Dynamic Stackelberg Games

Given a dynamic system controlled by two players
\[ x = f(x_1, u_1, u_2), \quad x(t_0) = x_0 \]
and
\[ J_1 = \frac{\partial}{\partial t} \int_{t_0}^{t_f} l_1 (x, u_1, u_2) dt + g_1 (x(t_f)) \]
\[ J_2 = \frac{\partial}{\partial t} \int_{t_0}^{t_f} l_2 (x, u_1, u_2) dt + g_2 (x(t_f)) \]

Each player has a performance criterion. Each player minimizes its own criterion. P1 is the leader, and P2 is the follower.

This is the game with sequential decision making. The leader solves the control optimization problem by treating the optimization problem of the follower as an additional constraint.

**Follower's Optimization Problem**

\[ H_2 = l_2 (x, u_1, u_2) + p_2^T f(x_1, u_1, u_2) \]

\[ \begin{cases} \dot{x} = \frac{\partial H_2}{\partial p_2} = f(x_1, u_1, u_2), & x(t_0) = x_0 \\ \dot{p}_2 = -\frac{\partial H_2}{\partial x} - \frac{\partial H_2}{\partial u_1} \frac{\partial u_1}{\partial x}, & p(t_f) = \frac{\partial g_2 (x(t_f))}{\partial x} \\ 0 = \frac{\partial H_2}{\partial u_2} \end{cases} \]

These are the necessary conditions for optimality (minimality), as derived in the context.
of Hash games. We will assume that both players use feedback strategies, hence
\[ \frac{\partial u_1}{\partial x} = 0 \] and \[ \frac{\partial u_2}{\partial x} \neq 0 \]

(1) Basically defines the reaction of P2 to any strategy \( u_1 \) of P1.

**LEADER'S OPTIMIZATION PROBLEM**

\[ \tilde{H}_1 = L_1 + p_1^T f + \left( \frac{\partial H_2 + 2 \partial H_2}{\partial u_1} \frac{\partial u_1}{\partial x} \right) \tilde{u} + \frac{\partial H_2}{\partial u_2} \tilde{u} \]

the necessary optimality conditions for the leader are:

\[
\begin{cases}
\dot{x} = \frac{\partial \tilde{H}_1}{\partial \tilde{u}_1} = f_1(x, u_1, u_2), \quad x(t_0) = x_0 \\
\dot{p}_1 = -\frac{\partial \tilde{H}_1}{\partial x} - \frac{\partial H_2}{\partial u_2} \frac{\partial u_2}{\partial x}, \quad p_1(t_0) = \frac{\partial g_1}{\partial u_1} \\
\dot{p}_2 = -\frac{\partial \tilde{H}_1}{\partial \tilde{u}_2} = -\frac{\partial H_2}{\partial u_2} + \frac{\partial H_2}{\partial u_1} \frac{\partial u_1}{\partial x}, \quad p_2(t_0) = \frac{\partial g_2}{\partial u_2} \\
\dot{\tilde{\eta}} = -\frac{\partial \tilde{H}_1}{\partial \tilde{u}_2} = -\frac{\partial H_2}{\partial u_2} - \frac{2}{\partial u_2} \left( \frac{\partial H_2}{\partial u_1} + \frac{\partial H_2}{\partial u_2} \right) \tilde{u} + \frac{\partial H_2}{\partial u_2} \tilde{u}_2 \tilde{u}_2 \\
\end{cases}
\]

\( \tilde{\eta}(t_0) = 0 \)

\( \tilde{\eta} = \frac{\partial H_2}{\partial u_2} \)

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Substituting \( u_1, u_2, \tilde{\eta} \) in (2) we get the two-point boundary value problem of the form
\[ x = x, \quad x(t_0) = x_0 \]
\[ \eta = \eta, \quad \eta(t_0) = 0 \]
\[ p_1 = p_2, \quad p_1(t_f) = \Omega_2^{p_2} (t_f) \]
\[ p_2 = p_3, \quad p_2(t_f) = \Omega_2^{p_3} (t_f) \]

