5. Reflection and Transmission

5.1 Propagation Matrices

In this chapter, we consider uniform plane waves incident normally on material interfaces. Using the boundary conditions for the fields, we will relate the forward-backward fields on one side of the interface to those on the other side, expressing the relationship in terms of a $2\times2$ matching matrix.

If there are several interfaces, we will propagate our forward-backward fields from one interface to the next with the help of a $2\times2$ propagation matrix. The combination of a matching and a propagation matrix relating the fields across different interfaces will be referred to as a transfer or transition matrix.

We begin by discussing propagation matrices. Consider an electric field that is linearly polarized in the $x$-direction and propagating along the $z$-direction in a lossless (homogeneous and isotropic) dielectric. Setting $E(z) = \hat{x} E_x(z) = \hat{x} \hat{E}(z)$ and $H(z) = \hat{y} H_x(z) = \hat{y} H(z)$, we have from Eq. (2.2.6):

$$E(z) = E_0 e^{-jkz} + E_0 e^{jkz} = E_+(z) + E_-(z)$$

$$H(z) = \frac{1}{\eta} [E_0 e^{-jkz} - E_0 e^{jkz}] = \frac{1}{\eta} [E_+(z) - E_-(z)]$$

where the corresponding forward and backward electric fields at position $z$ are:

$$E_+(z) = E_0 e^{-jkz}$$

$$E_-(z) = E_0 e^{jkz}$$

(5.1.2)

We can also express the fields $E_\pm(z)$ in terms of $E(z), H(z)$. Adding and subtracting the two equations (5.1.1), we find:

$$E_+(z) = \frac{1}{2} [E(z) + \eta H(z)]$$

$$E_-(z) = \frac{1}{2} [E(z) - \eta H(z)]$$

(5.1.3)

Eqs. (5.1.1) and (5.1.3) can also be written in the convenient matrix forms:

$$\begin{bmatrix} E \\ H \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \eta^{-1} & -\eta^{-1} \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}, \quad \begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & \eta \\ 1 & -\eta \end{bmatrix} \begin{bmatrix} E \\ H \end{bmatrix}$$

(5.1.4)

Two useful quantities in interface problems are the wave impedance at $z$:

$$Z(z) = \frac{E(z)}{H(z)}$$

(wave impedance) (5.1.5)

and the reflection coefficient at position $z$:

$$\Gamma(z) = \frac{E_-(z)}{E_+(z)}$$

(reflection coefficient) (5.1.6)

Using Eq. (5.1.3), we have:

$$\Gamma = \frac{E_0 \frac{1}{\eta} (E - \eta H)}{E_0 \frac{1}{\eta} (E + \eta H)} = \frac{E - \eta}{E + \eta} = \frac{Z - \eta}{Z + \eta}$$

Similarly, using Eq. (5.1.1) we find:

$$Z = \frac{E_+}{H} = \frac{E_+ + E_-}{\eta (E_+ - E_-)} = \eta \frac{1 + \frac{E_-}{E_+}}{1 - \frac{E_-}{E_+}} = \eta \frac{1 + \Gamma}{1 - \Gamma}$$

Thus, we have the relationships:

$$Z(z) = \eta \frac{1 + \Gamma(z)}{1 - \Gamma(z)} \quad \implies \quad \Gamma(z) = \frac{Z(z) - \eta}{Z(z) + \eta}$$

(5.1.7)

Using Eq. (5.1.2), we find:

$$\Gamma(z) = \frac{E_-(z)}{E_+(z)} = \frac{E_0 e^{jkz}}{E_0 e^{-jkz}} = \Gamma(0) e^{2jkz}$$

where $\Gamma(0) = E_0--/E_0+$ is the reflection coefficient at $z = 0$. Thus,

$$\Gamma(z) = \Gamma(0) e^{2jkz}$$

(propagation of $\Gamma$) (5.1.8)

Applying (5.1.7) at $z$ and $z = 0$, we have:

$$\frac{Z(z) - \eta}{Z(z) + \eta} = \Gamma(z) = \Gamma(0) e^{2jkz} = \frac{Z(0) - \eta}{Z(0) + \eta} e^{2jkz}$$

This may be solved for $Z(z)$ in terms of $Z(0)$, giving after some algebra:

$$Z(z) = \eta \frac{Z(0) - j\eta \tan k z}{\eta - jZ(0) \tan k z}$$

(propagation of $Z$) (5.1.9)
5.1. Propagation Matrices

The reason for introducing so many field quantities is that the three quantities \( E_1(z), E_2(z), \Gamma(z) \) have simple propagation properties, whereas \( E(z), H(z), Z(z) \) do not. On the other hand, \( E(z), H(z), Z(z) \) match simply across interfaces, whereas \( E_1(z), E_2(z), \Gamma(z) \) do not.

Eqs. (5.1.1) and (5.1.2) relate the field quantities at location \( z \) to the quantities at \( z = 0 \). In matching problems, it proves more convenient to be able to relate these quantities at two arbitrary locations.

Fig. 5.1.1 depicts the quantities \( E(z), H(z), E_1(z), E_2(z), Z(z), \Gamma(z) \) at the two locations \( z_1 \) and \( z_2 \) separated by a distance \( l = z_2 - z_1 \). Using Eq. (5.1.2), we have for the forward field at these two positions:

\[
E_{2+} = E_0 e^{-jkz_2}, \quad E_{1+} = E_0 e^{-jkz_1} = E_{0+} e^{-jk(z_1+l)} = e^{jkl} E_{2+}
\]

\[
\begin{bmatrix}
E_{1+} \\
E_{2+}
\end{bmatrix}
= e^{jkl}
\begin{bmatrix}
E_{2+} \\
E_{1+}
\end{bmatrix}
\]

\[
E_{1+} = e^{jkl} E_{2+}, \quad E_{1-} = e^{-jkl} E_{2-}
\]

and in matrix form:

\[
\begin{bmatrix}
E_{1+} \\
E_{1-}
\end{bmatrix}
= \begin{bmatrix}
e^{jkl} & 0 \\
0 & e^{-jkl}
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
E_{2-}
\end{bmatrix}
\] (propagation matrix)

We will refer to this as the propagation matrix for the forward and backward fields. It follows that the reflection coefficients will be related by:

\[
\Gamma_1 = \frac{E_{1+}}{E_{2+}} = \frac{E_2 e^{-jkl}}{E_2 e^{jkl}} = e^{-2jkl}, \quad \text{or,}
\]

\[
\Gamma_1 = \Gamma_2 e^{-2jkl}
\] (reflection coefficient propagation) (5.1.12)

Using the matrix relationships (5.1.4) and (5.1.11), we may also express the total electric and magnetic fields \( E_1, H_1 \) at position \( z_1 \) in terms of \( E_2, H_2 \) at position \( z_2 \):

\[
\begin{bmatrix}
E_{1+} \\
H_{1+}
\end{bmatrix}
= \begin{bmatrix}
e^{jkl} & 0 \\
0 & e^{-jkl}
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
H_{2+}
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
\eta^{-1} & -\eta^{-1}
\end{bmatrix}
\begin{bmatrix}
e^{jkl} & 0 \\
0 & e^{-jkl}
\end{bmatrix}
\begin{bmatrix}
1 & \eta \\
1 & -\eta
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
H_{2+}
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
1 & 1 \\
\eta^{-1} & -\eta^{-1}
\end{bmatrix}
\begin{bmatrix}
e^{jkl} & 0 \\
0 & e^{-jkl}
\end{bmatrix}
\begin{bmatrix}
1 & \eta \\
1 & -\eta
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
H_{2+}
\end{bmatrix}
\]

which gives after some algebra:

\[
\begin{bmatrix}
E_{1+} \\
H_{1+}
\end{bmatrix}
= \begin{bmatrix}
\cos kl & \eta \sin kl \\
\eta^{-1} \sin kl & \cos kl
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
H_{2+}
\end{bmatrix}
\] (propagation matrix) (5.1.13)

Writing \( \eta = \eta_0 / n \), where \( n \) is the refractive index of the propagation medium, Eq. (5.1.13) can written in following form, which is useful in analyzing multilayer structures and is common in the thin-film literature [632, 634, 638, 649]:

\[
\begin{bmatrix}
E_{1+} \\
H_{1+}
\end{bmatrix}
= \begin{bmatrix}
\cos \delta & \eta \sin \delta \\
\eta^{-1} \sin \delta & \cos \delta
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
H_{2+}
\end{bmatrix}
\] (propagation matrix) (5.1.14)

where \( \delta \) is the propagation phase constant, \( \delta = kl = k_0 n l = 2\pi n l / \lambda_0 \), and \( nl \) the optical length. Eqs. (5.1.13) and (5.1.5) imply for the propagation of the wave impedance:

\[
Z_1 = \frac{E_{1+}}{H_{1+}} = \frac{E_2 \cos kl + jn \eta \sin kl}{\eta \cos kl + jn E_2 \sin kl}
\]

which gives:

\[
Z_1 = \frac{Z_2 \cos kl + jn \eta \sin kl}{\eta \cos kl + jZ_2 \sin kl}
\] (impedance propagation) (5.1.15)

It can also be written in the form:

\[
Z_1 = \eta \frac{Z_2 \cos kl + jn \eta \sin kl}{\eta + jZ_2 \tan kl}
\] (impedance propagation) (5.1.16)

A useful way of expressing \( Z_1 \) is in terms of the reflection coefficient \( \Gamma_2 \). Using (5.1.7) and (5.1.12), we have:

\[
Z_1 = \eta \frac{1 + \Gamma_1}{1 - \Gamma_1} = \eta \frac{1 + \Gamma_2 e^{-2jkl}}{1 - \Gamma_2 e^{-2jkl}} \quad \text{or,}
\]

\[
Z_1 = \eta \frac{1 + \Gamma_2 e^{-2jkl}}{1 - \Gamma_2 e^{-2jkl}}
\] (5.1.17)

We mention finally two special propagation cases: the half-wavelength and the quarter-wavelength cases. When the propagation distance is \( l = \lambda / 2 \), or any integral multiple thereof, the wave impedance and reflection coefficient remain unchanged. Indeed, we have in this case \( kl = 2\pi n / \lambda = \pi / 2 \) and \( 2kl = \pi \). It follows from Eq. (5.1.12) that \( \Gamma_1 = \Gamma_2 \) and hence \( Z_1 = Z_2 \).

If on the other hand \( l = \lambda / 4 \), or any odd integral multiple thereof, then \( kl = \pi / 2 \) and \( 2kl = \pi \). The reflection coefficient changes sign and the wave impedance inverts:

\[
\Gamma_1 = \Gamma_2 e^{-2jkl} = \Gamma_2 e^{-j\pi} = -\Gamma_2 \quad \Rightarrow \quad Z_1 = \eta \frac{1 + \Gamma_1}{1 - \Gamma_1} = \eta \frac{1 - \Gamma_2}{1 + \Gamma_2} = \eta \frac{1}{Z_2 / \eta} = \eta^2 Z_2
\]
5.2. Matching Matrices

Thus, we have in the two cases:

\[
\begin{align*}
\frac{l}{2} & \Rightarrow Z_1 = Z_2, \quad \Gamma_1 = \Gamma_2 \\
\frac{l}{4} & \Rightarrow Z_1 = \frac{\eta_2}{Z_2}, \quad \Gamma_1 = -\Gamma_2
\end{align*}
\]

(5.1.18)

5.2 Matching Matrices

Next, we discuss the matching conditions across dielectric interfaces. We consider a planar interface (taken to be the xy-plane at some location z) separating two dielectric/conducting media with (possibly complex-valued) characteristic impedances \(\eta, \eta'\), as shown in Fig. 5.2.1.\(^1\)

\[
\begin{array}{|c|c|c|}
\hline
\eta & \eta' \\
\hline
E_+ & E'_+ \\
E_- & E'_- \\
\hline
\rho, \tau & \rho', \tau' \\
\hline
\end{array}
\]

Fig. 5.2.1 Fields across an interface.

