Section 4.11: Translation Operators

Common mathematical operations such as rotating or translating coordinates are handle by operators in the quantum theory. Previous sections in this chapter show that states transform by the application of a single unitary operator whereas "operators" transform through a similarity transformation. The translation through the spatial coordinate x provides a standard example. Every operation in physical space has a corresponding operation in the Hilbert space.

Topic 4.11.1: The Exponential Form of the Translation Operator

Let $\hat{x}$ and $\hat{p}$ be the position operator and an operator defined in terms of a derivative

$$\hat{p} = \frac{1}{i}\frac{\partial}{\partial x}$$

which is the "position" representation of $\hat{p}$ and $i = \sqrt{-1}$. The position representation of $\hat{x}$ is x. The operator $\hat{p}$ is Hermitian (note that $\hat{p}$ is the momentum operator from quantum theory except the $\hbar$ has been left out of the definition given above). The coordinate kets satisfy $\hat{x}|x\rangle = x|x\rangle$ and the operators satisfy $[\hat{x},\hat{p}] = [\hat{x}\hat{p} - \hat{p}\hat{x}] = i$ as can be easily verified

$$[\hat{x},\hat{p}]f(x) = [\hat{x}\hat{p} - \hat{p}\hat{x}]f(x) = x\frac{1}{i}\frac{\partial}{\partial x}f - \frac{1}{i}\frac{\partial}{\partial x}(xf) = x\frac{1}{i}\frac{\partial}{\partial x}f - \frac{x}{i}\frac{\partial f}{\partial x} - \frac{1}{i}f = if$$

comparing both sides, we see that the *operator* equation $[\hat{x},\hat{p}] = i$ holds. The commutator being non-zero defines the so-called conjugate variables. The translation operator uses products of the conjugate variables. The operator $\hat{p}$ is sometimes called the generator of translations. The Hamiltonian is the generator of translations in time.

This topic shows that the exponential $\hat{T}(\eta) = e^{-i\eta\hat{p}}$ translates the coordinate system according to

$$\hat{T}(\eta)f(x) = e^{-i\eta\hat{p}}f(x) = f(x-\eta)$$

where $\hat{p} = \frac{1}{i}\frac{\partial}{\partial x}$. The proof starts by working with a small displacement ξ_k and calculating the Taylor expansion about the point "x"

$$f(x+\xi_k) \cong f(x) + \frac{\partial f(x)}{\partial x}\xi_k + \ldots = \left(1 + \xi_k\frac{\partial}{\partial x} + \ldots\right)f(x)$$

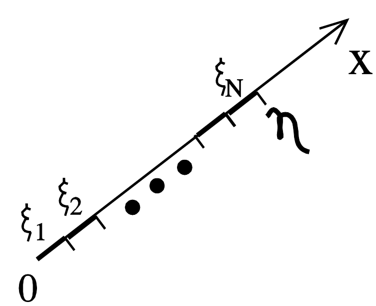

Figure 4.11.1: The total translation is divided into smaller translations

Substituting the operator for the derivative

$$\hat{p} = \frac{1}{i}\frac{\partial}{\partial x}$$

gives

$$f(x+\xi_k) = \left(1+\xi_k \frac{\partial}{\partial x} + \ldots\right) f(x) = \left(1 + i\xi_k \hat{p} + \ldots\right) f(x) = \exp\left(+i\xi_k \hat{p}\right) f(x)$$

Now, by repeated application of the infinitesimal translation operator, we can build up the entire length η

$$f(x+\eta) = \prod_k \exp\left(+i\xi_k \hat{p}\right) f(x) = \exp\left(\sum_k i\xi_k \hat{p}\right) f(x) = \exp\left(i\eta\hat{p}\right) f(x)$$

So the exponential with the momentum operator provides a translation. Replacing η with –η we obtain

$$\hat{T}(\eta) f(x) = e^{-i\eta\hat{p}} f(x) = f(x-\eta)$$

Note that the translation operator is unitary $\hat{T}^{+} = \hat{T}^{-1}$ for η real since $\hat{p}$ is Hermitian. Also note that $\hat{T}^{+}(-\eta) = \hat{T}(\eta)$.

