GAME THEORY

1928 von Neumann published a paper on game theory. That year marks the beginning of mathematical game theory.

1924 von Borel introduced the concept of game theory

1944 von Neumann and Morgenstern published the book "Theory of Games and Economic Behavior"

1957 Luce and Raiffa (Columbia Univ. Statistics group) published the book "Games and Decisions", Wiley

Hence, the foundation of game theory was established within mathematics, statistics, and economy.

Another source come from engineering applications. 1965, Isaacs published "Differential Games"

This book was actually written 1951-1955, but forbidden for publication due to potential military applications.

The central theme are zero-sum differential games

\[ \min_{m} \max_{n} (\text{distance between two objects}) \]
The third source to games is the optimal control theory rapidly evolving by the end of sixties and during the 1970's. These are mostly dynamic games:
1) cooperative games (Pareto strategies)
2) conflict games
   2a) simultaneous decision making (Nash strategies)
   2b) sequential decision making (Stackelberg strategy)
3) zero-sum differential games.

Development of game theory in mathematics, statistics, economy, politics, sociology, ecology, biology, hydrology, engineering, and military.

Important for EE students and researchers: applications of games to networking, communications, and signal processing. Comeback of zero-sum games to control (since H-infinity optimization (optimization of system transfer function in the frequency domain = zero-sum differential game problem).

1994: John Nash Jr. got the Nobel Prize for economics, together with John Harsanyi and Reinhard Selten - both game theorists in economics.
Journals:

1) Game Theory (Math - too much theoretical and mathematical)
2) JOTA (Math & Eng - friendly to grad. student)
3) Games and Economic Behavior (Econony)
4) IEEE Transactions on Control
5) Automatica
6) Recently IEEE Trans. on Communications
   IEEE J. Selected Areas in Communications
   IEEE Trans. on Signal Processing
   Journal of ACM

Games and Related Course at Rutgers
16: 220: 546 Topics in Game Theory (Economics)
   (Prof. E. Friedman and R. McLean)

no graduate course on games in Mathematics and Statistics Department.

16: 198: 524 Linear Programming (Coop. Sci. Dept.)
16: 540: 514 Deterministic Models in Industrial Eng (Linear Programming)
16: 540: 615 Nonlinear Programming
16: 198: 524 Nonlinear Programming Algorithms
16: 332: 510 Synthesis of Optimal Control Systems
   Dynamic Programming - Dynamic Optimization

Heed Linear and Nonlinear Programming for Static Games
Heed Dynamic Optimization for Dynamic Games
CLASSIC GAMES

1) EXTENSIVE FORM
2) NORMAL FORM
3) MATRIX FORM

MODERN GAMES

4) IN EUCLIDEAN SPACES (finite dimensional $E^n$)
   a) zero-sum games (antagonistic games)
   b) constant games (Nash and Stackelberg)
   c) cooperative games (Pareto)

5) GAMES IN FUNCTIONAL SPACES
   a) differential games (zero-sum games)
   b) constant games
   c) cooperative games
   d) coalition games
   e) team theory

All of the above can be deterministic and stochastic.

Schools with strong programs in games:
- Univ. of Tennessee (Eng.)
- Princeton Univ. (Math - in the past)
- Columbia Univ. (Stats - in the past)

These days Lazar teaches a course in EE Dept.
at Columbia "Resource Allocation and Networking Games". He's former student Dr. Kotteris teaches
the structural course at NYU in the Computer Sci. Dept.
EXTENSIVE FORM (DESCRIPTION)

This approach originated in the work of Von Neumann. It is used to describe games with finite number of strategies (moves). These games are called FINITE GAMES. They can be deterministic (with full information) or stochastic (with partial information).

DETERMINISTIC FINITE GAMES:
1) The order of strategies (moves) is precisely defined.
2) The game has a finite number of moves.
3) At every move, the number of available alternatives is finite.
4) Every player (decision maker) has complete information about the game.

Such games are represented by GAME TREES (graph).

A connected graph is composed of a finite number of nodes connected by branches. A connected graph without closed loop of branches is called a tree.

- Terminal nodes (end points)
- H-nodes
- H-1 branches
- Distinguished node (start of the game)
Deterministic Finite Game is described by
1) A finite tree with a distinguished node.
2) Utility (loss) function assigned to every terminal node

\[ x_j \left( x_{i_1}, x_{i_2}, \ldots, x_{i_n} \right) \quad j = 1, 2, \ldots, k = \text{number of terminal nodes} \]

loss of player \( n \) at node \( j \)

3) Partition of internal nodes into \( S_1, S_2, \ldots, S_m \) subsets indicating who is supposed to make a move at the given node.

\[(2,5,1) (3,8) (2,4,7) (4,5,4) (3,7,4) (2,5,8) (3,1,3) (2,5,7)\]

In the case of stochastic games we can do partitioning into \( S_0, S_1, S_2, \ldots, S_m \).
So nodes determine their moves randomly.

A strategy of player \( i \) determines only one branch at each node at which the player \( i \) is supposed to make a move.
Example:
Two piles of stones, with respectively $n$ and $m$ stones

- $m$-stones
- $n$-stones

Game rule: Take either the same number of stones from each pile or an arbitrary number of stones from one of the piles.

The winner is the player who clears the last stone.

The players take turns.

The maximum number of moves is $n + m$.

The players have complete information about the game.

$\Rightarrow$ finite deterministic game.

Take case $n = 3$ and $m = 2$ and draw the tree.

- $I$ player
  - $(-)$
  - $(+)$
  - $(+)$
  - $(+)$
  - $(+)$

$II$ player
  - $(1, 0)$
  - $(2, 1)$
  - $(2, 0)$
  - $(3, 1)$
  - $(3, 2)$
  - $(0, 2)$

$I$ player
  - $(3, 2) =$ state of the game

$(-) =$ loss for player $I$, $(+)$ = win for player $I$
Looks simple. But, what about $n=1119$, $m=235$ or $n=23$, $m=36$ to make the game simpler. Who is going to win this game and what is the winning strategy?

Available strategies

![Diagram](attachment:diagram.png)

Apparentely $(2,0)$, $(0,1)$, $(3,2)$

be the player who has to make a move wins.

Also $(1,2)$ the player who makes the move is going to loose. Also, using the tree can be shown that $(3,5) \Rightarrow$ the player who has to make the move is going to loose.

Solution: (tough to find)

$$L = \left( \left[ \frac{1+\sqrt{5}}{2} \right], \left[ \frac{3+\sqrt{5}}{2} \right] \right), \text{ for } \kappa = 1, 2. \text{ } \left[ \frac{\cdot}{\cdot} \right] \text{ denotes an integer part operation.}$$

determines a discrete "surface" on which the player who has to make the move loses the game.
"Loosing Surface"

\[ L = \{ (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), (12, 20), (14, 23), (16, 26), (17, 28), (19, 29), (21, 34), (22, 36) \} \]

Hence, if the state of the game \((n, m) \in L\)

the player who has to make the move loses.

If \((n, m) \notin L\) then the player who has to make the move can bring it to \(L\) and assures himself of winning.

(Ex) \(n = 25, \ m = 36 \) \(\Rightarrow\) take 3 stones from the first pile \(\Rightarrow (22, 36) \in L\)

\(n = 149, \ m = 235\)

\(k = 90 \Rightarrow \left\lfloor \frac{3 + \sqrt{5} \times 90}{2} \right\rfloor = 235.623 = 235 \quad \text{and} \quad \left\lfloor \frac{1 + \sqrt{5} \times 90}{2} \right\rfloor = 145\)