Because the normally incident fields are tangential to the interface plane, the boundary conditions require that the total electric and magnetic fields be continuous across the two sides of the interface:

\[
\begin{align*}
E &= E' \\
H &= H'
\end{align*}
\] (continuity across interface) (5.2.1)

In terms of the forward and backward electric fields, Eq. (5.2.1) reads:

\[
E_+ + E_- = E'_+ + E'_- \\
\frac{1}{\eta}(E_+ - E_-) = \frac{1}{\eta'}(E'_+ - E'_-)
\] (5.2.2)

Eq. (5.2.2) may be written in a matrix form relating the fields \(E_+\) on the left of the interface to the fields \(E'_+\) on the right:

\[
\begin{bmatrix} E_+ \\ E_-
\end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & \rho \\ \rho & 1
\end{bmatrix} \begin{bmatrix} E'_+ \\ E'_-
\end{bmatrix}
\] (matching matrix) (5.2.3)

\(\frac{1}{\tau} = \frac{1}{\eta'} = \frac{\eta}{\eta' + \eta} \) (5.2.4)

where \(\{\rho, \tau\} \) and \(\{\rho', \tau'\} \) are the elementary reflection and transmission coefficients from the left and from the right of the interface, defined in terms of \(\eta, \eta'\) as follows:

\[
\begin{align*}
\rho &= \frac{\eta' - \eta}{\eta' + \eta} \\
\tau &= \frac{2\eta}{\eta' + \eta} \\
\rho' &= \frac{\eta - \eta'}{\eta + \eta'} \\
\tau' &= \frac{2\eta'}{\eta + \eta'}
\end{align*}
\] (5.2.5)

Writing \(\eta = \eta_0/\eta\) and \(\eta' = \eta_0/\eta'\), we have in terms of the refractive indices:

\[
\begin{align*}
\rho &= \frac{n - n'}{n + n'} \\
\tau &= \frac{2n}{n + n'} \\
\rho' &= \frac{n' - n}{n' + n} \\
\tau' &= \frac{2n'}{n' + n}
\end{align*}
\] (5.2.6)

These are also called the Fresnel coefficients. We note various useful relationships:

\[
\tau = 1 + \rho, \quad \rho' = -\rho, \quad \tau' = 1 + \rho' = 1 - \rho, \quad \tau \tau' = 1 - \rho^2
\] (5.2.7)

In summary, the total electric and magnetic fields \(E, H\) match simply across the interface, whereas the forward/backward fields \(E_\pm\) are related by the matching matrices of Eqs. (5.2.3) and (5.2.4). An immediate consequence of Eq. (5.2.1) is that the wave impedance is continuous across the interface:

\[
Z = \frac{E}{H} = \frac{E'}{H'} = Z'
\]

On the other hand, the corresponding reflection coefficients \(\Gamma = E_-/E_+\) and \(\Gamma' = E'_-E'_+\) match in a more complicated way. Using Eq. (5.1.7) and the continuity of the wave impedance, we have:

\[
\eta \frac{1 + \Gamma}{1 - \Gamma} = Z = Z' = \eta' \frac{1 + \Gamma'}{1 - \Gamma'}
\]

which can be solved to get:

\[
\Gamma = \frac{\rho + \Gamma'}{1 + \rho \Gamma'} \quad \text{and} \quad \Gamma' = \frac{\rho' + \Gamma}{1 + \rho' \Gamma}
\]

The same relationship follows also from Eq. (5.2.3):

\[
\begin{align*}
\Gamma &= \frac{E_-}{E_+} = \frac{1}{\tau} \frac{\rho E'_+ + E'_-}{E'_+ + \rho E'_-} = \frac{\rho + \Gamma'}{1 + \rho' \Gamma'} \\
\Gamma' &= \frac{E'_-}{E'_+} = \frac{1}{\tau'} \frac{\rho' E'_+ + E'_-}{E'_+ + \rho' E'_-} = \frac{\rho + \Gamma}{1 + \rho \Gamma'}
\end{align*}
\]
To summarize, we have the matching conditions for $Z$ and $\Gamma$:

$$Z = \zeta' \quad \Rightarrow \quad \Gamma = \frac{\rho + \Gamma'}{1 + \rho \Gamma'} \quad \Rightarrow \quad \Gamma = \frac{\Gamma' + \Gamma}{1 + \rho' \Gamma}$$ \hspace{1cm} (5.2.9)

Two special cases, illustrated in Fig. 5.2.1, are when there is only an incident wave on the interface from the left, so that $E_+ = 0$, and when the incident wave is only from the right, so that $E_+ = 0$. In the first case, we have $\Gamma = E_+ / E_-' = 0$, which implies $Z' = \eta' (1 + \Gamma') / (1 - \Gamma') = \eta'$. The matching conditions give then:

$$Z = \zeta' = \eta', \quad \Gamma = \frac{\rho + \Gamma'}{1 + \rho \Gamma'} = \rho$$

The matching matrix (5.2.3) implies in this case:

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} E_+ \\ 0 \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} E' \\ \rho E_+ \end{bmatrix}$$

Expressing the reflected and transmitted fields $E_-, E_+'$ in terms of the incident field $E_+$, we have:

$$\begin{align*}
E_- &= \rho E_+ \\
E_+' &= \tau E_+
\end{align*}$$ \hspace{1cm} (left-incident fields) \hspace{1cm} (5.2.10)

This justifies the terms reflection and transmission coefficients for $\rho$ and $\tau$. In the right-incident case, the condition $E_+ = 0$ implies for Eq. (5.2.4):

$$\begin{bmatrix} E_+ \\ E_-' \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & \rho' \\ \rho' & 1 \end{bmatrix} \begin{bmatrix} 0 \\ E_- \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 0 \\ \rho' E_- \end{bmatrix}$$

These can be rewritten in the form:

$$\begin{align*}
E_+' &= \rho' E_- \\
E_- &= \tau' E_-
\end{align*}$$ \hspace{1cm} (right-incident fields) \hspace{1cm} (5.2.11)

which relates the reflected and transmitted fields $E_+, E_-'$ to the incident field $E_+$. In this case $\Gamma = E_+/E_-' = \infty$ and the third of Eqs. (5.2.9) gives $\Gamma' = E_+ / E_+' = 1/\rho'$, which is consistent with Eq. (5.2.11).

When there are incident fields from both sides, that is, $E_+, E_+'$, we may invoke the linearity of Maxwell’s equations and add the two right-hand sides of Eqs. (5.2.10) and (5.2.11) to obtain the outgoing fields $E_+', E_-'$ in terms of the incident ones:

$$\begin{align*}
E_+' &= \tau E_+ + \rho' E_- \\
E_- &= \rho E_+ + \tau' E_-
\end{align*}$$ \hspace{1cm} (5.2.12)

This gives the scattering matrix relating the outgoing fields to the incoming ones:

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \begin{bmatrix} \tau & \rho' \\ \rho & \tau' \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}$$ \hspace{1cm} (scattering matrix) \hspace{1cm} (5.2.13)

Using the relationships Eq. (5.2.8), it is easily verified that Eq. (5.2.13) is equivalent to the matching matrix equations (5.2.3) and (5.2.4).

## 5.3 Reflected and Transmitted Power

For waves propagating in the $z$-direction, the time-averaged Poynting vector has only a $z$-component:

$$P = \frac{1}{2} \text{Re}(\mathbf{k} \times \mathbf{E}^* \mathbf{H}^*) = \frac{1}{2} \text{Re}(\mathbf{E} \times \mathbf{H}^*)$$

A direct consequence of the continuity equations (5.2.1) is that the Poynting vector is conserved across the interface. Indeed, we have:

$$P = \frac{1}{2} \text{Re}(\mathbf{E}^* \mathbf{H}^*) = \frac{1}{2} \text{Re}(\mathbf{E'} \mathbf{H'}^*) = P'$$ \hspace{1cm} (5.3.1)

In particular, consider the case of a wave incident from a lossless dielectric $\eta$ onto a lossy dielectric $\eta'$. Then, the conservation equation (5.3.1) reads in terms of the forward and backward fields (assuming $E'_- = 0$):

$$P = \frac{1}{2} \eta (|E_+|^2 - |E_-|^2) = \text{Re} \left( \frac{1}{2\eta} |E'_+|^2 \right)$$

The left hand-side is the difference of the incident and the reflected power and represents the amount of power transmitted into the lossy dielectric per unit area. We saw in Sec. 2.6 that this power is completely dissipated into heat inside the lossy dielectric (assuming it is infinite to the right.) Using Eqs. (5.2.10), we find:

$$P = \frac{1}{2\eta} |E_+|^2 |1 - |\rho|^2| = \text{Re} \left( \frac{1}{2\eta} \right) |E'_+|^2 |\tau|^2$$ \hspace{1cm} (5.3.2)

This equality requires that:

$$\frac{1}{\eta} |1 - |\rho|^2| = \text{Re} \left( \frac{1}{\eta} \right) |\tau|^2$$ \hspace{1cm} (5.3.3)

This can be proved using the definitions (5.2.5). Indeed, we have:

$$\frac{\eta}{\eta'} = \frac{1 - |\rho|^2}{1 + |\rho|^2} \Rightarrow \text{Re} \left( \frac{\eta}{\eta'} \right) = \frac{1 - |\rho|^2}{|1 + |\rho|^2|} = \frac{1 - |\rho|^2}{|\tau|^2}$$

which is equivalent to Eq. (5.3.3), if $\eta$ is lossless (i.e., real.) Defining the incident, reflected, and transmitted powers by

$$P_{in} = \frac{1}{2\eta} |E_+|^2$$

$$P_{ref} = \frac{1}{2\eta} |E_-|^2 = \frac{1}{2\eta} |E_+|^2 |\rho|^2 = P_{in} |\rho|^2$$

$$P_{tr} = \text{Re} \left( \frac{1}{2\eta} \right) |E'_+|^2 = \text{Re} \left( \frac{1}{2\eta} \right) |E_+|^2 |\tau|^2 = P_{in} \text{Re} \left( \frac{\eta}{\eta'} \right) |\tau|^2$$

Then, Eq. (5.3.2) reads $P_{tr} = P_{in} - P_{ref}$. The **power reflection and transmission coefficients**, also known as the **reflectance** and **transmittance**, give the percentage of the incident power that gets reflected and transmitted:
5.3. Reflected and Transmitted Power

\[ \frac{P_{\text{ref}}}{P_{\text{in}}} = |\rho|^2, \quad \frac{P_{\text{tr}}}{P_{\text{in}}} = 1 - |\rho|^2 = \text{Re} \left( \frac{\rho}{\eta} \right) |\tau|^2 = \text{Re} \left( \frac{n_1^*}{n_0} \right) |\tau|^2 \]  
(5.3.4)

If both dielectrics are lossless, then \( \rho, \tau \) are real-valued. In this case, if there are incident waves from both sides of the interface, it is straightforward to show that the net power moving towards the z-direction is the same at either side of the interface:

\[ P = \frac{1}{2\eta} (|E_+|^2 - |E_-|^2) = \frac{1}{2\eta} (|E'_+|^2 - |E'_-|^2) = P' \]  
(5.3.5)

This follows from the matrix identity satisfied by the matching matrix of Eq. (5.2.3):

\[ \frac{1}{\tau^2} \left[ \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] = \frac{\eta}{\eta'} \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \]

If \( \rho, \tau \) are real, then we have with the help of this identity and Eq. (5.2.3):

\[ P = \frac{1}{2\eta} (|E_+|^2 - |E_-|^2) = \frac{1}{2\eta} \left[ E'_+^* E'_- \right] \left[ \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ E'_- \right] \]

\[ = \frac{1}{2\eta} \frac{\eta}{\eta'} \left[ E'_+^* E'_- \right] \left[ \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] \left[ E'_- \right] \]

Example 5.3.1: Glasses have a refractive index of the order of \( n = 1.5 \) and dielectric constant \( \epsilon = n^2 \epsilon_0 = 2.25\epsilon_0 \). Calculate the percentages of reflected and transmitted powers for visible light incident on a planar glass interface from air.

Solution: The characteristic impedance of glass will be \( \eta = \eta_0/n \). Therefore, the reflection and transmission coefficients can be expressed directly in terms of \( n \), as follows:

\[ \rho = \frac{n-1}{n+1} \rho_0 = \frac{n^{-1}-1}{n^{-1}+1}, \quad \tau = 1 + \rho = \frac{2}{1+n} \]

For \( n = 1.5 \), we find \( \rho = -0.2 \) and \( \tau = 0.8 \). It follows that the power reflection and transmission coefficients will be

\[ |\rho|^2 = 0.04, \quad 1 - |\rho|^2 = 0.96 \]

That is, 4% of the incident power is reflected and 96% transmitted.

