The extensive game form described by a game tree displays explicitly the evolvement of the game. However, the description by itself becomes complex in the case of many players, many alternatives, and many stages.

The normal game form suppresses unnecessary information about game evolvement and expresses the utility (loss) functions explicitly in terms of players' strategies.

\[ J_i = J_i (u_1, u_2, ..., u_N) \]
\[ C = 1, 2, ..., N \]
\[ J_i = \text{loss function (utility function, performance)} \]
\[ u_i = \text{strategy of player } i \]

(Example)

All possible strategies are

\[ LL, LR, RL, RR \]
\[ u_1 \quad u_2 \]
\[ \uparrow \quad \uparrow \]
\[ u_1 \quad u_2 \]

Hence

\[ J_1 = J_1 (u_1, u_2) \]
\[ J_2 = J_2 (u_1, u_2) \]

Rational behavior of each player is to minimize his/her own losses assuming that the other player is doing the same.
\[ J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2) \]
\[ J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2) \]

Such a strategy \((u_1^*, u_2^*)\), assuming that it exists, is called the \textit{equilibrium} strategy.

In general, for \(N\)-players, the loss functions are defined by
\[ J_c = J_c(u_1, u_2, \ldots, u_N), \quad c = 1, 2, \ldots, N \]
with the equilibrium strategy satisfying
\[ J_c(u_1^*, u_2^*, \ldots, u_N^*) \leq J_c(u_1^*, u_2^*, \ldots, u_N^*) \]

For \(N = 2\), normal form games are called \textit{the matrix games} since they can be described using matrices.

<table>
<thead>
<tr>
<th>Player 1</th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>(1,2)</td>
<td>(4,0)</td>
</tr>
<tr>
<td>R</td>
<td>(3,2)</td>
<td>(3,3)</td>
</tr>
</tbody>
</table>

\[ \begin{array}{c|cc}
\text{P1} & \text{L} & \text{R} \\
\hline
\text{L} & J_1 = 1 & J_1 = 4 \\
& J_2 = 2 & J_2 = 0 \\
\text{R} & J_1 = 5 & J_1 = 3 \\
& J_2 = 0 & J_2 = 3 \\
\end{array} \]
GAMES IN EUCLIDEAN SPACES
(Infinite games)

1) ZERO-SUM GAMES
   a) simultaneous decision making \(\Rightarrow\) saddle point solution
   b) sequential decision making \(\Rightarrow\) minmax or maxmin solution

2) CONFLICT GAMES
   a) simultaneous decision making \(\Rightarrow\) Nash strategies
   b) sequential decision making \(\Rightarrow\) Stackelberg

3) COOPERATIVE GAMES \(\Rightarrow\) Pareto strategies

All games in \(\mathbb{E}^n\) spaces are described by a static state equation

\[ f(x, u) = 0 \]

\(x \in \mathbb{R}^n = \) state of the game vector
\(u \in \mathbb{R}^m\) with \(u = (u_1, u_2, \ldots, u_n)\)
\(u_i \in \mathbb{R}^{m_i} = \) strategy of player \(i\)

In addition, each player has a cost function (utility function, cost, performance criterion)

\[ J_i(x, u) \quad i = 1, 2, \ldots, N \]

Assuming that \(f(x, u) = 0 \Rightarrow x = \varphi(u)\)

\[ J_i(\varphi(u), u) = J_i(u_1, u_2, \ldots, u_n) \]

Note that none of the players has complete control over the game outcome.
1) **Zero-sum Games** \((J_2 = -J_1)\)

a) **Saddle point strategies**

From:
\[
\begin{align*}
J_1(u_1^*, u_2^*) & \leq J_1(u_1, u_2^*) \quad (4) \\
J_2(u_1^*, u_2^*) & \leq J_2(u_1, u_2^*)
\end{align*}
\]

and the fact that \(J_2 = -J_1\), and
\[
J_2(u_1^*, u_2^*) \leq J_2(u_1, u_2) \quad \Rightarrow \quad -J_1(u_1^*, u_2^*) \leq -J_1(u_1, u_2)
\]

\[
\Rightarrow \quad J_1(u_1^*, u_2^*) \geq J_1(u_1, u_2)
\]

We get from (4)

\[
\begin{align*}
J_1(u_1^*, u_2) & \leq J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*) 
\end{align*}
\]

which represents the saddle point condition.

Apparentely in this zero-sum game, the player 1 is minimizer and the player 2 is the maximizer.

It exists if \(J_1(u_1, u_2)\) is convex with respect to \(u_1\) and concave with respect to \(u_2\).