Section 4.10: A Relation Between Unitary and Hermitian Operators

As previously discussed, a Hermitian operator $\hat{H}: V \to V$ has the property that $\hat{H} = \hat{H}^{+}$. This section shows how unitary operators can be expressed in the form $\hat{u} = e^{i\hat{H}}$ where $\hat{H}$ is a Hermitian operator.

We can show that the operator $\hat{u} = e^{i\hat{H}}$ is unitary by showing $\hat{u}^{+}\hat{u} = 1$

$$\hat{u}^{+}\hat{u} = \left(e^{i\hat{H}}\right)^{+}\left(e^{i\hat{H}}\right) = e^{-i\hat{H}^{+}}e^{i\hat{H}} = e^{-i\hat{H}}e^{i\hat{H}} = e^{0} = 1$$

This is a one line proof, but a few steps need to be explained as follows:

(1) A function of an operator $f\left(\hat{A}\right)$ must be interpreted as a Taylor expansion. Therefore, we define the *exponential of an operator* to be shorthand notation for a Taylor series expansion in that operator. Recall that the Taylor series expansion of an exponential has the form

$$e^{ax} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^{n} e^{ax}}{\partial x^{n}}\bigg|_{x=0} x^{n} = 1 + \frac{\partial}{\partial x}\left(e^{ax}\right)_{x=0} x + \cdots = 1 + ax + \frac{a^{2}}{2}x^{2} + \cdots$$

In analogy, the exponential of an operator $\hat{H}$ (or equivalently of a matrix $\underline{H}$) can be written as

$$e^{i\underline{H}t} = \underline{1} + \left(i\underline{H}\right)t + \frac{\left(i\underline{H}\right)^{2}}{2}t^{2} + \ldots$$

(2) We wrote $e^{-i\hat{H}}e^{i\hat{H}} = e^{i\left(\hat{H}-\hat{H}\right)} = e^{0} = 1$. As shown in Section 4.6, $e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}}$ when $\left[\hat{A},\hat{B}\right] = 0$. This condition is satisfied because $\left[\hat{H},\hat{H}\right] = \hat{H}\hat{H} - \hat{H}\hat{H} = 0$.

Example 4.10.1: Find the unitary matrix corresponding to $e^{i\underline{H}}$ where

$$\underline{H} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

Solution: First note that the matrix H is Hermitian $\underline{H} = \underline{H}^{+}$

$$\underline{u} = e^{i\underline{H}} = \exp\left\{ i\begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix} \right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} + \frac{i^{2}}{2!}\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}^{2} + \ldots = \begin{bmatrix} e^{i\,0.1} & 0 \\ 0 & e^{i\,0.2} \end{bmatrix}$$