Example 5.3.2: A uniform plane wave of frequency \( f \) is normally incident from air onto a thick conducting sheet with conductivity \( \sigma \), and \( \epsilon = \epsilon_0 \), \( \mu = \mu_0 \). Show that the proportion of power transmitted into the conductor (and then dissipated into heat) is given approximately by

\[ \frac{P_{\text{tr}}}{P_{\text{in}}} = \frac{4R_s}{\eta_0} = \sqrt{\frac{8\omega \epsilon_0}{\sigma}} \]

Calculate this quantity for \( f = 1 \) GHz and copper \( \sigma = 5.8 \times 10^7 \) Siemens/m.

Solution: For a good conductor, we have \( \sqrt{\omega \epsilon_0/\sigma} \ll 1 \). It follows from Eq. (2.8.4) that \( R_s/\eta_0 = \sqrt{\omega \epsilon_0/2\sigma} \ll 1 \). From Eq. (2.8.2), the conductor’s characteristic impedance is \( \eta_c = R_s (1 + j) \). Thus, the quantity \( \eta_c/\eta_0 = (1 + j) R_s/\eta_0 \) is also small. The reflection and transmission coefficients \( \rho, \tau \) can be expressed to first-order in the quantity \( \eta_c/\eta_0 \) as follows:

\[ \tau = \frac{2\eta_c}{\eta_0 + \eta_0} = \frac{2\eta_c}{\eta_0}, \quad \rho = \tau - 1 = -1 + \frac{2\eta_c}{\eta_0} \]

Similarly, the power transmission coefficient can be approximated as

\[ 1 - |\rho|^2 = 1 - |\tau - 1|^2 = 1 - 1 - |\tau|^2 + 2 \text{Re}(\tau) = 2 \text{Re}(\tau) = 2 \frac{2 \text{Re}(\eta_c)}{\eta_0} = \frac{4R_s}{\eta_0} \]

where we neglected \( |\tau|^2 \) as it is second-order in \( \eta_c/\eta_0 \). For copper at 1 GHz, we have \( \sqrt{\omega \epsilon_0/2\sigma} = 2.19 \times 10^{-5} \), which gives \( R_s = \eta_0 \sqrt{\omega \epsilon_0/2\sigma} = 377 \times 2.19 \times 10^{-5} = 0.0082 \Omega \). It follows that \( 1 - |\rho|^2 = 4R_s/\eta_0 = 8.76 \times 10^{-5} \).

This represents only a small power loss of 8.76\% percent and the sheet acts as very good mirror at microwave frequencies.

On the other hand, at optical frequencies, e.g., \( f = 600 \) THz corresponding to green light with \( \lambda = 500 \) nm, the exact equations (2.6.3) yield the value for the characteristic impedance of the sheet \( \eta_c = 6.3924 + 6.3888i \) \( \Omega \) and the reflection coefficient \( \rho = -0.9661 + 0.0328i \). The corresponding power loss is \( 1 - |\rho|^2 \leq 0.065 \), or 6.5 percent. Thus, metallic mirrors are fairly lossy at optical frequencies.

Example 5.3.3: A uniform plane wave of frequency \( f \) is normally incident from air onto a thick conductor with conductivity \( \sigma \), and \( \epsilon = \epsilon_0 \), \( \mu = \mu_0 \). Determine the reflected and transmitted electric and magnetic fields to first-order in \( \eta_c/\eta_0 \) and in the limit of a perfect conductor (\( \eta_c = 0 \)).

Solution: Using the approximations for \( \rho \) and \( \tau \) of the previous example and Eq. (5.2.10), we have for the reflected, transmitted, and total electric fields at the interface:

\[ E_+ = \rho E_+ = \left( -1 + \frac{2\eta_c}{\eta_0} \right) E_+ \]

\[ E'_+ = \tau E_+ = \frac{2\eta_c}{\eta_0} E_+ \]

\[ E = E_+ + E'_+ = \frac{2\eta_c}{\eta_0} E_+ = E'_+ = E' \]

For a perfect conductor, we have \( \sigma \rightarrow \infty \) and \( \eta_c/\eta_0 \rightarrow 0 \). The corresponding total tangential electric field becomes zero \( E = E' = 0 \), and \( \rho = -1, \tau = 0 \). For the magnetic fields, we need to develop similar first-order approximations. The incident magnetic field intensity is \( H_+ = E_+/\eta_0 \). The reflected field becomes to first order:

\[ H_- = \frac{1}{\eta_0} E_+ = -\frac{1}{\eta_0} \rho E_+ = -\rho H_+ = \left( -1 + \frac{2\eta_c}{\eta_0} \right) H_+ \]

Similarly, the transmitted field is
5.4 Single Dielectric Slab

\[ H' = \frac{1}{\eta_c} E'_1 - \frac{1}{\eta_c} \tau E'_s = \frac{\eta_0}{\eta_c} \tau H_s = \frac{\eta_0}{\eta_c} \frac{2 \eta}{\eta_c + \eta_0} H_s = \frac{2 \eta_0}{\eta_c + \eta_0} H_s = 2 \left( 1 - \frac{\eta_0}{\eta_c} \right) H_s \]

The total tangential field at the interface will be:

\[ H = H_+ + H_- = 2 \left( 1 - \frac{\eta_0}{\eta_c} \right) H_s = H'_s + H' = H' \]

In the perfect conductor limit, we find \( H = H' = 2H_+ \). As we saw in Sec. 2.6, the fields just inside the conductor, \( E'_s, H'_s \), will attenuate while they propagate. Assuming the interface is at \( z = 0 \), we have:

\[ E'_+(z) = E'_1 e^{-\alpha z} e^{-j\beta z}, \quad H'_+(z) = H'_1 e^{-\alpha z} e^{-j\beta z} \]

where \( \alpha = \beta = (1 - j)/\delta \), and \( \delta \) is the skin depth \( \delta = \sqrt{\omega \mu \sigma / Z} \). We saw in Sec. 2.6 that the effective surface current is equal in magnitude to the magnetic field at \( z = 0 \), that is, \( J_s = H'_s \). Because of the boundary condition \( H = H' = H'_s \), we obtain the result \( J_s = H_s \), or vectorially, \( J_s = H \times \hat{n} = \hat{n} \times H \), where \( \hat{n} \) is the outward normal to the conductor.

This result provides a justification of the boundary condition \( J_s \) at \( H \) at an interface with a perfect conductor. \( \Box \)

5.4 Single Dielectric Slab

Multiple interface problems can be handled in a straightforward way with the help of the matching and propagation matrices. For example, Fig. 5.4.1 shows a two-interface problem with a dielectric slab \( \eta_1 \) separating the semi-infinite media \( \eta_a \) and \( \eta_b \).

![Fig. 5.4.1 Single dielectric slab.](image)

Let \( l_1 \) be the width of the slab, \( k_1 = \omega / c_1 \) the propagation wavenumber, and \( \lambda_1 = 2\pi / k_1 \) the corresponding wavelength within the slab. We have \( \lambda_1 = \lambda_0 / \eta_1 \), where \( \lambda_0 \) is the free-space wavelength and \( \eta_1 \) the refractive index of the slab. We assume the incident field is from the left medium \( \eta_a \), and thus, in medium \( \eta_b \) there is only a forward wave.

Let \( \rho_1, \rho_2 \) be the elementary reflection coefficients from the left sides of the two interfaces, and let \( \tau_1, \tau_2 \) be the corresponding transmission coefficients:

\[ \rho_1 = \frac{\eta_1 - \eta_a}{\eta_1 + \eta_a}, \quad \rho_2 = \frac{\eta_b - \eta_1}{\eta_b + \eta_1}, \quad \tau_1 = 1 + \rho_1, \quad \tau_2 = 1 + \rho_2 \]

To determine the reflection coefficient \( \Gamma_1 \) into medium \( \eta_a \), we apply Eq. (5.2.9) to relate \( \Gamma_1 \) to the reflection coefficient \( \Gamma'_1 \) at the right-side of the first interface. Then, we propagate to the left of the second interface with Eq. (5.1.12) to get:

\[ \Gamma_1 = \frac{\rho_1 + \rho'_1}{1 + \rho_1 \rho'_1} = \frac{\rho_1 + \rho'_2 e^{-2jk_1 l_1}}{1 + \rho_1 \rho'_2 e^{-2jk_1 l_1}} \]

At the second interface, we apply Eq. (5.2.9) again to relate \( \Gamma_2 \) to \( \Gamma'_2 \). Because there are no backward-moving waves in medium \( \eta_b \), we have \( \Gamma'_2 = 0 \). Thus,

\[ \Gamma_2 = \frac{\rho_2 + \rho'_2}{1 + \rho_2 \rho'_2} = \rho_2 \]

We finally find for \( \Gamma_1 \):

\[ \Gamma_1 = \frac{\rho_1 + \rho'_2 e^{-2jk_1 l_1}}{1 + \rho_1 \rho'_2 e^{-2jk_1 l_1}} \]

This expression can be thought of as function of frequency. Assuming a lossless medium \( \eta_1 \), we have \( 2k_1 l_1 = \omega (2l_1 / c_1) = \omega T \), where \( T = 2l_1 / c_1 = 2(\eta_1 l_1) / c_0 \) is the two-way travel time delay through medium \( \eta_1 \). Thus, we can write:

\[ \Gamma_1(\omega) = \frac{\rho_1 + \rho_2 e^{-j\omega T}}{1 + \rho_1 \rho_2 e^{-j\omega T}} \]

This can also be expressed as a \( z \)-transform. Denoting the two-way travel time delay in the \( z \)-domain by \( Z^{-1} = e^{-j\omega T} = e^{-2jk_1 l_1} \), we may rewrite Eq. (5.4.4) as the first-order digital filter transfer function:

\[ \Gamma_1 (z) = \frac{\rho_1 + \rho_2 z^{-1}}{1 + \rho_1 \rho_2 z^{-1}} \]

An alternative way to derive Eq. (5.4.3) is working with wave impedances, which are continuous across interfaces. The wave impedance at interface-2 is \( Z_2 = Z'_2 \), but \( Z'_2 = \eta_b \) because there is no backward wave in medium \( \eta_b \). Thus, \( Z_2 = \eta_b \). Using the propagation equation for impedances, we find:

\[ Z_1 = Z_1' = \frac{Z_2 + jn_1 \tan k_1 l_1}{n_1 + jZ_2 \tan k_1 l_1} = \frac{\eta_b + jn_1 \tan k_1 l_1}{\eta_1 + jn_1 \eta_b \tan k_1 l_1} \]

Inserting this into \( \Gamma_1 = (Z_1 - \eta_a) / (Z_1 + \eta_a) \) gives Eq. (5.4.3). Working with wave impedances is always more convenient if the interfaces are positioned at half- or quarter-wavelength spacings.
5.4. Single Dielectric Slab

If we wish to determine the overall transmission response into medium \( \eta_b \), that is, the quantity \( T = \frac{E_2}{E_1} \), then we must work with the matrix formulation. Starting at the left interface and successively applying the matching and propagation matrices, we obtain:

\[
\begin{bmatrix}
E_{1+} \\
E_{1-}
\end{bmatrix} = \frac{1}{t_1} \begin{bmatrix}
1 & \rho_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
E_{1+}' \\
E_{1-}'
\end{bmatrix} = \frac{1}{t_1} \begin{bmatrix}
1 & \rho_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
e^{ik_1 l_1} & 0 \\
0 & e^{-ik_1 l_1}
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
E_{2-}
\end{bmatrix}
\]

\[
\begin{bmatrix}
E_{1+} \\
E_{1-}
\end{bmatrix} = \frac{1}{t_1} \begin{bmatrix}
1 & \rho_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
e^{ik_1 l_1} & 0 \\
0 & e^{-ik_1 l_1}
\end{bmatrix}
\begin{bmatrix}
\rho_2 & 1 \\
1 & \rho_2
\end{bmatrix}
\begin{bmatrix}
E_{2+}' \\
E_{2-}'
\end{bmatrix}
\]

where we set \( E_{2-}' = 0 \) by assumption. Multiplying the matrix factors out, we obtain:

\[
E_{1+} = \frac{e^{ik_1 l_1}}{\tau_1 \tau_2} (1 + \rho_1 \rho_2 e^{-2ik_1 l_1}) E_{2+}'
\]

\[
E_{1-} = \frac{e^{ik_1 l_1}}{\tau_1 \tau_2} (\rho_1 + \rho_2 e^{-2ik_1 l_1}) E_{2+}'
\]

These may be solved for the reflection and transmission responses:

\[
\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{\rho_1 + \rho_2 e^{-2ik_1 l_1}}{1 + \rho_1 \rho_2 e^{-2ik_1 l_1}}
\]

\[
T = \frac{E_{2+}'}{E_{1+}} = \frac{\tau_1 \tau_2 e^{-ik_1 l_1}}{1 + \rho_1 \rho_2 e^{-2ik_1 l_1}}
\]

(5.4.6)

The transmission response has an overall delay factor of \( e^{-j\omega T/2} \), representing the one-way travel time delay through medium \( \eta_1 \).