Topic 4.11.2: Translation of the Position Operator

This topic show that

$$\hat{T}^{+}(\eta)\hat{x}\,\hat{T}(\eta) = \hat{x} - \eta$$

where $\hat{T}(\eta) = e^{-i\eta\hat{p}}$. This is easy to show using the operator expansion theorem in Section 4.6

$$e^{\eta\hat{A}}\hat{B}e^{-\eta\hat{A}} = \hat{B} + \frac{\eta}{1!}\left[\hat{A},\hat{B}\right] + \frac{\eta^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \ldots$$

Using $\hat{A} = i\hat{p}$ and the commutation relations $\left[\hat{x},\hat{p}\right] = i$, we find

$$e^{i\eta\hat{p}}\hat{x}e^{-i\eta\hat{p}} = \hat{x} + \frac{\eta}{1!}\left[i\hat{p},\hat{x}\right] + \frac{\eta^2}{2!}\left[i\hat{p},\left[i\hat{p},\hat{x}\right]\right] + \ldots = \hat{x} - \eta$$

Topic 4.11.3: Translation of the Position-Coordinate Ket

The position-coordinate ket $|x\rangle$ is an eigenvector of the position operator $\hat{x}$

$$\hat{x}|x\rangle = x|x\rangle$$

What position-coordinate ket $|\phi\rangle$ is an eigenvector of the translated operator

$$\hat{T}^{+}(\eta)\hat{x}\,\hat{T}(\eta) = \hat{x} - \eta$$

that is, what is the state $|\phi\rangle = \hat{T}^{+}(\eta)|x\rangle$? The eigenvector equation for the translated operator $\hat{x}_T = \hat{T}^{+}\hat{x}\,\hat{T}$ is

$$\hat{x}_T|\phi\rangle = \hat{T}^{+}(\eta)\hat{x}\,\hat{T}(\eta)|\phi\rangle = \left[\hat{T}^{+}(\eta)\hat{x}\,\hat{T}(\eta)\right]\hat{T}^{+}(\eta)|x\rangle = \hat{T}^{+}(\eta)\,\hat{x}|x\rangle = x\,\hat{T}^{+}(\eta)\,|x\rangle = x|\phi\rangle$$

However, we know the translated operator is $\hat{x}_T = \hat{x} - \eta$ and therefore the previous equation provides

$$x|\phi\rangle = \hat{x}_T|\phi\rangle = (\hat{x}-\eta)|\phi\rangle = (\phi-\eta)|\phi\rangle$$

Comparing both sides, we see $\phi = x + \eta$ which therefore shows that the translated position vector is

$$|\phi\rangle = \hat{T}^{+}(\eta)|x\rangle = |x + \eta\rangle$$

Topic 4.11.4: Example Using the Dirac Delta Function

Show that

$$|\phi\rangle = \hat{T}^{+}(\eta)|x'\rangle = |x' + \eta\rangle$$

using the fact that the position-ket represents the Dirac Delta function in Hilbert space

$$|x'\rangle \equiv |\delta(\bullet - x')\rangle$$

where "$\bullet$" represents the missing variable. If "x" is a coordinate on the x-axis then

$$\langle x|x'\rangle \equiv \int_{-\infty}^{\infty} d\varsigma \; \delta(\varsigma - x)\, \delta(\varsigma - x') = \delta(x - x')$$

Applying the translation operator in the x-representation

$$\langle x|\hat{T}(\eta)|x'\rangle = e^{-i\eta\hat{p}_x} \langle x|x'\rangle = e^{-i\eta\hat{p}_x} \delta(x - x') = \delta(x - \eta - x') = \langle x|x' + \eta\rangle$$

Evidently

$$\hat{T}(\eta)|x'\rangle = |x' + \eta\rangle$$