Section 4.13: Dyadic Notation

This section develops the dyadic notation for the second rank tensor. We will see that it is equivalent to writing a 2-D matrix. Studies in solid state often use dyadic quantities to describe the effective mass of an electron or hole. For example, formulas relating the acceleration of a particle $\vec{a}$ to the applied force $\vec{F}$ have the form

$$\vec{F} = \overleftrightarrow{m} \cdot \vec{a} \tag{4.13.1}$$

where the dyadic quantity $\overleftrightarrow{m}$ represents the effective mass. This equation says that an applied force can produce an acceleration in a direction other than parallel to the force.

A dyad can be written in terms of components for example

$$\overleftrightarrow{A} = \sum_{ij} A_{ij} \hat{e}_i \hat{e}_j \tag{4.13.2}$$

where the unit vector $\hat{e}_i$ can be one of the basis vectors $\{\hat{x}, \hat{y}, \hat{z}\}$ for a 3-D space, and the $\hat{e}_i\hat{e}_j$ symbol places the unit vectors next to each other without an operator separating them.

Example 4.13.1: Find $\overleftrightarrow{A} \cdot \vec{v}$ for $\overleftrightarrow{A} = 1\hat{e}_1\hat{e}_1 + 2\hat{e}_3\hat{e}_2 + 3\hat{e}_2\hat{e}_3$ and $\vec{v} = 4\hat{e}_1 + 5\hat{e}_2 + 6\hat{e}_3$

Solution:

$\overleftrightarrow{A} \cdot \vec{v} = \left(1\hat{e}_1\hat{e}_1 + 2\hat{e}_3\hat{e}_2 + 3\hat{e}_2\hat{e}_3\right) \cdot \left(4\hat{e}_1 + 5\hat{e}_2 + 6\hat{e}_3\right) = 4\hat{e}_1 + 10\hat{e}_3 + 18\hat{e}_2 = 4\hat{x} + 18\hat{y} + 10\hat{z}$

The coefficients in Equation 4.13.2 can be arranged in a matrix. This means that a 3x3 matrix provides an alternate representation of the second rank tensor and the dyad. The matrix elements can easily be seen to be

$$\hat{e}_a \cdot \overleftrightarrow{A} \cdot \hat{e}_b = \sum_{ij} A_{ij}\ \hat{e}_a \cdot \hat{e}_i \hat{e}_j \cdot \hat{e}_b = \sum_{ij} A_{ij}\ \delta_{ai}\delta_{jb} = A_{ab} \tag{4.13.3}$$

The procedure should remind you of Dirac notation for the matrix discussed in Chapter 5.

The unit dyad can be written as

$$\overleftrightarrow{1} = \sum_i \hat{e}_i \hat{e}_i \tag{4.13.4}$$

Applying the definition of the matrix elements in Equation 4.13.3 shows the unit dyad produces the unit matrix.

Example 4.13.2: Show that if $\overleftrightarrow{1} = \overleftrightarrow{A}$ then $A_{ab} = \delta_{ab}$

Solution: Operate with $\hat{e}_a$ on the left and $\hat{e}_b$ on the right to find

$$\hat{e}_a \cdot \overleftrightarrow{1} \cdot \hat{e}_b = \hat{e}_a \cdot \overleftrightarrow{A} \cdot \hat{e}_b = \hat{e}_a \cdot \left(\sum_{ij} A_{ij} \hat{e}_i \hat{e}_j \right) \cdot \hat{e}_b = \sum_{ij} A_{ij} \delta_{ai} \delta_{jb} = A_{ab}$$

Now let's discuss the inverse of a dyad. Suppose

$$\overleftrightarrow{1} = \overleftrightarrow{A} \cdot \overleftrightarrow{B} \tag{4.13.5}$$

then we can show that $\overleftrightarrow{B} = \overleftrightarrow{A}^{-1}$ where $\overleftrightarrow{A} = \sum_{ii'} A_{ii'} \hat{e}_i \hat{e}_{i'}$ and $\overleftrightarrow{B} = \sum_{jj'} B_{jj'} \hat{e}_j \hat{e}_{j'}$. Operating on the left of Equation 4.13.5 with $\hat{e}_a$ and on the right by $\hat{e}_b$ produces

$$\delta_{ab} = \hat{e}_a \cdot \left(\sum_{ii'} A_{ii'} \hat{e}_i \hat{e}_{i'} \cdot \sum_{jj'} B_{jj'} \hat{e}_j \hat{e}_{j'} \right) \cdot \hat{e}_b = \sum_{ii'jj'} A_{ii'} B_{jj'} \hat{e}_a \cdot \hat{e}_i \hat{e}_{i'} \cdot \hat{e}_j \hat{e}_{j'} \cdot \hat{e}_b$$

The dot products produce may Kronecker delta functions.

$$\delta_{ab} = \sum_{ii'jj'} A_{ii'} B_{jj'} \delta_{ai} \delta_{i'j} \delta_{j'b} = \sum_j A_{aj} B_{jb}$$

which shows the *matrices* $\underline{A}$ and $\underline{B}$ must be inverses.