For convenience, we summarize the match-and-propagate equations relating the field quantities at the left of interface-1 to those at the left of interface-2. The forward and backward electric fields are related by the transfer matrix:

\[
\begin{bmatrix}
E_{1+} \\
E_{1-}
\end{bmatrix} = \frac{1}{t_1} \begin{bmatrix}
1 & \rho_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
e^{ik_1 l_1} & 0 \\
0 & e^{-ik_1 l_1}
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
E_{2-}
\end{bmatrix}
\]

\[
\begin{bmatrix}
E_{1+} \\
E_{1-}
\end{bmatrix} = \frac{1}{t_1} \begin{bmatrix}
1 & \rho_1 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
e^{ik_1 l_1} & \rho_1 e^{-j\omega T/2} \\
\rho_1 e^{j\omega T/2} & e^{-j\omega T/2}
\end{bmatrix}
\begin{bmatrix}
E_{2+} \\
E_{2-}
\end{bmatrix}
\]

(5.4.7)

The reflection responses are related by Eq. (5.4.2):

\[
\Gamma_1 = \frac{\rho_1 + \Gamma_2 e^{-2ik_1 l_1}}{1 + \rho_1 \Gamma_2 e^{-2ik_1 l_1}}
\]

(5.4.8)

The total electric and magnetic fields at the two interfaces are continuous across the interfaces and are related by Eq. (5.1.13):

\[
\begin{bmatrix}
E_1 \\
H_1
\end{bmatrix} = \begin{bmatrix}
\cos k_1 l_1 & jn_1 \sin k_1 l_1 \\
jn_1^{-1} \sin k_1 l_1 & \cos k_1 l_1
\end{bmatrix}
\begin{bmatrix}
E_2 \\
H_2
\end{bmatrix}
\]

(5.4.9)

Eqs. (5.4.7)-(5.4.9) are valid in general, regardless of what is to the right of the second interface. There could be a semi-infinite uniform medium or any combination of multiple slabs. These equations were simplified in the single-slab case because we assumed that there was a uniform medium to the right and that there were no backward-moving waves.

For lossless media, energy conservation states that the energy flux into medium \( \eta_1 \) must equal the energy flux out of it. It is equivalent to the following relationship between \( \Gamma \) and \( T \), which can be proved using Eq. (5.4.6):

\[
\frac{1}{\eta_a} (1 - |\Gamma_1|^2) = \frac{1}{\eta_b} |T|^2
\]

(5.4.10)

Thus, if we call \( |\Gamma_1|^2 \) the reflectance of the slab, representing the fraction of the incident power that gets reflected back into medium \( \eta_a \), then the quantity

\[
1 - |\Gamma_1|^2 = \frac{\eta_a}{\eta_b} |T|^2 = \frac{\eta_a}{\eta_b} |T|^2
\]

(5.4.11)

will be the transmittance of the slab, representing the fraction of the incident power that gets transmitted through into the right medium \( \eta_b \). The presence of the factors \( \eta_a, \eta_b \) can be understood as follows:

\[
\frac{T_{\text{transmitted}}}{T_{\text{incident}}} = \frac{1}{2\eta_a} |T'|^2 = \frac{\eta_a}{\eta_b} |T|^2
\]

5.5 Reflectionless Slab

The zeros of the transfer function (5.4.5) correspond to a reflectionless interface. Such zeros can be realized exactly only in two special cases, that is, for slabs that have either half-wavelength or quarter-wavelength thickness. It is evident from Eq. (5.4.5) that a zero will occur if

\[
2 \nu_1 l_1 = \nu \pi, \quad \text{or that } 2 \nu_1 l_1 = \nu \pi \quad \text{is an integer. This gives the}
\]

condition

\[
z = e^{j2k_1 l_1} = -\frac{\rho_2}{\rho_1}
\]

(5.5.1)

Because the right-hand side is real-valued and the left-hand side has unit magnitude, this condition can be satisfied only in the following two cases:

\[
z = e^{j2k_1 l_1} = 1, \quad \rho_2 = -\rho_1, \quad \text{(half-wavelength thickness)}
\]

\[
z = e^{j2k_1 l_1} = -1, \quad \rho_2 = \rho_1, \quad \text{(quarter-wavelength thickness)}
\]

The first case requires that \( 2k_1 l_1 \) be an integral multiple of \( 2\pi \), that is, \( 2k_1 l_1 = 2m\pi \), where \( m \) is an integer. This gives the half-wavelength condition \( l_1 = m\lambda_1 / 2 \), where \( \lambda_1 \) is the wavelength in medium-1. In addition, the condition \( \rho_2 = -\rho_1 \) requires that:

\[
\frac{\eta_b - \eta_1}{\eta_b + \eta_1} = \rho_2 = -\rho_1 = \frac{\eta_a - \eta_1}{\eta_a + \eta_1} \Rightarrow \eta_a = \eta_b
\]

that is, the media to the left and right of the slab must be the same. The second possibility requires \( e^{j2k_1 l_1} = -1 \), or that \( 2k_1 l_1 \) be an odd multiple of \( \pi \), that is, \( 2k_1 l_1 =\)
5.5. Reflectionless Slab

\((2m + 1)\pi\), which translates into the *quarter-wavelength* condition \(l_1 = (2m + 1)\lambda_1/4\). Furthermore, the condition \(\rho_2 = \rho_1\) requires:

\[
\frac{\eta_\alpha - \eta_t}{\eta_b + \eta_t} = \rho_2 = \rho_1 = \frac{\eta_\alpha - \eta_b}{\eta_1 + \eta_a} \quad \Leftrightarrow \quad \eta_1^2 = \eta_a\eta_b
\]

To summarize, a reflectionless slab, \(\Gamma_1 = 0\), can be realized only in the two cases:

- **half-wave:** \(l_1 = m\frac{\lambda_1}{2}\), \(\eta_1\) arbitrary, \(\eta_a = \eta_b\)
- **quarter-wave:** \(l_1 = (2m + 1)\frac{\lambda_1}{4}\), \(\eta_1 = \sqrt{\eta_a\eta_b}\), \(\eta_a, \eta_b\) arbitrary

(5.5.2)

An equivalent way of stating these conditions is to say that the **optical length** of the slab must be a half or quarter of the *free-space* wavelength \(\lambda_0\). Indeed, if \(n_1\) is the refractive index of the slab, then its optical length is \(n_1l_1\), and in the half-wavelength case we have \(n_1l_1 = n_1m\lambda_1/2 = m\lambda_0/2\), where we used \(\lambda_1 = \lambda_0/n_1\). Similarly, we have \(n_1l_1 = (2m + 1)\lambda_0/4\) in the quarter-wavelength case. In terms of the refractive indices, Eq. (5.5.2) reads:

- **half-wave:** \(n_1l_1 = m\frac{\lambda_0}{2}\), \(n_1\) arbitrary, \(n_a = n_b\)
- **quarter-wave:** \(n_1l_1 = (2m + 1)\frac{\lambda_0}{4}\), \(n_1 = \sqrt{n_a n_b}\), \(n_a, n_b\) arbitrary

(5.5.3)

The reflectionless matching condition can also be derived by working with wave impedances. For half-wavelength spacing, we have from Eq. (5.1.18) \(Z_1 = Z_2 = \eta_t\). The condition \(\Gamma_1 = 0\) requires \(Z_1 = \eta_a\), thus, matching occurs if \(\eta_a = \eta_b\). Similarly, for the quarter-wavelength case, we have \(Z_1 = \eta_1^2/\eta_2 = \eta_1^2/\eta_b = \eta_a\).

We emphasize that the reflectionless response \(\Gamma_1 = 0\) is obtained only at certain slab widths (half- or quarter-wavelength), or equivalently, at certain operating frequencies. These operating frequencies correspond to \(\omega T = 2m\pi\), or, \(\omega T = (2m + 1)\pi\), that is, \(\omega = 2m\pi T = m\omega_0\), or, \(\omega = (2m + 1)\omega_0/2\), where we defined \(\omega_0 = 2\pi T\).

The dependence on \(l_1\) or \(\omega\) can be seen from Eq. (5.4.5). For the half-wavelength case, we substitute \(\rho_2 = -\rho_1\) and for the quarter-wavelength case, \(\rho_2 = \rho_1\). Then, the reflection transfer functions become:

\[
\Gamma_1(z) = \frac{\rho_1(1 - z^{-1})}{1 - \rho_1^2 z^{-1}}, \quad \text{(half-wave)}
\]

\[
\Gamma_1(z) = \frac{\rho_1(1 + z^{-1})}{1 + \rho_1^2 z^{-1}}, \quad \text{(quarter-wave)}
\]

(5.5.4)

where \(Z = e^{2jk_1l_1} = e^{i\omega T}\). The magnitude-square responses then take the form:

\[
|\Gamma_1|^2 = \frac{2\rho_1^2 (1 - \cos(2k_1l_1))}{1 - 2\rho_1^2 \cos(2k_1l_1) + \rho_1^4}, \quad \text{(half-wave)}
\]

\[
|\Gamma_1|^2 = \frac{2\rho_1^2 (1 + \cos(2k_1l_1))}{1 + 2\rho_1^2 \cos(2k_1l_1) + \rho_1^4}, \quad \text{(quarter-wave)}
\]

(5.5.5)

These expressions are periodic in \(l_1\) with period \(\lambda_1/2\), and periodic in \(\omega\) with period \(\omega_0 = 2\pi T\). In DSP language, the slab acts as a digital filter with sampling frequency \(\omega_0\). The maximum reflectivity occurs at \(z = -1\) and \(z = 1\) for the half- and quarter-wavelength cases. The maximum squared responses are in either case:

\[
|\Gamma_1|^2_{\text{max}} = \frac{4\rho_1^2}{(1 + \rho_1^2)^2}
\]

Fig. 5.5.1 shows the magnitude responses for the three values of the reflection coefficient: \(|\rho_1| = 0.9, 0.7, \text{and 0.5}\). The closer \(\rho_1\) is to unity, the narrower are the reflectionless notches.

![Fig. 5.5.1](image)

Fig. 5.5.1 Reflection responses \(|\Gamma(\omega)|^2\). (a) \(|\rho_1| = 0.9\), (b) \(|\rho_1| = 0.7\), (c) \(|\rho_1| = 0.5\).

It is evident from these figures that for the same value of \(\rho_1\), the half- and quarter-wavelength cases have the same notch widths. A standard measure for the width is the 3-dB width, which for the half-wavelength case is twice the 3-dB frequency \(\omega_3\), as shown in Fig. 5.5.1 for the case \(|\rho_1| = 0.5\). The frequency \(\omega_3\) is determined by the 3-dB half-power condition:

\[
|\Gamma_1(\omega_3)|^2 = \frac{1}{2} |\Gamma_1|^2_{\text{max}}
\]

or, equivalently:

\[
\frac{2\rho_1^2 (1 - \cos(\omega_3 T))}{1 - 2\rho_1^2 \cos(\omega_3 T) + \rho_1^4} = \frac{1}{2} \frac{4\rho_1^2}{(1 + \rho_1^2)^2}
\]

Solving for the quantity \(\cos(\omega_3 T) = \cos(\Delta \omega T/2)\), we find:

\[
\cos(\Delta \omega T/2) = \frac{2\rho_1^2}{1 + \rho_1^4} \quad \Leftrightarrow \quad \tan(\Delta \omega T/2) = \frac{1 - \rho_1^2}{1 + \rho_1^2}
\]

(5.5.6)

If \(\rho_1^2\) is very near unity, then \(1 - \rho_1^2\) and \(\Delta \omega\) become small, and we may use the approximation \(\tan \Delta x = \Delta x\) to get:

\[
\frac{\Delta \omega T}{4} \approx \frac{1 - \rho_1^2}{1 + \rho_1^4} \approx \frac{1 - \rho_1^2}{2}
\]
5.5. Reflectionless Slab

Determine the reflection coefficients of half- and quarter-wave slabs that do not necessarily satisfy the impedance conditions of Eq. (5.5.2). Fig. 5.5.2 shows the corresponding transmittances $1 - |\Gamma_1(\omega)|^2$ of the slabs. The transmission response acts as a periodic bandpass filter. This is the simplest example of a so-called Fabry-Perot interference filter or Fabry-Perot resonator. Such filters find application in the spectroscopic analysis of materials. We discuss them further in Chap. 6.

