A time-invariant realization

\[ x = \begin{bmatrix} 3 \lambda & -3 \lambda^2 & \lambda^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \]

\[ y = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} x \]

4.26 \[ g(t, x) = \sin \pi e^{-t} e^{-t} \cos \pi t \]

A time-varying realization

\[ x = \alpha x + e^t \cos t u(t) \]

\[ y = \sin t e^{-t} x \]

Because \( g(t, x) \) cannot be expressed as \( g(t-e) \), it cannot be realized as a linear time-invariant equation.

Chapter 5

5.1 The transfer function from \( u \) to \( y \) is

\[ \hat{g}(s) = \frac{5 \frac{t}{s + \frac{1}{s}}}{s^2 + 1} = \frac{5}{s^2 + 1} \]

If \( u(t) = \sin t \), then

\[ \hat{y}(s) = \hat{g}(s) \hat{u}(s) = \frac{5}{s^2 + 1} \frac{1}{s^2 + 1} = \frac{5}{(s^2 + 1)^2} \]

and \( y(t) = 0.5 \sin t \)

which is not bounded. Thus the network is not BIBO stable.

5.2 \[ \hat{g}(s) = \int_0^\infty g(t) e^{-st} dt \]

For \( s = \alpha + j \omega \), if \( \alpha > 0 \), then

\[ |e^{-st}| = |e^{-\alpha t}||e^{-j\omega t}| = e^{-\alpha t} \leq 1 \]

for all \( t \). If the system is BIBO stable, then \( \int_0^\infty |g(t)| < \infty \). Thus we have, for \( \text{Re } s > 0 \),

\[ |\hat{g}(s)| \leq \int_0^\infty |g(t)||e^{-st}| dt \leq \int_0^\infty |g(t)| dt < \infty \]

\[ \int_0^\infty |g(t)| dt = \int_0^\infty \frac{1}{1 + \pi^2} dt = \ln(1 + \pi^2) \]

Thus the system is not BIBO stable.

For \( g(t) = e^{-t} \), we have

\[ \hat{g}(s) = \int [g(t)] = \frac{1}{(s + 1)^2} \]

all the poles have negative real parts, thus the system is BIBO stable.

5.4. \[ \hat{g}(s) = \frac{e^{-st}}{s+1} \]

\[ \text{irrational function of } s \]

\[ \hat{g}(t) = L^{-1}[\hat{g}(s)] = \begin{cases} e^{-(t-2)} & \text{for } t \geq 2 \\ 0 & \text{for } t < 2 \end{cases} \]

\[ \int_0^\infty |g(t)| dt = \int_2^\infty e^{-(t-2)} dt = \int_0^\infty e^{-t} dt \]

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It is BIBO stable.

\[ g(s) = \frac{5 - s}{s + 1} \]

Thus the system is BIBO stable.

5.5 If \( r(t) = \delta(t) \), then
\[
\hat{g}(s) = \frac{5 - s}{s + 1}
\]

If \( u(t) = 3 \), then \( y(t) \rightarrow \hat{g}(s) \cdot 3 = -6 \)

If \( u(t) = 2t \), then
\[
y(t) \rightarrow \hat{g}(s) \cdot \min(2t + 4, \frac{4}{s+1}) \]

5.7 The matrix has eigenvalues 0, 1, 3; thus the equation is not asymptotically stable. If the repeated eigenvalue 0 is a simple root of the minimal polynomial or, equivalently, has only Jordan blocks of size 1, then the equation is marginally stable. We compute the eigenvector associated with \( \lambda = 0 \):

\[
(A - \lambda I) v = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0
\]

which yields the linearly independent eigenvectors \( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \). Thus the Jordan form of \( A \) is

\[
\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

5.11 The matrix has eigenvalues 0, 1, 3; thus the equation is not asymptotically stable. We compute the eigenvector associated with \( \lambda = 0 \):

\[
(A - \lambda I) v = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0
\]

It has only one linearly independent
eigenvector. Thus its Jordan form is
\[ \hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \]

It has one Jordan block with order 2, thus the equation is not marginally stable.

5.12 The matrix has eigenvalues 0.9, 1, 1; thus the discrete-time system is not asymptotically stable. Its Jordan form is
\[ \hat{A} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

Thus it is marginally stable.

