Section 4.14: Minkowski Space

The study of the matter-light interaction often starts with the Lagrangian or Hamiltonian formulation. The tensor notation commonly found with studies of special relativity provides a compact, simplifying notation in many cases of interest. However, to be useful for special relatively, the notation must also accurately account for the psuedo inner product for Minkowski space necessary to make the speed of light independent of the observer. Refer to the companion volume for an introduction to the special relativity.

Minkowski space has four dimensions with coordinates (x_0, x_1, x_2, x_4) where for special relativity, the first coordinate is related to the time t. Rather than defining the inner product as $\langle v|w\rangle = \sum_n v_n w_n$, the inner product has the form

$$\langle v|w\rangle = v_0 w_0 - (v_1 w_1 + v_2 w_2 + v_3 w_3) \tag{4.14.1}$$

Based on this definition, the inner product for Minkowski space does not satisfy all the properties of the inner product. In particular, the psuedo inner product in Equation 4.14.1 does not require the vectors v and w to be zero when the inner product has the value of zero. The theory of relativity uses two types of notation. In the first, Minkowski 4-vectors use an imaginary number "i" to make the "inner product" appear similar to Euclidean inner products. In the second, a "metric" matrix is defined along with specialized notation. Additionally, a constant multiplies the time coordinate t in order to give it the same units as the spatial coordinates.

One variant of the 4-vector notation uses an imaginary "i" with the time coordinate $x_\mu = (ict, x, y, z) = (ict, \vec{r})$. The constant c, the speed of light, converts the time t into a distance. The psuedo inner product of the vector with itself then has the form

$$x_\mu x_\mu \equiv \sum_{\mu=1}^{4} x_\mu x_\mu = (ict, \vec{r}) \cdot (ict, \vec{r}) = -c^2 t^2 + x^2 + y^2 + z^2 \tag{4.14.2}$$

The imaginary number $i = \sqrt{-1}$ makes the calculation of length look like Pythagorean's theorem but produces the same result as for the psuedo inner product in Equation 4.14.1. Notice the "Einstein repeated summation convention" where repeated indices indicate a summation. The indices appear as subscripts. Notice this psuedo inner product does not require x_μ to be zero when $x_\mu x_\mu = 0$.

As an alternate notation, the imaginary number can be removed by using a "metric" matrix. As is conventional, we use natural units with the speed of light c=1 and $\hbar$=1 for convenience. The various constants can be reinserted if desired.

We represent the basic 4-vector with the index in the upper position. For example, we can represent the space-time 4-vector in component form as

$$x^\mu = (t, x, y, z) = (t, \vec{r}) \tag{4.14.3}$$

where time t comprises the $\mu = 0$ component. Notice the conventional order of the components. The position of the index is significant. To take a psuedo inner product, we could try writing $x^\mu x^\mu = t^2 + x^2 + \ldots$ where we have used a repeated index convention. However, the result needs an extra minus sign. Instead, if we write

$$x_\mu = (t, -\vec{r}) \tag{4.14.4}$$

then the summation beomes $x_\mu x^\mu = (t, -\vec{r}) \cdot (t, \vec{r}) = t^2 - r^2$ where the "extra" minus sign appears. Again the position of the index is important. Apparently, lowering an index places a minus sign on the spatial part of the 4-vector.

A *metric* (matrix) provides a better method of tracking the minus signs. Consider the following metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\mu\nu} \tag{4.14.5}$$

Ordinary matrix multiplication then produces

$$x_\mu = g_{\mu\nu} x^\nu \tag{4.14.6a}$$

Notice the form of this result and the fact that we sum over the ν index by the summation convention. We can also write

$$x^\mu = g^{\mu\nu} x_\nu \tag{4.14.6b}$$

Therefore to take a psuedo inner product, we write

$$x_\mu x^\mu = \left(g_{\mu\nu} x^\nu\right) x^\mu = (t, -\vec{r}) \cdot (t, \vec{r}) = t^2 - r^2 \tag{4.14.7}$$

The metric given here is the "West Coast" metric since it became most common on the west coast of the U.S. The east coast metric contains a minus sign on the time component and the rest have a "+" sign.

Derivatives naturally have lower indices.

$$\partial_\mu = (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial t}, \nabla\right) = \Box \tag{4.14.8}$$

Notice the location of the indices. The upper-index case gives

$$\partial^\mu = g^{\mu\nu} \partial_\nu = (\partial_0, -\partial_1, -\partial_2, -\partial_3) = \left(\frac{\partial}{\partial t}, -\nabla\right) \tag{4.14.9}$$

Let's consider a few examples. The complex plane wave has the form

$$e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)} = e^{-i\left(\omega t - \vec{k}\cdot\vec{r}\right)} = e^{-ik_\mu x^\mu}$$

where $k^\mu = (\omega, \vec{k})$. Also notice that the wave equation

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\psi = 0$$

can be written as

$$\partial_\mu \partial^\mu \psi = 0$$

Just keep in mind the repeated index convention.

As a note, any valid theory must transform correctly. The inner product is relativistically correct since it is invariant with respect to Lorentz Transformations.