Example 5.5.1: Determine the reflection response is given in general by Eq. (5.4.6). For the half-wavelength case, we have $\Delta\omega = \Delta\omega_T$, Eq. (5.5.6)

\[
\eta_b - \eta_a = \frac{\eta_a - \eta_n}{\eta_n + \eta_a}
\]

This is the same as if the slab were absent. For this reason, half-wavelength slabs are sometimes referred to as absentee layers. Similarly, in the quarter-wavelength case, we have $e^{j\rho_1 l_1}$ and find:

\[
\Gamma_1 = \frac{\rho_1 - \rho_2}{1 - \rho_1 \rho_2} = \frac{\eta_1 - \eta_n \eta_b}{\eta_1 + \eta_n \eta_b} = \frac{n_a n_b - n_1^2}{n_a n_b + n_1^2}
\]

The slab becomes reflectionless if the conditions (5.5.2) are satisfied.

Example 5.5.2: Antireflection Coating. Determine the refractive index of a quarter-wave antireflection coating on a glass substrate with index 1.5.

Solution: From Eq. (5.5.3), we have with $n_a = 1$ and $n_b = 1.5$:

\[
n_1 = \sqrt{n_a n_b} = \sqrt{1.5} = 1.22
\]

The closest refractive index that can be obtained is that of cryolite (Na$_3$AlF$_6$) with $n_1 = 1.35$ and magnesium fluoride (MgF$_2$) with $n_1 = 1.38$. Magnesium fluoride is usually preferred because of its durability. Such a slab will have a reflection coefficient as given by the previous example:

\[
\Gamma_1 = \frac{\rho_1 - \rho_2}{1 - \rho_1 \rho_2} = \frac{n_1^2 - n_a n_b}{n_1^2 + n_a n_b} = \frac{n_a n_b - n_1^2}{n_a n_b + n_1^2} = 1.5 - 1.38^2/1.5 + 1.38^2 = 0.118
\]

with reflectance $|\Gamma|^2 = 0.014$, or 1.4 percent. This is to be compared to the 4 percent reflectance of uncoated glass that we determined in Example 5.3.1.

Fig. 5.5.3 shows the reflectance $|\Gamma(\lambda)|^2$ as a function of the free-space wavelength $\lambda$. The reflectance remains less than one or two percent in the two cases, over almost the entire visible spectrum.

The slabs were designed to have quarter-wavelength thickness at $\lambda_0 = 550$ nm, that is, the optical length was $n_1 l_1 = \lambda_0/4$, resulting in $l_1 = 112.71$ nm and 99.64 nm in the two cases of $n_1 = 1.22$ and $n_1 = 1.38$. Such extremely thin dielectric films are fabricated by means of a thermal evaporation process [632,634].

The MATLAB code used to generate this example was as follows:

```matlab
n = [1, 1.22, 1.50]; l = 1/4; refractive indices and optical length
lambda = linspace(400,700,101) / 550; visible spectrum wavelengths
Gamma = multidiel(n, l, lambda); reflection response of slab
```

which gives the approximation:

\[
\Delta\omega_T = 2(1 - \rho_1^2)
\]

This is a standard approximation for digital filters relating the 3-dB width of a pole peak to the radius of the pole [49]. For any desired value of the bandwidth $\Delta\omega$, Eq. (5.5.6) or (5.5.7) may be thought of as a design condition that determines $\rho_1$.

\[
\Delta\omega_T = \frac{\Delta\omega}{\sin^2(\Delta\omega_T/2)}
\]

where the $\mathcal{F}$ is called the finesse in the Fabry-Perot context and is defined by:

\[
\mathcal{F} = \frac{4\rho_1^2}{(1 - \rho_1^2)^2}
\]

The finesse is a measure of the peak width, with larger values of $\mathcal{F}$ corresponding to narrower peaks. The connection of $\mathcal{F}$ to the 3-dB width (5.5.6) is easily found to be:

\[
\tan\left(\frac{\Delta\omega_T}{4}\right) = \frac{1 - \rho_1^2}{1 + \rho_1^2} = \frac{1}{\sqrt{1 + \mathcal{F}}}
\]

Quarter-wavelength slabs may be used to design anti-reflection coatings for lenses, so that all incident light on a lens gets through. Half-wavelength slabs, which require that the medium be the same on either side of the slab, may be used in designing radar domes (radomes) protecting microwave antennas, so that the radiated signal from the antenna goes through the radome wall without getting reflected back towards the antenna.
where we used $\rho_1 = (1 - n)/(1 + n)$. This explains why glass windows do not exhibit a frequency-selective behavior as predicted by Eq. (5.5.5). For $n = 1.5$, we find $1 - |T_1|^2 = 0.9231$, that is, 92.31% of the incident light is transmitted through the plate.

The same expressions for the average reflectance and transmittance can be obtained by summing incoherently all the multiple reflections within the slab, that is, summing the multiple reflections of power instead of field amplitudes. The timing diagram for such multiple reflections is shown in Fig. 5.6.1.

Indeed, if we denote by $p_r = \rho_1^2$ and $p_t = 1 - p_r = 1 - \rho_1^2$, the power reflection and transmission coefficients, then the first reflection of power will be $p_r$. The power transmitted through the left interface will be $p_t$ and through the second interface $p_t^2$ (assuming the same medium to the right.) The reflected power at the second interface will be $p_t^2 p_r$ and will come back and transmit through the left interface giving $p_t^2 p_r$.

Similarly, after a second round trip, the reflected power will be $p_t^2 p_r^2$, while the transmitted power to the right of the second interface will be $p_t^2 p_r^2$, and so on. Summing up all the reflected powers to the left and those transmitted to the right, we find:

$$|T_1|^2 = p_r + p_t^2 p_r + p_t^4 p_r + p_t^6 p_r + \cdots = p_r + \frac{p_t^2 p_r}{1 - p_t^2} = \frac{2p_r}{1 + p_t}$$

$$1 - |T_1|^2 = p_t^2 + p_t^4 p_t^2 + p_t^6 p_t^2 + \cdots = \frac{p_t^2}{1 - p_t^2} = 1 - p_t$$

where we used $p_t = 1 - p_r$. These are equivalent to Eqs. (5.5.10).}

Example 5.5.4: Radomes. A radome protecting a microwave transmitter has $\epsilon = 4\epsilon_0$ and is designed as a half-wavelength reflectionless slab at the operating frequency of 10 GHz. Determine its thickness.

Next, suppose that the operating frequency is 1% off its nominal value of 10 GHz. Calculate the percentage of reflected power back towards the transmitting antenna.

Determine the operating bandwidth as that frequency interval about the 10 GHz operating frequency within which the reflected power remains at least 30 dB below the incident power.

Solution: The free-space wavelength is $\lambda_0 = c_0/f_0 = 30$ GHz cm/10 GHz = 3 cm. The refractive index of the slab is $n = 2$ and the wavelength inside it, $\lambda_1 = \lambda_0/n = 3/2 = 1.5$ cm. Thus, the slab thickness will be the half-wavelength $l = \lambda_1/2 = 0.75$ cm, or any other integral multiple of this.

Assume now that the operating frequency is $\omega = \omega_0 + \delta\omega$, where $\omega_0 = 2\pi f_0 = 2\pi/\tau$. Denoting $\delta = \delta\omega/\omega_0$, we can write $\omega = \omega_0(1 + \delta)$. The numerical value of $\delta$ is very small, $\delta = 1\% = 0.01$. Therefore, we can do a first-order calculation in $\delta$. The reflection coefficient $\rho_1$ and reflection response $T$ are:

$$\rho_1 = \frac{\eta - \eta_0}{\eta + \eta_0} = 0.5 - \frac{1}{3} = \frac{1}{3}, \quad T_1(\omega) = \frac{\rho_1 (1 - z^{-1})}{1 - \rho_1^2 z^{-1}} = \frac{\rho_1 (1 - e^{-j\omega T})}{1 - \rho_1^2 e^{-j\omega T}}$$

where we used $\eta = \eta_0/n = \eta_0/2$. Noting that $\omega T = \omega_0 T (1 + \delta) = 2\pi (1 + \delta)$, we can expand the delay exponential to first-order in $\delta$:

$$z^{-1} = e^{-j\omega T} = e^{-2\pi j(1+\delta)} = e^{-2\pi j\delta} = e^{-2\pi j\delta} - 1 - 2\pi j\delta$$
Thus, the reflection response becomes to first-order in $\delta$:

$$ G_1 = \frac{\rho_1 (1 - (1 - 2\pi j \delta))}{1 - \rho_1 (1 - 2\pi j \delta)} = \frac{\rho_1 2\pi j \delta}{1 - \rho_1 + \rho_1^2 2\pi j \delta} = \rho_1 2\pi j \delta $$

where we replaced the denominator by its zeroth-order approximation because the numerator is already first-order in $\delta$. It follows that the power reflection response will be:

$$ |G_1|^2 = \frac{\rho_1^2 (2\pi j \delta)^2}{(1 - \rho_1^2)^2} $$

Evaluating this expression for $\delta = 0.01$ and $\rho_1 = -1/3$, we find $|G|^2 = 0.00049$, or 0.049 percent of the incident power gets reflected. Next, we find the frequency about $\omega_0$ at which the reflected power is $A = 30$ dB below the incident power. Writing again, $\omega = \omega_0 + \delta \omega = \omega_0 (1 + \delta)$ and assuming $\delta$ is small, we have the condition:

$$ |G_1|^2 = \frac{\rho_1^2 (2\pi j \delta)^2}{(1 - \rho_1^2)^2} $$

Evaluating this expression, we find $\delta = 0.0134$, or $\delta \omega = 0.0134 \omega_0$. The bandwidth will be twice that, $\Delta \omega = 2 \delta \omega = 0.0268 \omega_0$, or in Hz, $\Delta f = 0.0268 \Omega_0 = 268$ MHz. 

**Example 5.5.5:** Because of manufacturing imperfections, suppose that the actual constructed thickness of the above radome is 1% off the desired half-wavelength thickness. Determine the percentage of reflected power in this case.

**Solution:** This is essentially the same as the previous example. Indeed, the quantity $\delta = \omega T = 2k l_1 = 2 \omega l_1 / c_1$ can change either because of $\omega$ or because of $l_1$. A simultaneous infinitesimal change (about the nominal value $\theta_0 = \omega_0 T = 2\pi$) will give:

$$ \delta \theta = 2 \delta \omega l_1 / c_1 = 2 \omega_0 l_1 / c_1 = \delta \omega_0 / \omega_0 + \delta l_1 / l_1 $$

In the previous example, we varied $\omega$ while keeping $l_1$ constant. Here, we vary $l_1$, while keeping $\omega$ constant, so that $\delta = \delta l_1 / l_1$. Thus, we have $\delta \theta = \delta \omega = 2 \pi \delta$. The corresponding delay factor becomes approximately $z^{-1} = e^{-j \theta} = e^{-j (2 \pi \omega_0 \delta) / c_1} = e^{-j \delta} = 1 - 2 \pi j \delta$. The resulting expression for the power reflection response is identical to the above and its numerical value is the same if $\delta = 0.01$. 

**Example 5.5.6:** Because of weather conditions, suppose that the characteristic impedance of the medium outside the above radome is 1% off the impedance inside. Calculate the percentage of reflected power in this case.

**Solution:** Suppose that the outside impedance changes to $\eta = \eta_0 + \delta \eta$. The wave impedance at the outer interface will be $Z = \eta = \eta_0 + \delta \eta$. Because the slab length is still a half-wavelength, the wave impedance at the inner interface will be $Z_1 = Z_2 = \eta_0 + \delta \eta$. It follows that the reflection response will be:

$$ G_1 = \frac{Z_1 - \eta_0}{Z_1 + \eta_0} = \frac{\eta_0 + \delta \eta - \eta_0}{\eta_0 + \delta \eta + \eta_0} = \frac{\delta \eta}{2 \eta_0 + \delta \eta} = \frac{\delta \eta}{2 \eta_0} $$

where we replaced the denominator by its zeroth-order approximation in $\delta \eta$. Evaluating at $\delta \eta / \eta_0 = 1% = 0.01$, we find $G_1 = 0.005$, which leads to a reflected power of $|G|^2 = 2.5 \times 10^{-5}$, or 0.0025 percent.