5.13 The matrix has eigenvalues 0.9, 0, 0, 1, and 1, and its Jordan form, as in Prob. 5.11, is
\[ \hat{A} = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \]

Thus the equation is not marginally stable, nor asymptotically stable.

5.14 \[ A = \begin{bmatrix} 0 & 0 \\ -0.5 & -1 \end{bmatrix} \] Select \( N = 2 \).
\[ \begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_3 & m_2 \end{bmatrix} + \begin{bmatrix} m_1 & m_2 \\ m_3 & m_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

Equating the (1,1) entry:
(1.1): \(-0.5m_3 - 0.5m_2 = -1 \Rightarrow m_3 = 1\)
(2.2): \(m_3 - m_2^2 - m_1 - m_2 = 1 \Rightarrow m_3 = 1.5\)
(1.2): \(-0.5m_3 + m_2 - m_1 = 0 \Rightarrow m_1 = 1.75\)

\[ M = \begin{bmatrix} 1.75 & 1 \\ 1 & 1.5 \end{bmatrix} \]

Leading principal minors 1.75 > 0
1.75x1.5 - 1x1 > 0

\( M \) is positive definite. Thus all eigenvalues of \( A \) have negative real parts.

5.15 \[ \begin{bmatrix} m_1 & m_3 \\ m_2 & m_4 \end{bmatrix} - \begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_3 & m_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]

(1,1): \( m_1 - 0.25m_3 = 1 \)
(1,2): \(-0.5m_2 + 1.5m_3 = 0 \)
(2,2): \( m_3 - m_1 + 2m_2 - m_3 = 1 \)

From these, we can obtain \( m_3 = 1.6, m_2 = 4.8, m_1 = 2.2. \)

\[ M = \begin{bmatrix} 2.2 & 1.6 \\ 1.6 & 4.8 \end{bmatrix} \]

Thus all eigenvalues of \( A \) have magnitude less than 1. As a check, the eigenvalues of \( A \) are \(-0.5 \pm j0.5\). Both have negative real parts and have magnitudes less than 1.

5.16 Let \( A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \). Then
\[ \lambda' \lambda = -\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]

It is clear that all eigenvalues of \( A \) have negative real parts and \( N \) is positive semidefinite. We compute
\[ \begin{bmatrix} N \lambda A \\ N \lambda a^2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0 \]
\[ \begin{bmatrix} \lambda A_1 & \lambda A_2 & \lambda A_3 \\ \lambda^2 A_4 & \lambda^2 A_5 & \lambda^2 A_6 \\ \lambda^2 A_7 & \lambda^2 A_8 & \lambda^2 A_9 \end{bmatrix} \]

det \( Q \) = \( a_1 a_3 a_2 \det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \)

\( = a_1 a_3 a_2 \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \)

\( = a_1 a_3 (A_2 - A_1) (A_3 - A_1) \det \begin{bmatrix} 1 & 0 \\ 0 & -A_2^{-1} A_1 \end{bmatrix} \)

\( = a_1 a_3 (A_2 - A_1) (A_3 - A_1) \det \begin{bmatrix} 1 & 0 \\ 0 & A_3^{-1} - A_1 \end{bmatrix} \)

\( \det Q \) is nonzero if \( a \neq 0 \) and \( A_2 \) are distinct. Thus \( Q \) has rank 3 and \( M \) is positive definite. (Corollary 5.5)
5.17 \( M = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \) is not positive definite because \( \text{tr} M = 1 \neq 0 \). Its eigenvalues are 1 and 2. \( H = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \) is not positive definite because
\[
\begin{bmatrix} 0.5805 & -0.8142 \\ -0.8142 & -0.5805 \end{bmatrix}
\]

is not positive definite.

Its leading principal minors are 2 and \((2 \times 1 - 1.9 \times 1) = 0.1\); both are positive. Therefore, the assertions do not hold. Because
\[
\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}
\]

we may check the positive definiteness of \( H \) by forming the symmetric matrix \( \frac{1}{2} (M + M^T) \) and then check \( \tilde{H} \). For example, we form
\[
\tilde{H} = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ -0.5 & 3 \end{bmatrix}
\]

It is not positive definite because its leading principal minors are 0 and -0.25. Similarly, we form
\[
\tilde{H}_2 = \begin{bmatrix} 2 & 1.45 \\ 1.45 & 1 \end{bmatrix}
\]

It is not positive definite because its leading principal minors are 2 and \( 2 \times 1 - (1.45)^2 = -0.1025 \).