**Linear-Quadratic Stackelberg**

\[ x = A x + B_1 u_1 + B_2 u_2, \quad x(t_0) = x_0 \]
\[ J_1 = \frac{1}{2} \int_{t_0}^{t_f} (x^T Q x + u_1^T R_1 u_1 + u_2^T R_2 u_2) dt + \frac{1}{2} x(t_f)^T F_2 x(t_f) \]
\[ J_2 = \frac{1}{2} \int_{t_0}^{t_f} (x^T Q x + u_1^T R_1 u_1 + u_2^T R_2 u_2) dt + \frac{1}{2} x(t_f)^T F_2 x(t_f) \]

\[ q_1, q_2, q_1, q_2 \geq 0 \quad \text{and symmetric} \]
\[ R_1 = R_1^T > 0, \quad R_2 = R_2^T > 0 \]
\[ R_2 = R_2^T > 0, \quad q_2 = q_2^T > 0 \]

**The Followers' Problem:**

\[ J_2 = \frac{1}{2} (x^T Q x + u_1^T R_1 u_1 + u_2^T R_2 u_2) + q_2^T (A x + B_1 u_1 + B_2 u_2) \]

**Followers' Necessary Conditions:**

\[ x = A x + B_1 u_1 + B_2 u_2, \quad x(t_0) = x_0 \]
\[ \dot{p}_2 = -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial u_1} \frac{\partial u_1}{\partial x} = -q_2 x - (B_1 p_2 + B_2 u_1) \frac{\partial u_1}{\partial x} \]
\[ 0 = \frac{\partial H}{\partial u_2} = R_2 u_2 + B_2 p_2 \Rightarrow u_2 = -B_2^{-1} R_2 u_2 \]

Assume that \[ \frac{\partial u_1}{\partial x} = 0 \Rightarrow \text{open-coop strategy for the leader} \]

For \[ \frac{\partial u_1}{\partial x} \neq 0 \Rightarrow \text{closed-coop Stackelberg strategy for the leader} \]
The reader's problem:

\[ H_1 = \frac{1}{2} \left( x^T R_1 x + u_1^T R_{11} u_1 + u_2^T R_{22} u_2 \right) + p_1^T (A x + B_1 u_1 + B_2 u_1) \]

\[ + \left( -\frac{\partial p_1}{\partial x} - \frac{\partial p_1}{\partial u_1} \right) y^T + \frac{\partial p_1}{\partial u_2} : \]

\[ := -p_2 x - A^T p_2 \]

\[ p_{22} u_2 + B_2^T p_2 = 0 \]

\[ H_1 = \frac{1}{2} \left( x^T R_1 x + u_1^T R_{11} u_1 + p_2^T B_2 R_{22}^{-1} B_2^T p_2 \right) - (p_2 x + A^T p_2) \]

So far we have

\[
\begin{cases}
\dot{x} &= A x - B_2 R_{22}^{-1} B_2^T p_2 + B_1 u_1 \\
\dot{p}_2 &= -p_2 x - A^T p_2
\end{cases}
\]

And

\[
J_1 = \frac{1}{2} \int_0^T \left( x^T R_1 x + u_1^T R_{11} u_1 + p_2^T B_2 R_{22}^{-1} B_2^T p_2 \right) dt + \frac{1}{2} x^T(0) P_1 x(0)
\]

(3) and (4) represent an augmented linear system

\[
\begin{bmatrix}
\dot{x} \\
\dot{p}_2
\end{bmatrix} =
\begin{bmatrix}
A & -B_2 R_{22}^{-1} B_2^T \\
-\sigma_2 & -A^T
\end{bmatrix}
\begin{bmatrix}
x \\
p_2
\end{bmatrix}
+ \begin{bmatrix}
B_1 \\
0
\end{bmatrix} u_1
\]

And augmented quadratic performance criterion

\[
J_1 = \frac{1}{2} \int_0^T \left( \begin{bmatrix}
x \\
p_2
\end{bmatrix}^T \begin{bmatrix}
\phi_1 & 0 \\
0 & \phi_2
\end{bmatrix} \begin{bmatrix}
x \\
p_2
\end{bmatrix} + \frac{1}{2} \left( \begin{bmatrix}
L & 0 \\
0 & L
\end{bmatrix} \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} \right)^T \right) dt + \frac{1}{2} x^T(0) P_1 x(0)
\]

\[
\Rightarrow \quad \eta_1 = -D_1 \begin{bmatrix}
B_1 \\
0
\end{bmatrix} K \begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix}
\]

\[ A^T K + KA + Q - K B R_{11} B^T K = 0 \]

Note: \( K \) is doubled.