### 5.6 Time-Domain Reflection Response

We conclude our discussion of the single slab by trying to understand its behavior in the time domain. The z-domain reflection transfer function of Eq. (5.4.5) incorporates the effect of all multiple reflections that are set up within the slab as the wave bounces back and forth at the left and right interfaces. Expanding Eq. (5.4.5) in a partial fraction expansion and then in power series in $z^{-1}$ gives:

$$ G_1 (z) = \frac{\rho_1 + \rho_2 z^{-1}}{1 - \rho_1 \rho_2 z^{-1}} = 1 - \frac{1}{\rho_1} \cdot \frac{(1 - \rho_1^2)}{1 + \rho_1 + \rho_1 \rho_2 z^{-1}} = \rho_1 + \sum_{n=1}^{\infty} (1 - \rho_1^2) (-\rho_1)^{n-1} \rho_2^n z^{-n} $$

Using the reflection coefficient from the right of the first interface, $\rho_1' = -\rho_1$, and the transmission coefficients $\tau_1 = 1 + \rho_1$ and $\tau_1' = 1 + \rho_1' = 1 - \rho_1$, we have $\tau_1 \tau_1' = 1 - \rho_1^2$. Then, the above power series can be written as a function of frequency in the form:

$$ G_1 (\omega) = \rho_1 + \sum_{n=1}^{\infty} \tau_1 \tau_1' (\rho_1')^{n-1} \rho_2^n \delta (t - nT) $$

where we set $z^{-1} = e^{-j \omega T}$. It follows that the time-domain reflection impulse response, that is, the inverse Fourier transform of $G_1 (\omega)$, will be the sum of discrete impulses:

$$ G_1 (t) = \rho_1 \delta (t) + \sum_{n=1}^{\infty} \tau_1 \tau_1' (\rho_1')^{n-1} \rho_2^n \delta (t - nT) $$

This is the response of the slab to a forward-moving impulse striking the left interface at $t = 0$, that is, the response to the input $G_1 (t) = \delta (t)$. The first term $\rho_1 \delta (t)$ is the impulse immediately reflected at $t = 0$ with the reflection coefficient $\rho_1$. The remaining terms represent the multiple reflections within the slab. Fig. 5.6.1 is a timing diagram that traces the reflected and transmitted impulses at the first and second interfaces.

![Fig. 5.6.1 Multiple reflections building up the reflection and transmission responses.](image-url)

The input pulse $\delta (t)$ gets transmitted to the inside of the left interface and picks up a transmission coefficient factor $\tau_1$. In $T/2$ seconds this pulse strikes the right interface...
and causes a reflected wave whose amplitude is changed by the reflection coefficient \( \rho_2 \) into \( \tau_1 \rho_2 \).

Thus, the pulse \( \tau_1 \rho_2 \delta (t - T/2) \) gets reflected backwards and will arrive at the left interface \( T/2 \) seconds later, that is, at time \( t = T \). A proportion \( \tau'_1 \) of it will be transmitted through to the left, and a proportion \( \tau'_1 \) will be re-reflected towards the right. Thus, at time \( t = T \), the transmitted pulse into the left medium will be \( \tau_1 \tau'_1 \rho_2 \delta (t - T) \), and the re-reflected pulse will be \( \tau_1 \rho'_1 \rho_2 \delta (t - T - T/2) \).

The re-reflected pulse will travel forward to the right interface, arriving there at time \( t = 3T/2 \) getting reflected backwards picking up a factor \( \rho_2 \). This will arrive at the left at time \( t = 2T/2 \). The part transmitted to the left will be now \( \tau_1 \tau'_1 \rho'_2 \rho_2 \delta (t - 2T) \), and the part re-reflected to the right will be \( \tau_1 \tau'_1 \rho'_2 \rho_2 \delta (t - 2T) \). And so on, after the nth round trip, the pulse transmitted to the left will be \( \tau_1 \tau'_1 \rho_2 \delta (t - nT) \). The sum of all the reflected pulses will be \( \Gamma (t) \) of Eq. (5.6.1).

In a similar way, we can derive the overall transmission response to the right. It is seen in the figure that the transmitted pulse at time \( t = nT + (T/2) \) will be \( \tau_1 \tau_2 (\rho'_1)^n \rho_2 \). Thus, the overall transmission impulse response will be:

\[
\mathcal{T} (t) = \sum_{n=0}^{\infty} \tau_1 \tau_2 (\rho'_1)^n \rho_2 \delta (t - nT - T/2)
\]

It follows that its Fourier transform will be:

\[
\mathcal{T} (\omega) = \sum_{n=0}^{\infty} \tau_1 \tau_2 (\rho'_1)^n \rho_2 e^{-j\omega T/2}
\]

which sums up to Eq. (5.6.2):

\[
\mathcal{T} (\omega) = \frac{\tau_1 \tau_2 e^{-j\omega T/2}}{1 - \rho'_1 \rho_2 e^{-j\omega T/2}} = \frac{\tau_1 \tau_2 e^{-j\omega T/2}}{1 + \rho_1 \rho_2 e^{-j\omega T/2}}
\]

For an incident field \( E_{1+} (t) \) with arbitrary time dependence, the overall reflection response of the slab is obtained by convolving the impulse response \( \Gamma (t) \) with \( E_{1+} (t) \). This follows from the linear superposition of the reflection responses of all the frequency components of \( E_{1+} (t) \), that is,

\[
E_{1-} (t) = \int_{-\infty}^{\infty} \Gamma (\omega) E_{1+} (\omega) e^{j\omega t} d\omega,
\]

where \( E_{1+} (t) = \int_{-\infty}^{\infty} E_{1+} (\omega) e^{j\omega t} d\omega \).

Then, the convolution theorem of Fourier transforms implies that:

\[
E_{1-} (t) = \int_{-\infty}^{\infty} \Gamma (\omega) E_{1+} (\omega) e^{j\omega t} d\omega = \int_{-\infty}^{\infty} \Gamma (t') E_{1+} (t - t') dt'
\]

Inserting (5.6.1), we find that the reflected wave arises from the multiple reflections of \( E_{1+} (t) \) as it travels and bounces back and forth between the two interfaces:

\[
E_{1-} (t) = \rho_1 E_{1+} (t) + \sum_{n=1}^{\infty} \tau_1 \tau'_1 (\rho'_1)^{n-1} \rho_2 \delta E_{1+} (t - nT)
\]

For a causal waveform \( E_{1+} (t) \), the summation over \( n \) will be finite, such that at each time \( t \geq 0 \) only the terms that have \( t - nT \leq 0 \) will be present. In a similar fashion, we find for the overall transmitted response into medium \( n_2 \):

\[
E_{2+} (t) = \int_{-\infty}^{\infty} \mathcal{T} (t') E_{1+} (t - t') dt' = \sum_{n=0}^{\infty} \tau_1 \tau_2 (\rho'_1)^n \rho_2 \delta E_{1+} (t - nT - T/2)
\]

5.7 Two Dielectric Slabs

Next, we consider more than two interfaces. As we mentioned in the previous section, Eqs. (5.4.7)–(5.4.9) are general and can be applied to all successive interfaces. Fig. 5.7.1 shows three interfaces separating four media. The overall reflection response can be calculated by successive application of Eq. (5.4.8):

\[
\Gamma_1 = \frac{\rho_1 + \rho_1 e^{-2jk_1 l_1}}{1 + \rho_1 \rho_2 e^{-2jk_1 l_1}}; \quad \Gamma_2 = \frac{\rho_2 + \rho_3 e^{-2jk_2 l_2}}{1 + \rho_2 \rho_3 e^{-2jk_2 l_2}}
\]

![Fig. 5.7.1 Two dielectric slabs.](image-url)
The fields $E_1, H_1$ are obtained by successively applying Eq. (5.4.9):

$$
\begin{bmatrix}
E_1 \\
H_1
\end{bmatrix} = \begin{bmatrix}
\cos k_1 l_1 & j \eta_1 \sin k_1 l_1 \\
-\eta_1 j \eta_1^{-1} \sin k_1 l_1 & \cos k_1 l_1
\end{bmatrix} \begin{bmatrix}
E_2 \\
H_2
\end{bmatrix} = \begin{bmatrix}
\cos k_2 l_2 & j \eta_2 \sin k_2 l_2 \\
-\eta_2 j \eta_2^{-1} \sin k_2 l_2 & \cos k_2 l_2
\end{bmatrix} \begin{bmatrix}
E_3 \\
H_3
\end{bmatrix}
$$

But at interface 3, $E_3 = E_3' = E_3''$, and $H_3 = Z_3^{-1} E_1 = \eta_b^{-1} E_3'$, because $Z_3 = \eta_b$. Therefore, we can obtain the fields $E_1, H_1$ by the matrix multiplication:

$$
\begin{bmatrix}
E_1 \\
H_1
\end{bmatrix} = \begin{bmatrix}
\cos k_1 l_1 & j \eta_1 \sin k_1 l_1 \\
-\eta_1 j \eta_1^{-1} \sin k_1 l_1 & \cos k_1 l_1
\end{bmatrix} \begin{bmatrix}
\cos k_2 l_2 & j \eta_2 \sin k_2 l_2 \\
-\eta_2 j \eta_2^{-1} \sin k_2 l_2 & \cos k_2 l_2
\end{bmatrix} \begin{bmatrix}
1 \\
\eta_b^{-1}
\end{bmatrix} E_3' = E_3''
$$

Because $Z_3$ is the ratio of $E_1$ and $H_1$, the factor $E_3''$ cancels out and can be set equal to unity.

Example 5.7.1: Determine $\Gamma_1$ if both slabs are quarter-wavelength slabs. Repeat if both slabs are half-wavelength and when one is half- and the other quarter-wavelength.

Solution: Because $l_1 = \lambda_1 / 4$ and $l_2 = \lambda_2 / 4$, we have $2 k_1 l_1 = 2 k_2 l_2 = \pi$, and it follows that $z_1 = z_2 = -1$. Then, Eq. (5.7.1) becomes:

$$
\Gamma_1 = \frac{\rho_1 - \rho_2 - \rho_3 (\rho_1 \rho_2 + \rho_3)}{1 - \rho_1 \rho_2 - \rho_3 (\rho_1 \rho_2 + \rho_3)}
$$

A simpler approach is to work with wave impedances. Using $Z_3 = \eta_b$, we have:

$$
Z_1 = \frac{\eta_1}{Z_2} = \frac{\eta_1}{\eta_2 / Z_3} = \frac{\eta_1}{\eta_2} Z_3 = \frac{\eta_1}{\eta_2} \eta_b
$$

Inserting this into $\Gamma_1 = (Z_1 - \eta_a) / (Z_1 + \eta_a)$, we obtain:

$$
\Gamma_1 = \frac{\eta_1 \eta_b - \eta_2 \eta_a}{\eta_1 \eta_b + \eta_2 \eta_a}
$$

The two expressions for $\Gamma_1$ are equivalent. The input impedance $Z_1$ can also be obtained by matrix multiplication. Because $k_1 l_1 = k_2 l_2 = \pi / 2$, we have $k_1 l_1 = 0$ and $k_2 l_1 = 1$ and the propagation matrices for $E_1, H_1$ take the simplified form:

$$
\begin{bmatrix}
E_1 \\
H_1
\end{bmatrix} = \begin{bmatrix}
0 & j \eta_1 \\
-\eta_1 j \eta_1^{-1} & 0
\end{bmatrix} \begin{bmatrix}
0 & j \eta_2 \\
-\eta_2 j \eta_2^{-1} & 0
\end{bmatrix} \begin{bmatrix}
1 \\
\eta_b^{-1}
\end{bmatrix} E_3' = E_3''
$$

The ratio $E_1 / H_1$ gives the same answer for $Z_1$ as above. When both slabs are half-wavelength, the impedances propagate unchanged: $Z_1 = Z_2 = Z_3$, but $Z_1 = \eta_b$.

If $\eta_1$ is half- and $\eta_2$ quarter-wavelength, then $Z_1 = Z_2 = \eta_2 / Z_3 = \eta_2 / \eta_b$. And, if the quarter-wavelength is first and the half-wavelength second, $Z_1 = \eta_1 / Z_2 = \eta_1 / Z_3 = \eta_1 / \eta_b$.

