Section 4.1 Vector and Hilbert Spaces

Linear algebra starts with the definition of the vector space. An inner product space consists of a vector space with an inner product defined on it. The Hilbert space often refers to an inner product space of functions. However, this section uses the term Hilbert and inner product spaces interchangeably.

Topic 4.1.1: Definition of Vector Space

A vector space consists of a set F with a defined binary operation "+" and a scalar multiplication over the field of numbers $\mathscr{N}$ such that (assuming f, f_1, f_2 are in F and α, β are in $\mathscr{N}$),

Closure:	f_1+f_2 is in F and αf is in F
Associative:	$(f_1+f_2)+f_3 = f_1+(f_2+f_3)$
Commutative:	$f_1+f_2 = f_2+f_1$
Zero:	There exists a zero vector $\mathscr{O}$ such that $\mathscr{O} + f = f$
Negatives:	For every f in F, there exists (-f) in F such that $f+(-f) = \mathscr{O}$
SM Associative:	$(\alpha\beta)f = \alpha(\beta f)$
SM Distributive:	$\alpha(f_1+f_2)=\alpha f_1+\beta f_2$
SM Distributive:	$(\alpha+\beta)f = \alpha f + \beta f$
SM Unit:	$1f = f$

If "F" is a set of functions then "F" is sometimes called a function space. For complex functions F, the number field $\mathscr{N}$ must be the set of complex numbers $\mathscr{C}$ while, for real functions F, the number field $\mathscr{N}$ consists of the real numbers $\mathscr{R}$. For example, if F represents the set of real functions but the number field consists of complex numbers, then objects such as $c_1f(x)$ (where c_1 is complex) cannot be in the original vector space because the function $g(x)=c_1f(x)$ has complex values. Therefore, for this example, closure cannot be satisfied contrary to the requirements of the definition for the vector space.

Topic 4.1.2: Inner Product, Norm, and Metric

An *inner product* $\langle \bullet | \bullet \rangle$ in a (real or complex) vector space F is a scalar valued function that maps $F \times F \to C$ (where C is the set of complex numbers) with the properties

1. $\langle f | g \rangle = \langle g | f \rangle^*$ with f,g elements in F and where "*" denotes complex conjugate
2. $\langle \alpha f + \beta g | h \rangle = \alpha^* \langle f | h \rangle + \beta^* \langle g | h \rangle$ and $\langle h | \alpha f + \beta g \rangle = \alpha \langle h | f \rangle + \beta \langle h | g \rangle$ where f,g,h are element of F and α,β are elements in the complex number field $\mathscr{C}$.
3. $\langle f | f \rangle \geq 0$ for all vectors f. The inner product can be zero $\langle f | f \rangle = 0$ if and only if $f = 0$ (except at possibly a few points).

The *norm* or "length" of a vector f is defined to be $\| f \| = \langle f | f \rangle^{1/2}$.

A *metric* d(f,g) is a relation between two elements f and g of a set F such that

1. $d(f,g) \geq 0$ and d=0 only when f=g (except at possibly a few points for $C_p[a,b]$. Recall that two functions are equal only when f(x)=g(x) for all "x" in the domain of definition
2. d(f,g)=d(g,f)
3. $d(f,g) \leq d(f,h)+d(h,g)$ where h is any third element of F

The metric measures the distance between two elements of the space. The properties of the inner product are very similar to those of the metric. In fact, if d(f,g) is a metric then it can be written as

$$d(f,g) = \langle f-g | f-g \rangle^{1/2}$$

Consider $\mathscr{R}^2$ which is the set of Euclidean vectors in the x-y plane. Assume $\vec{r}_1$ and $\vec{r}_2$ are two vectors in R^2 with $\vec{r}_1 = x_1\tilde{x} + y_1\tilde{y}$ and $\vec{r}_2 = x_2\tilde{x} + y_2\tilde{y}$. Simple vector analysis provides the relations

Inner product	$\langle \vec{r}_1 \mid \vec{r}_2 \rangle = \vec{r}_1 \cdot \vec{r}_2 = x_1 x_2 + y_1 y_2$
Norm	$\|\vec{r}_1\| = \langle \vec{r}_1 \mid \vec{r}_1 \rangle^{1/2} = ({x_1}^2 + {y_1}^2)^{1/2}$
Metric	$d(\vec{r}_1, \vec{r}_2) = \|\vec{r}_1 - \vec{r}_2\| = [(\vec{r}_1 - \vec{r}_2)\cdot(\vec{r}_1 - \vec{r}_2)]^{1/2} = \sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2}$

The inner product can be defined for functions as

Inner product
$$\langle f | g \rangle = \int_a^b dx\ f(x)^* g(x)$$

Norm
$$\|f(x)\| = \langle f | f \rangle^{1/2} = \int_a^b dx\ f(x)^* f(x) = \int_a^b dx\ |f(x)|^2$$

Example 4.1.1: Find the length of f(x)=x for $x \in [-1,1]$

$$\|f\| = \langle f | f \rangle^{1/2} = \left[\int_{-1}^{1} dx\ x^* x\right]^{1/2} = \left[\int_{-1}^{1} dx\ x \cdot x\right]^{1/2} = \left[\int_{-1}^{1} dx\ x^2\right]^{1/2} = \sqrt{\frac{2}{3}}$$

where we used the fact that f(x)=x is real. If we were to divide the function by the norm and write $g(x) = f(x)/\|f\|$ then the length of g(x) would be unity. In general, we normalize a function f(x) to one by dividing by the norm of f(x).

Topic 4.1.3: Hilbert Space

We define a Hilbert Space H to be a vector space with an inner product defined on the space. Some books reserve the term "Hilbert space" for vector spaces of *functions* with an inner product; they sometimes denote the inner product by (f_1,f_2). For function spaces, the functions must be square integrable in the sense that the following integral must exist for $f \in H$

$$\int_a^b dx\,|f(x)|^2$$

Sometimes the term "inner product space" refers to a vector space (regardless of whether its a Euclidean or function space) having a defined inner product. This book doesn't make any distinction between the function or Euclidean vector spaces and assumes all of the inner products exist (such as the previous integral).