Lemma: \( v = \int_0^\infty e^{At}Qe^{At}dt \)

then \( \begin{array}{c} \sum \frac{A^TV + VA + Q = 0}{} 
\end{array} \)

Proof: Start with
\[ \frac{d}{dt}(e^{At}Qe^{At}) = A^Te^{At}Qe^{At} + e^{At}Qe^{At}A \]

and integrate from 0 to +\( \infty \)
\[ \int_0^\infty d(e^{At}Qe^{At}) = 0 = A^T \int_0^\infty (e^{At}Qe^{At})dt + (e^{At}Qe^{At})dt + \]

Hence
\[ -Q = A^TV + VA \]

\[ \text{ZERO-SUM DIFFERENTIAL GAMES} \]

\[ x = f(x, u, v) \] \hspace{1cm} x(t_0) = x_0 \]

\[ J(x(t_0), t_0) = \sum_{t_0}^{\infty} L(x, u, v) dt + g(x(t_f)) \]

\( \text{P1 unleads to minimize} \ J, \text{hence his goal is to find} \ u \ \text{such that} \)
\[ \min u \left\{ \sum_{t_0}^{\infty} L(x, u, v) dt + g(x(t_f)) \right\} \]

\( \text{is achieved. The other player maximizes} \ J, \text{that is} \)
\[ \max v \left\{ \sum_{t_0}^{\infty} L(x, u, v) dt + g(x(t_f)) \right\} \]

Let us define the optimal performance value
\[ J^*(x(t_0), t_0) = \min u \max v \left\{ \sum_{t_0}^{\infty} L(x, u, v) dt + g(x(t_f)) \right\} \]

\[ = \sum_{t_0}^{\infty} L(x^*, u^*, v^*) dt + g(x^*(t_f)) \]
Let $v$ uses his optimal strategy from $t_0$ to $t$, then

$$J^+(x(t_0), t_0) = \min_u \left\{ \int_{t_0}^{t+\Delta t} L(x,u,v^*) \, dt + \int_{t_0}^{t+\Delta t} L(x^+, u^*, \nu^*) \, dt + g(x^*(t)) \right\}$$

we can expand $J^+(x(t_0), t_0)$ into a Taylor series, which reads to

$$J^+(x(t_0), t_0) = \min_u \left\{ \int_{t_0}^{t+\Delta t} L(x,u,v^*) \, dt + J^+(x(t_0), t_0) + \frac{\partial J^+}{\partial t} \Delta t + \frac{\partial J^+}{\partial x} (x(t_0), t_0) ^T \times [x(t+\Delta t) - x(t)] + \cdots \right\}$$

for $\Delta t$ small, we can also approximate the integral by $L(x,u,v^*) \Delta t$, so that

$$J^+(x(t_0), t_0) = \min_u \left\{ L(x,u,v^*) \Delta t + J^+(x(t_0), t_0) + \frac{\partial J^+}{\partial t} \Delta t + \frac{\partial J^+}{\partial x} (x(t_0), t_0) ^T \times [x(t+\Delta t) - x(t)] \right\}$$

In the limit when $\Delta t \to 0$, we have

$$- \frac{\partial J^+}{\partial t} = \min_u \left\{ L(x,u,v^*) + \left( \frac{\partial J^+}{\partial x} (x(t_0), t_0) ^T \times [x(t+\Delta t) - x(t)] \right) \right\}$$

since $t_0$ is any initial time we can take $t_0 = t$ and $x(t_0)$ is any initial state so that

$$- \frac{\partial J^+}{\partial t} = \min_u \left\{ L(x^*, u,v^*) + \left( \frac{\partial J^+}{\partial x} (x(t), t) ^T \times [x(t+\Delta t) - x(t)] \right) \right\}$$

This is the Isaacs equation for zero-sum differential games. Note it was derived at the beginning of the 1950s.
Summed, for $P_x$ the Isaacs equation is

$$-\frac{\partial J^*}{\partial t} = \min_{u,v} \{ L(x,u,v) + (\frac{\partial J^*}{\partial x})^T f(x,u,v,v) \}$$

