooperate the game solution can be along the dash-dotted line that connects $J_{4_{1_1}}$ and $J_{4_{1_2}}$ (absolute minimum of $J_1$ and $J_2$).
Pareto Strategies (Cooperative Game)

\[ J_2 = f(J_1) \]

\[ J_1 = J_2 \]

\[ u_{p_1} \]

\[ u_{p_2} \]

\[ (J_{1M}, J_{2M}) \]

Hash equilibrium

It can be seen from the Hash strategy diagram that both players can minimize losses if they decide to cooperate and optimize

\[ J = \delta_1 J_1 + \delta_2 J_2, \quad \delta_1 + \delta_2 = 1, \quad \delta_1, \delta_2 > 0 \]

Pareto strategy is any strategy that minimizes losses of both (all) players.

\[ J_i (U) \leq J_i (U') \quad \forall i = 1, 2 \ldots \]

There are infinitely many Pareto strategies (any one that brings the system on or above the \( J_2 = f(J_1) \) curve and below \( J_1 \leq J_{1M} \) and \( J_2 \leq J_{2M} \).

Once we reach the curve \( J_2 = f(J_1) \) we cannot move any more without increasing one of the \( J \)'s. At that point we are done. Which point to choose on the \( J_2 = f(J_1) \) curve? This leads to different strategies like \( P_{mm} \) (minimax Pareto) and \( P_e \) (equalization Pareto)