The corresponding reflection coefficient $\Gamma_1$ is in the three cases:

$$
\Gamma_1 = \frac{\eta_b - \eta_a}{\eta_b + \eta_a}, \quad \Gamma_1 = \frac{\eta_2 - \eta_0 \eta_b}{\eta_2 + \eta_0 \eta_b}, \quad \Gamma_1 = \frac{\eta_1 - \eta_0 \eta_b}{\eta_1 + \eta_0 \eta_b}
$$

These expressions can also be derived by Eq. (5.7.1), or by the matrix method.

---

5.8 Reflection by a Moving Boundary

Reflection and transmission by moving boundaries, such as reflection from a moving mirror, introduce Doppler shifts in the frequencies of the reflected and transmitted waves. Here, we look at the problem of normal incidence on a dielectric interface that is moving with constant velocity $v$ perpendicularly to the interface, that is, along the $z$-direction as shown in Fig. 5.8.1. Additional examples may be found in [474–492]. The case of oblique incidence is discussed in Sec. 7.12.

The frequency dependence of Eq. (5.7.1) arises through the factors $z_1, z_2$, which can be written in the forms: $z_1 = e^{j \omega T_1}$ and $z_2 = e^{j \omega T_2}$, where $T_1 = 2l_1 / c_1$ and $T_2 = 2l_2 / c_2$ are the two-way travel time delays through the two slabs.

A case of particular interest arises when the slabs are designed to have the equal travel-time delays so that $T_1 = T_2 = T$. Then, defining a common variable $z = z_1 = z_2 = e^{j \omega T}$, we can write the reflection response as a second-order digital filter transfer function:

$$
\Gamma_1(z) = \frac{\rho_1 + \rho_2 (1 + \rho_1 \rho_3) z^{-1} + \rho_3 z^{-2}}{1 + \rho_2 (1 + \rho_3) z^{-1} + \rho_1 \rho_3 z^{-2}}
$$

In the next chapter, we discuss further the properties of such higher-order reflection transfer functions arising from multilayer dielectric slabs.

---

Fig. 5.8.1 Reflection and transmission at a moving boundary.

The dielectric is assumed to be non-magnetic and lossless with permittivity $\epsilon$. The left medium is free space $\epsilon_0$. The electric field is assumed to be in the $x$-direction and thus, the magnetic field will be in the $y$-direction. We consider two coordinate frames, the fixed frame $S$ with coordinates $(t, x, y, z)$, and the moving frame $S'$ with $(t', x', y', z')$. The two sets of coordinates are related by the Lorentz transformation equations (K.1) of Appendix K.

We are interested in determining the Doppler-shifted frequencies of the reflected and transmitted waves, as well as the reflection and transmission coefficients as measured in the fixed frame $S$. 

[Diagram Fig. 5.8.1]
5.8. Reflection by a Moving Boundary

The procedure for solving this type of problem—originally suggested by Einstein in his 1905 special relativity paper [474]—is to solve the reflection and transmission problem in the moving frame $S'$ with respect to which the boundary is at rest, and then transform the results back to the fixed frame $S$ using the Lorentz transformation properties. In the fixed frame $S$, the fields to the left and right of the interface will have the forms:

$$\left\{ \begin{array}{ll}
E_x = E_x e^{i(\omega t - k_z z)} & \text{left} \\
H_y = H_y e^{i(\omega t - k_z z)} & \text{right}
\end{array} \right. \quad \left\{ \begin{array}{ll}
E_x = E_x e^{i(\omega t - k_z z)} & \text{left} \\
H_y = H_y e^{i(\omega t - k_z z)} & \text{right}
\end{array} \right. \quad (5.8.1)
$$

where $\omega$, $\omega_r$, $\omega_t$, and $k_i$, $k_r$, $k_t$ are the frequencies and wavenumbers of the incident, reflected, and transmitted waves measured in $S$. Because of Lorentz invariance, the propagation phases remain unchanged in the frames $S$ and $S'$, that is,

$$\begin{align*}
\phi_i &= \omega t - k_i z = \omega' t' - k' z' = \phi'_i \\
\phi_r &= \omega_r t + k_r z = \omega' t' + k'_ z' = \phi'_r \\
\phi_t &= \omega_t t - k_t z = \omega' t' - k'_ z' = \phi'_t
\end{align*} \quad (5.8.2)
$$

In the frame $S'$ where the dielectric is at rest, all three frequencies are the same and set equal to $\omega'$. This is a consequence of the usual tangent boundary conditions applied to the interface at rest. Note that $\phi_r$ can be written as $\phi_r = \omega_r t = (-k_r) z$ implying that the reflected wave is propagating in the negative $z$-direction. In the rest frame $S'$ of the boundary, the wavenumbers are:

$$k'_i = \omega' / c', \quad k'_r = k'_z = \omega' / \sqrt{\epsilon_0} = n \omega' / c \quad (5.8.3)
$$

where $c$ is the speed of light in vacuum and $n = \sqrt{\epsilon_0/c}$ is the refractive index of the dielectric at rest. The frequencies and wavenumbers in the fixed frame $S$ are related to those in $S'$ by applying the Lorentz transformation of Eq. (K.14) to the frequency-wavenumber four-vectors $(\omega/c, 0, 0, k_i)$, $(\omega_r/c, 0, 0, -k_r)$, and $(\omega_t/c, 0, 0, k_t)$:

$$\begin{align*}
\omega &= \gamma (\omega' + \beta c k_i') = \omega' \gamma (1 + \beta) \\
k_i &= k'_i = \omega' / c = \omega' / c (1 + \beta) \\
\omega_r &= \gamma (\omega' + \beta c (-k'_r)) = \omega' \gamma (1 - \beta) \\
-k_r &= k'_r + \beta c k'_i = -\omega' \gamma (1 - \beta) \\
\omega_t &= \gamma (\omega' + \beta c k'_t) = -\omega' \gamma (1 + \beta n) \\
k_t &= k'_t + \beta c k'_i = \omega' \gamma (n + \beta)
\end{align*} \quad (5.8.4)
$$

where $\beta = \nu/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$. Eliminating the primed quantities, we obtain the Doppler-shifted frequencies of the reflected and transmitted waves:

$$\begin{align*}
\omega_r &= \omega \frac{1 - \beta}{1 + \beta} \\
\omega_t &= \omega \frac{1 + \beta n}{1 + \beta}
\end{align*} \quad (5.8.5)
$$

The phase velocities of the incident, reflected, and transmitted waves are:

$$\nu_i = \frac{\omega}{k_i} = c, \quad \nu_r = \frac{\omega_r}{k_r} = c, \quad \nu_t = \frac{\omega_t}{k_t} = \frac{1 + \beta n}{n + \beta} \quad (5.8.6)
$$

These can also be derived by applying Einstein’s velocity addition theorem of Eq. (K.8). For example, we have for the transmitted wave:

$$\nu_t = \frac{\nu_d + \nu}{1 + \nu_d \nu/c^2} = \frac{c/n + \nu}{1 + (c/n) \nu/c^2} = \frac{1 + \beta n}{n + \beta}$$

where $\nu_d = c/n$ is the phase velocity within the dielectric at rest. To first-order in $\beta = \nu/c$, the phase velocity within the moving dielectric becomes:

$$\nu_t = \frac{1 + \beta n}{n + \beta} \approx \frac{c}{n} + \nu \left(1 - \frac{1}{n^2}\right)$$

The second term is known as the “Fresnel drag.” The quantity $n_i = (n + \beta) / (1 + \beta n)$ may be thought of as the “effective” refractive index of the moving dielectric as measured in the fixed system $S$.

Next, we derive the reflection and transmission coefficients. In the rest-frame $S'$ of the dielectric, the fields have the usual forms derived earlier in Sections 5.1 and 5.2:

$$\begin{align*}
E_x' &= E_x' (e^{i\phi'_i} + pe^{i\phi'_r}) \\
H_y' &= \frac{1}{\eta_0} E_x' (e^{i\phi'_i} - pe^{i\phi'_r})
\end{align*} \quad (5.8.7)
$$

where $
 = \eta_0 / \eta_0'$, $\rho = \eta_1 / \eta_0 = 1 - n / n_0$, $\tau = 1 + \rho = 2 / (1 + n_0)$. The primed fields can be transformed to the fixed frame $S$ using the inverse of the Lorentz transformation equations (K.31), that is,

$$\begin{align*}
E_x &= y(E_x' + \beta c B_y') = y(E_x' + \beta \eta_0 H_y') \\
H_y &= y(H_y' + c B_x' - \beta E_x') = y(H_y' + c \beta E_x')
\end{align*} \quad (5.8.8)
$$

where we replaced $B_y' = \mu_0 H_y'$, $c \mu_0 = \eta_0$, and $D_x' = c E_x'$ (of course, $c = \epsilon_0$ in the left medium). Using the invariance of the propagation phases, we find for the fields at the left side of the interface:

$$E_x = y[E_x' (e^{i\phi'_i} + pe^{i\phi'_r}) + \beta E_x' (e^{i\phi'_i} - pe^{i\phi'_r})] = E_x'[1 + \beta e^{i\phi'_i} + \rho (1 - \beta) e^{i\phi'_r}] \quad (5.8.9)
$$

Similarly, for the right side of the interface we use the property $\eta_0 / \eta_0 = n$ to get:

$$E_x = y[\tau E_x' e^{i\phi'_i} + \beta n \tau E_x' e^{i\phi'_r}] = y \tau E_x'[1 + \beta n e^{i\phi'_r}] \quad (5.8.10)
$$

Comparing these with Eq. (5.8.1), we find the incident, reflected, and transmitted electric field amplitudes:

$$E_i = y E_x' (1 + \beta), \quad E_r = \rho y E_x' (1 - \beta), \quad E_t = \tau y E_x' (1 + \beta n) \quad (5.8.11)$$

5. Reflection and Transmission
from which we obtain the reflection and transmission coefficients in the fixed frame $S$:

$$\frac{E_r}{E_i} = \rho \frac{1 - \beta}{1 + \beta}, \quad \frac{E_t}{E_i} = \tau \frac{1 + \beta n}{1 - \beta n}$$  (5.8.12)

The case of a perfect mirror is also covered by these expressions by setting $\rho = -1$ and $\tau = 0$. Eq. (5.8.5) is widely used in Doppler radar applications. Typically, the boundary (the target) is moving at non-relativistic speeds so that $\beta = v/c \ll 1$. In such case, the first-order approximation of (5.8.5) is adequate:

$$f_r = f (1 - 2\beta) = f (1 - 2\frac{\beta}{c}) \Rightarrow \Delta f = -2\frac{\beta}{c}$$  (5.8.13)

where $\Delta f = f_r - f$ is the Doppler shift. The negative sign means that $f_r < f$ if the target is receding away from the source of the wave, and $f_r > f$ if it is approaching the source.

As we mentioned in Sec. 2.12, if the source of the wave is moving with velocity $v_a$ and the target with velocity $v_b$ (with respect to a common fixed frame, such as the ground), then one must use the relative velocity $v = v_b - v_a$ in the above expression:

$$\Delta f = \frac{f_r - f}{f} = \frac{2}{c} \frac{v_a - v_b}{v_a}$$  (5.8.14)

### 5.9 Problems

5.1 Fill in the details of the equivalence between Eq. (5.2.2) and (5.2.3), that is,

$$\frac{1}{\eta} (E_+ + E_-) = \frac{1}{\eta} (E'_+ - E'_-) \Rightarrow \left[ \frac{E_+}{E_-} \right] = \frac{1}{\rho} \left[ \begin{array}{c} 1 \\ \rho \\ 1 \\ \frac{1}{\rho} \end{array} \right] \left[ \begin{array}{c} E'_+ \\ E'_- \end{array} \right]$$

5.2 Fill in the details of the equivalences stated in Eq. (5.2.9), that is,

$$Z = Z' \Rightarrow \Gamma = \rho + \Gamma' \frac{1}{1 + \rho \Gamma'} \Rightarrow \Gamma' = \left( \frac{\rho}{1 + \rho \Gamma'} \right) \Gamma$$

Show that if there is no left-incident field from the right, then $\Gamma = \rho$, and if there is no right-incident field from the left, then, $\Gamma' = 1/\rho'$. Explain the asymmetry of the two cases.