5.18 \( A'M + MA + 2\mu M = -N \)
\( (A' + \mu I) M + M(A + \mu I) = -N \)

If \( N > 0 \) and \( M > 0 \), then all eigenvalues of \((A + \mu I)\) have magnitudes less than 0, i.e., \( \Re \lambda_i(A + \mu I) < 0 \).

Using Problem 5.19, we have
\[
\Re \lambda_i(A + \mu I) = \Re \lambda_i(A) + \mu. \text{ Thus }
\Re \lambda_i(A) + \mu < 0 \text{ or } \Re \lambda_i(A) < -\mu
\]

5.19 \((A' + \mu I) M + M(A + \mu I) = N \)
\( (A' + \mu I) M + M(A + \mu I) = N \)

If \( N > 0 \) and \( M > 0 \), then
\[
| \lambda_i(A + \mu I) | < 1
\]

Thus \( | \lambda_i(A) | < \rho \)

5.20 \( g(t, z) = e^{-t^2} [1 - t] \), \( \text{ for } t \geq 0 \)
\[
\int_t^\infty \left[ g(t, z) \right] dz = e^{-t^2} \int_t^\infty e^{-z} dz = e^{-t^2} t
\]

For \( t > 0 \), we have
\[
B = e^{t^2} \int_t^\infty e^{-z} dz = e^{t^2} \left( e^{-z} \right)_{z=t} < 1
\]

For \( t \geq t_o \), we have
\[
B = e^{t^2} \int_{t_o}^\infty e^{-z} dz = e^{t^2} \left[ e^{-z} \right]_{z=t_o} = e^{t^2} e^{-t_o} - e^{-t-o} < 0
\]

Thus the system is BIBO stable.

\[
y(t, z) = u(t) e^{-t^2}
\]

\[
\int_{t_o}^\infty g(t, z) dz = \int_{t_o}^t e^{-Z} dz = e^{-t} \int_{t_o}^t e^{z} dz
\]

\[
e^{-t} \left[ e^{z} \right]_{z=t_o} = 1 - e^{t-t_o} < 1
\]

For all \( t_o \) and \( t > t_o \). Thus the system is

\[
(5, 21) \quad \dot{x} = 2t x + u, \quad y = e^{-t^2} x
\]

This scalar equation, we have
\[
\phi(t, z) = e^{z} \int_{t_o}^t e^{z} dz = e^{t^2-t_o^2}
\]

\[
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\]
Thus $g(t, z) = e^{-z} \phi(t, z)$, if $1 = e^{-z} + e^{-z} = e^{-z}$

\[
\int_{t_0}^{t} \frac{g(t, z)}{z} \, dz = \int_{t_0}^{t} e^{-z} \, dz < \infty \text{ for all } t_0 \text{ and } t > t_0 \text{ because } e^{-z} < e^{-t_0} \text{ for } 1 > z > \infty.
\]

Thus the equation is BIBO stable.

Because $|\phi(t, z)| = e^{-z} - t^2 \to 0$ as $t \to \infty$, the equation is not marginally stable, nor asymptotically stable.

Using (4.70), we have

\[
\bar{A} = [P(t)A(t) + P(t)] P(t)
\]

\[
= [2te^{-t^2} - 2te^{-z^2}] e^{-t^2} = 0
\]

Thus the equivalent equation is

\[
\begin{align*}
\dot{x} &= 0.5x + e^{-t^2}u \\
y &= e^{-t^2}e^{-z^2} - \ddot{x}
\end{align*}
\]

\[
\bar{g}(t, z) = C(t) \phi(t, z) B(z) = 1 \times 1 \times e^{-z^2} = e^{-z^2}
\]

The impulse response remains unchanged.

Therefore the equation is BIBO stable.

The zero-input response is governed by the time-invariant equation

\[
\dot{x} = 0.5x
\]

with eigenvalue 0. Thus the equation is marginally stable, it is not asymptotically stable.

The transformation $P(t) = e^{-t^2}$ is not a Lyapunov transformation because $P(t) = e^{-t^2}$ is not bounded.

Therefore marginal and asymptotic stability criteria are not invariant under this transformation.