Putting these two equations together we have

$$-\frac{\partial J^*}{\partial t} = \max_u \min_v \{ L(x,u,v) + (\frac{\partial J^*}{\partial x})^T f(x,u,v,v) \}$$

or under the optimal controls $u^*$ and $v^*$ the Isaacs equation has the form

$$-\frac{\partial J^*}{\partial t} = L(x^*,u^*,v^*) + (\frac{\partial J^*}{\partial x})^T f(x^*,u^*,v^*)$$

Let us take the partial derivatives with respect to $x$ of the above (omitting $*$ for simplicity)

$$-\frac{\partial^2 J}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial x} + \frac{\partial J^*}{\partial x} f_x \right) + \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$

$$= 0$$

$$-\frac{\partial^2 J}{\partial x^2} + \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial L}{\partial x} + \frac{\partial J^*}{\partial x} f_x \right) + \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right)$$

Introduce the control variable $p = \frac{\partial J^*}{\partial x}$

$$-p^T = \frac{\partial}{\partial x} L + \frac{\partial}{\partial x} \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \left( L + p^T f \right)$$

$$-p = -\left( \frac{\partial H}{\partial x} \right)^T$$

Note that $p^T(f) = \frac{\partial f}{\partial x}$
Summary: In terms of Hamiltonian \( H = L + Pf \)
the necessary conditions for optimality (saddle point) are

\[
\dot{x} = \frac{\partial H}{\partial p} = f(x, u, y), \quad x(t_0) = 0
\]

\[
\dot{p} = -\frac{\partial H}{\partial x}^T \quad \text{or} \quad p(t_f) = g(x(t_f))
\]

\[
0 = \frac{\partial H}{\partial u}
\]

\[
0 = \frac{\partial H}{\partial y}
\]

Linear-Quadratic

\[
\dot{x} = Ax + Bu + B_2 y
\]

\[
J = \frac{1}{2} \int (x^T Q x + u^T R u - y^T P y) dt, \quad R > 0, \quad P > 0
\]

Form the Hamiltonian

\[
H = \frac{1}{2} (x^T Q x + u^T R u - y^T P y) + p^T (Ax + Bu + B_2 y)
\]

\[
\dot{x} = \frac{\partial H}{\partial p} = Ax + Bu + B_2 y, \quad x(t_0) = x_0
\]

\[
\dot{p} = -\frac{\partial H}{\partial x}^T p - Q x
\]

\[
p(t_f) = 0
\]

\[
0 = \frac{\partial H}{\partial u} = Ru + Bu^T p \Rightarrow u = -R_1 B_1^T p
\]

\[
0 = \frac{\partial H}{\partial y} = -R_2 y + B_2^T p \Rightarrow y = -R_2 B_2^T p
\]

\[
\dot{x} = Ax - \frac{B_1 R_1 B_1^T p + B_2 R_2 B_2^T p}{s_1} \quad \text{or} \quad x = Ax + (s_2 - s_1) p
\]

\[
\dot{p} = -Q x - A^T p
\]
This two point boundary value method can be solved by using the Riccati formalism:

\[ p(t) = P \times (t) \]
\[ \dot{p}(t) = P \times (t) \]
\[ -Q x - A^T p = P (A x + (s_1 - s_2) p) \]
\[ -Q x - A^T p x = P A x + P (s_1 - s_2) P x \]

Since this must hold for any \( x \), we have

\[ 0 = A^T P + PA + Q - P (s_1 - s_2) P \]

which represents the algebraic Riccati equation of zero-sum differential games (called the generalized algebraic Riccati equation).

Note that \( s_1 - s_2 \) is undefined, which makes this equation much more difficult for solving and analyzing than the standard algebraic Riccati equation.

Algorithm of (Li and Gayle, 1995, "New Trends in Dynamic Games and Applications, (ed) Olshner, Birnhauser, Boston" finds the solution of the above Riccati equation generalized:

\[ (A - s_2 \, p^{(c)})^T p^{(c)} + p^{(c)} (A - s_2 \, p^{(c)}) + (Q + p^{(c)} s_1 \, p^{(c)} + p^{(c)} s_2 \, p^{(c)}) = \]

Lyapunov equation for \( p^{(c)} \)

with

\[ A^T p^{(c)} + p^{(c)} A + Q - P^{(c)} s_2 \, P^{(c)} = 0 = \text{standard algebraic Riccati equation.} \]

The algorithm converges for any \((A, V_1, V_1)\) stabilizable-detectable (controllable-observable), assuming that the positive semi-definite stabilizing solution exists.