5.3 Let $\rho, \tau$ be the reflection and transmission coefficients from the left side of an interface and let $\rho', \tau'$ be those from the right, as defined in Eq. (5.2.5). One of the two media may be lossy, and therefore, its characteristic impedance and hence $\rho, \tau$ may be complex-valued. Show and interpret the relationships:

$$1 - |\rho|^2 = \text{Re} \left( \frac{\eta}{\eta'} \right) |\tau|^2 = \text{Re} (\tau^* \tau')$$

5.4 Show that the reflection and transmission responses of the single dielectric slab of Fig. 5.4.1 are given by Eq. (5.4.6), that is,

$$\Gamma = \frac{\rho_1 + \rho_2 e^{-j2k l}}{1 + \rho_1 \rho_2 e^{-j2k l}}, \quad \Gamma' = \frac{E_{2+}'}{E_{1+}} = \frac{\rho_1 \rho_2 e^{-j2k l}}{1 + \rho_1 \rho_2 e^{-j2k l}}$$

Moreover, using these expressions show and interpret the relationship:

$$\frac{1}{\eta_a} (1 - |\Gamma|^2) = \frac{1}{\eta_b} |\Gamma|^2$$

5.5 A 1-GHz plane wave is incident normally onto a thick copper plate ($\sigma = 5.8 \times 10^7$ S/m.) Can the plate be considered to be a good conductor at this frequency? Calculate the percentage of the incident power that enters the plate. Calculate the attenuation coefficient within the conductor and express it in units of dB/m. What is the penetration depth in mm?

5.6 With the help of Fig. 5.5.1, argue that the 3-dB width $\Delta\omega$ is related to the 3-dB frequency $\omega_f$ as $\Delta\omega = 2\omega_f$, where $\omega_f$ is the frequency for which the ratio of the reflected power to the incident power is 15 or 30 dB. Then, show that $\omega_f$ and $\Delta\omega$ are given by:

$$\cos \omega_f T = \pm \frac{2\rho_1^*}{1 + \rho_1^*}, \quad \tan \left( \frac{\Delta\omega T}{4} \right) = \frac{1 - \rho_1^*}{1 + \rho_1^*}$$

5.7 A fiberglass ($\epsilon = 4\epsilon_0$) radome protecting a microwave antenna is designed as a half-wavelength reflectionless slab at the operating frequency of 12 GHz.

a. Determine three possible thicknesses (in cm) for this radome.

b. Determine the 15-dB and 30-dB bandwidths in GHz about the 12 GHz operating frequency, defined as the widths over which the reflected power is 15 or 30 dB below the incident power.

5.8 A 5 GHz wave is normally incident from air onto a dielectric slab of thickness of 1 cm and refractive index of 1.5, as shown below. The medium to the right of the slab has an index of 2.25.

5.9 A single-frequency plane wave is incident obliquely from air onto a planar interface with a medium of permittivity $\epsilon = 2\epsilon_0$, as shown below. The incident wave has the following phasor form:

$$E(z) = \left( \frac{\mathbf{x} + \mathbf{z}}{\sqrt{2}} \right) e^{-jk(z-x)/\sqrt{2}}$$  (5.9.1)
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5.10 A uniform plane wave is incident normally on a planar interface, as shown below. The medium to the left of the interface is air, and the medium to the right is lossy with an effective complex permittivity \( \varepsilon_r \), complex wavenumber \( k = \beta - j\alpha = \omega \sqrt{\varepsilon_r \mu_r} \), and complex characteristic impedance \( \eta_c = \sqrt{\mu_0 \varepsilon_r} \). The electric field of the left and right of the interface has the following form:

\[
E_x = \begin{cases} 
E_0 e^{j k x} + \rho E_0 e^{j k z}, & z \leq 0 \\
\tau E_0 e^{-j k z}, & z \geq 0
\end{cases}
\]

where \( \rho, \tau \) are the reflection and transmission coefficients.

1. Determine the magnetic field at both sides of the interface.
2. Show that the Poynting vector only has a \( z \)-component, given as follows at the two sides of the interface:

\[
P = \frac{|E_0|^2}{2\eta_0} (1 - |\rho|^2), \quad P' = \frac{|E_0|^2}{2\omega \mu_0} \tau' |\tau|^2 e^{-2\alpha' z}
\]

3. Moreover show that \( P = P' \) at the interface, (i.e., at \( z = 0 \)).

5.11 Consider a lossy dielectric slab of thickness \( d \) and complex refractive index \( n_c = n_r - j n_i \) at an operating frequency \( \omega \), with air on both sides as shown below.

a. Let \( k = \beta - j\alpha = k_0 n_c \) and \( n_c = \eta_c/\eta_0 \) be the corresponding complex wavenumber and characteristic impedance of the slab, where \( k_0 = \omega \sqrt{\varepsilon_r \mu_r} \) and \( \eta_0 = \sqrt{\mu_0 \varepsilon_r} \). Show that the transmission response of the slab may be expressed as follows:

\[
T = \frac{1}{\cos kd + j \frac{1}{2} \left( n_c + \frac{1}{n_c} \right) \sin kd}
\]

b. At the cell phone frequency of 900 MHz, the complex refractive index of concrete is \( n_c = 2.5 - 0.14j \). Calculate the percentage of the transmitted power through a 20-cm concrete wall. How is this percentage related to \( T \) and why?

c. Is there anything interesting about the choice \( d = 20 \) cm? Explain.

5.12 Consider the slab of the previous problem. The tangential electric field has the following form in the three regions \( z \leq 0, 0 \leq z \leq d, \) and \( z \geq d \):

\[
E(z) = \begin{cases} 
e^{-j k x} + \Gamma e^{j k x}, & \text{if } z \leq 0 \\
e^{-j k x} + B e^{j k z}, & \text{if } 0 \leq z \leq d \\
e^{-j k (z - d)}, & \text{if } z \geq d
\end{cases}
\]

where \( k_0 \) and \( k \) were defined in the previous problem.

a. What are the corresponding expressions for the magnetic field \( H(z) \)?

b. Set up and solve four equations from which the four unknowns \( \Gamma, A, B, T \) may be determined.

c. If the slab is lossless and is designed to be a half-wave slab at the frequency \( \omega \), then what is the value of \( T \)?

d. If the slab is lossy with \( n_c = n_r - j n_i \) and is designed to be a half-wave slab with respect to the real part \( \beta \) of \( k \), that is, \( \beta d = \pi \), then, show that \( T \) is given by:

\[
T = -\frac{1}{\cosh \alpha d} + \frac{1}{2} \left( n_c + \frac{1}{n_c} \right) \sinh \alpha d
\]

5.13 Consider a two-layer dielectric structure as shown in Fig. 5.7.1, and let \( n_{a1}, n_{a2}, n_{b1}, n_{b2} \) be the refractive indices of the four media. Consider the four cases: (a) both layers are quarter-wave, (b) both layers are half-wave, (c) layer-1 is quarter- and layer-2 half-wave, and (d) layer-1 is half- and layer-2 quarter-wave. Show that the reflection coefficient at interface-1 is given by the following expressions in the four cases:

\[
\Gamma_1 = \frac{n_a n_b^2 - n_b n_a}{n_a n_b^2 + n_b n_a}, \quad \Gamma_1 = \frac{n_a n_b - n_b^2}{n_a n_b + n_b^2}, \quad \Gamma_1 = \frac{n_a n_b - n_b^2}{n_a n_b + n_b^2}, \quad \Gamma_1 = \frac{n_a n_b - n_b^2}{n_a n_b + n_b^2}
\]

5.14 Consider the lossless two-slab structure of Fig. 5.7.1. Write down all the transfer matrices relating the fields \( E_{z1} \), \( t = 1, 2, 3 \) at the left sides of the three interfaces. Then, show the energy conservation equations:

\[
\frac{1}{n_a} \left( |E_{z1}|^2 - |E_{z1}|^2 \right) = \frac{1}{n_1} \left( |E_{z2}|^2 - |E_{z2}|^2 \right) = \frac{1}{n_2} \left( |E_{z3}|^2 - |E_{z3}|^2 \right) = \frac{1}{n_b} |E_{z4}|^2
\]

5.15 An alternative way of representing the propagation relationship Eq. (5.11.12) is in terms of the hyperbolic \( w \)-plane variable defined in terms of the reflection coefficient \( \Gamma \), or equivalently, the wave impedance \( Z \) as follows:

\[
\Gamma = e^{-2w} \iff Z = \eta \coth(w) \quad (5.9.2)
\]

Show the equivalence of these expressions. Writing \( \Gamma_1 = e^{-2w_1} \) and \( \Gamma_2 = e^{-2w_2} \), show that Eq. (5.11.12) becomes equivalent to:

\[
w_1 = w_2 + jk \quad \text{(propagation in w-domain)} \quad (5.9.3)
\]

This form is essentially the mathematical (as opposed to graphical) version of the Smith chart and is particularly useful for numerical computations using MATLAB.
5.16 Plane A flying at a speed of 900 km/hr with respect to the ground is approaching plane B. Plane A’s Doppler radar, operating at the X-band frequency of 10 GHz, detects a positive Doppler shift of 2 kHz in the return frequency. Determine the speed of plane B with respect to the ground. [Ans. 792 km/hr.]

5.17 The complete set of Lorentz transformations of the fields in Eq. (5.8.8) is as follows (see also Eq. (K.31) of Appendix K):

\[
\begin{align*}
E_x &= y(E'_x + \beta c B'_y), \quad H_y = y(H'_y + c \beta D'_x), \\
D_x &= y(D'_x + \frac{1}{c} \beta H'_y), \quad B_y = y(B'_y + \frac{1}{c} \beta E'_x)
\end{align*}
\]

The constitutive relations in the rest frame \(S'\) of the moving dielectric are the usual ones, that is, \(B'_y = \mu H'_y\) and \(D'_x = \epsilon E'_x\). By eliminating the primed quantities in terms of the unprimed ones, show that the constitutive relations have the following form in the fixed system \(S\):

\[
\begin{align*}
D_x &= \frac{(1 - \beta^2) \epsilon E_x - \beta (n^2 - 1) H_y/c}{1 - \beta^2 n^2}, \\
B_y &= \frac{(1 - \beta^2) \mu H_y - \beta (n^2 - 1) E_x/c}{1 - \beta^2 n^2}
\end{align*}
\]

where \(n\) is the refractive index of the moving medium, \(n = \sqrt{\epsilon \mu / \epsilon_0 \mu_0}\). Show that for free space, the constitutive relations remain the same as in the frame \(S'\).

6. **Multilayer Structures**

Higher-order transfer functions of the type of Eq. (5.7.2) can achieve broader reflectionless notches and are used in the design of thin-film antireflection coatings, dielectric mirrors, and optical interference filters [632–694, 754–787], and in the design of broadband terminations of transmission lines [822–832]. They are also used in the analysis, synthesis, and simulation of fiber Bragg gratings [788–808], in the design of narrow-band transmission filters for wavelength-division multiplexing (WDM), and in other fiber-optic signal processing systems [818–821].

They are used routinely in making acoustic tube models for the analysis and synthesis of speech, with the layer recursions being mathematically equivalent to the Levinson lattice recursions of linear prediction [833–839]. The layer recursions are also used in speech recognition, disguised as the Schur algorithm.

They also find application in geophysical deconvolution and inverse scattering problems for oil exploration [840–849].

The layer recursions—known as the Schur recursions in this context—are intimately connected to the mathematical theory of lossless bounded real functions in the \(z\)-plane and positive real functions in the \(s\)-plane and find application in network analysis, synthesis, and stability [853–867].

6.1 **Multiple Dielectric Slabs**

The general case of arbitrary number of dielectric slabs of arbitrary thicknesses is shown in Fig. 6.1.1. There are \(M\) slabs, \(M + 1\) interfaces, and \(M + 2\) dielectric media, including the left and right semi-infinite media \(\eta_a\) and \(\eta_b\).

The incident and reflected fields are considered at the left of each interface. The overall reflection response, \(\Gamma = E_1 / E_{1+}\), can be obtained recursively in a variety of ways, such as by the propagation matrices, the propagation of the impedances at the interfaces, or the propagation of the reflection responses.

The elementary reflection coefficients \(\rho_i\) from the left of each interface are defined in terms of the characteristic impedances or refractive indices as follows:

\[
\rho_i = \frac{\eta_{i+1} - \eta_{i-1}}{\eta_i + \eta_{i-1}} = \frac{n_i - n_{i+1}}{n_{i-1} + n_i}, \quad i = 1, 2, \ldots, M + 1
\]