Chapter 4: Mathematical Foundations

Linear algebra is the natural mathematical language of quantum mechanics. For this reason, the present chapter starts with a review of Hilbert spaces for vectors and operators. We introduce vector and Hilbert spaces along with inner products and metrics. The Dirac notation is developed for the Euclidean vector spaces as a starting point for the concepts of complete orthonormal sets of vectors, closure, dual vector spaces and adjoint operators. The Dirac delta function in various forms and the principal part is introduced in Appendix 5 as an essential tool. The chapter turns to the main use of Dirac notation for function spaces; the concepts of norm, inner product and closure are discussed. Fourier, Cosine and Sine series are discussed as examples of expansions in complete orthonormal sets of functions.

Although Hilbert spaces are interesting mathematical objects with important physical applications, the study of linear algebra remains incomplete without a study of linear operators (i.e., linear transformations). In fact, the set of linear transformations itself forms a vector space and therefore has a basis set. The basis set for the operator is linked with the basis sets for the spaces that it operates between. The linear operator can be discussed as an abstract operator or through an isomorphism as a matrix or as a generalized expansion in operator space.

A Hermitian (a.k.a., *self*-adjoint) operator produces a basis set within a Hilbert space. The basis set comes from the eigenvector equation for the particular operator. The fact that a Hermitian operator produces a complete set (of orthornomal vectors) has special importance for quantum mechanics. Observables such as energy or momentum correspond to Hermitian operators. Complete sets make it must be possible to represent every possible result of a measurement of the observable by an object (vector) in the theory. The Hermitian operators have real eigenvalues which represent the results of the measurement.