5.4. MINIMUM-TIME PROBLEMS

\[ \begin{align*}
\dot{x} &= a(x, u, t) \\
|u(t)| &\leq 1, \quad t \in [t_0, t_f] \\
J(u) &= \int_{t_0}^{t_f} dt = t_f - t_0
\end{align*} \]

The objective is to transfer a system from an arbitrary initial state to a specified target state in minimum time.

can we do it?

see:
(Ex. 5.4-1)

To go,
1) The optimal control (if it exists) is maximum effort during the entire time interval, that is \( |u(t)| = 1 \)
2) For certain values of the initial conditions, the optimal control does not exist

REACHABLE STATES from \( x(t_0) = x_0 \) and \( t \in [t_0, t_f] \)

collection of all values \( x(t) \), with \( x(t_0) = x_0 \), is called the set of reachable states at \( t \).

It depends on \( x_0, t_0 \) and \( t \).

Notation \( R(t) \)
\( (\text{Ex. 5.4-2}) \)

\[
x = u \quad -1 \leq u(t) \leq 1
\]

\[
x(t) = x_0 + \int_{t_0}^{t} u(t) \, dt
\]

\[
\Rightarrow \quad x_0 - (t-t_0) \leq x(t) \leq x_0 + (t-t_0)
\]

\[
\mathcal{R}(t) = [x_0 - (t-t_0), x_0 + (t-t_0)]
\]

\[
\mathcal{R}(t) \cap \mathcal{S}(t)
\]

\[
\Rightarrow \quad \text{time-optimal control does not exist}
\]

\[
\mathcal{R}(t) \cap \mathcal{S}(t)
\]

\[
\Rightarrow \quad \text{time-optimal control exists}
\]

\( R(t) \) and \( S(t) \) must have at least one point in common.

\[
\mathcal{R}(t^*) \cap \mathcal{S}(t^*)
\]

\[
\left\{ \begin{array}{l}
\text{General theorems for the existence of the solution are not available}
\end{array} \right.
\]

\[
\text{solution at the earliest time } t^*
\]

\[
\text{when } S(t^*) \text{ and } R(t^*) \text{ meet}
\]
The Form of the Optimal Control for a Class of
- Minimum-Time Problems

\[ x = a(x, t) + B(x, t) u(t) \]

\[ M_{i-} \leq u_i \leq M_{i+}, \quad i = 1, 2, \ldots, m \quad t \in [t_0, t^*] \]

The Hamiltonian is

\[ H(x, u, p) = 1 + p^T (a(x, t) + B(x, t) u(t)) \]

From the minimum principle it is necessary that

\[ 1 + p^T (a(x^*, t) + B(x^*, t) u^*) \leq 1 + p^T (a(x^*(t), t) + B(x^*(t), t) u(t)) \]

or

\[ p^T B(x^*, t) u^* \leq p^T B(x^*(t), t) u(t) \]

\[ B = [b_1, b_2, \ldots, b_m] \]

\[ p^T B(x^*, t) u(t) = \sum_{i=1}^{m} p^T b_i (x^*, t) u_i(t) \]

Assuming that the control components are independent of one another we then must minimize

\[ p^T b_i(x^*) u_i(t) \]

\[ u^*_i(t) = \begin{cases} 
M_{i+} & \text{for } p^+ b_i (x^*(t), t) < 0 \\
M_{i-} & \text{for } p^- b_i (x^*(t), t) > 0 \\
\text{undetermined} & \text{for } p^T b_i (x^*(t), t) = 0 
\end{cases} \]

Singular Control
TIME INVAR iANT CONTROL:

\[ \dot{x} = A x + B u \quad \left| u_i(t) \right| < 1 \quad i = 1, 2, \ldots, m \]

Assuming that the system is completely controllable and normal, find a control (if one exists), which transfers the system from an arbitrary initial state \( x_0 \) to the final time \( x(t_f) = 0 \) in minimum time.

(Pontryagin et al. 1962 The Mathematical Theory of Optimal Processes)

EXISTENCE THEOREM: All eigenvalues of \( A \) have non-positive real parts \( \Rightarrow \) time-optimal control exists

UNIQUENESS THEOREM: If an extremal exists, the \( \mathcal{E} \) is unique

NUMBER OF SWITCHING:

If the eigenvalues of \( A \) are real, and a time optimal control exists, then each control component can switch at most \( (n-1) \) times.
5.24. \( x = 2x + u \)

\[ |u(t)| \leq 1 \]

minimum time problem

\( x = 2 \Rightarrow \) optimal control might not exist

\[ u = 1 + p(2x + u) \]

\[ \frac{\partial u}{\partial x} = p = -p \Rightarrow p = ce^{-t} \]

minimum principle

\[ 1 + p(x^*) u(x^*) \leq 1 + p_i^*(2x^* + u) \]

\[ p_i^* u^* \leq p_i^* u \]

\[ u^* = \begin{cases} +1 & \text{if } p_i^* < 0 \Rightarrow c_i e^{-t} < 0 \Rightarrow c_i < 0 \\ -1 & \text{if } p_i^* > 0 \Rightarrow c_i e^{-t} > 0 \Rightarrow c_i > 0 \end{cases} \]

\[ u^* = +1 \]

\[ x(t) = x_0 e^{2(t-t_0)} + \int_{t_0}^{t} e^{2(t-\tau)} d\tau \]

\[ x(t) = 0 = x_0 e^{2(t-t_0)} + e^{2(t-t_0)} \int_{t_0}^{t} e^{-2\tau} d\tau \]

\[ e^{2t_0} x_0 = -\frac{t}{2} e^{-2t_0} \int_{t_0}^{t} e^{-2\tau} d\tau = +\frac{1}{2} e^{-2t_0} \int_{t_0}^{t} e^{-2\tau} d\tau \Rightarrow -\frac{1}{2} < x_0 \leq 0 \]

\[ u^* = -1 \]

\[ e^{2t_0} x_0 = +\frac{t}{2} e^{-2t_0} \int_{t_0}^{t} e^{-2\tau} d\tau = -\frac{1}{2} e^{2t_0} \int_{t_0}^{t} e^{-2\tau} d\tau \Rightarrow 0 \leq x_0 < 1/2 \]

\[ 2x_0 = -e^{2t_0} + 1 \]

So \( u^* = -1 \) and \( u^* = +1 \)

\[ \{ -\frac{1}{2} < x_0 < \frac{1}{2} \} \]

For these initial states we can reach the origin

\[ e^{2t_0} x_0 = \frac{1}{2} (e^{2t_0} - e^{-2t_0}) \]

\[ x_0 = e^{-2t_0} - 1 \Rightarrow -2t_0 e^{-e} = e^n (2x_0 + 1) \Rightarrow \frac{t}{2} = -\frac{1}{2} e^n (2x_0 + 1) \]
b) If \( a_i = \lambda \leq 0 \), then an optimum contour exists that transfers any initial state \( x_0 \) to the origin.

For \( a_i > 0 \):

\[
\begin{align*}
\dot{x}_t &= a_i x_i + b_i u \\
|u(t)| &\leq 1.0 \quad b_i \neq 0 \\
u^*(t) &= \pm 1 \\
x_i(t) &= e^{a_i t} x_i + \int_0^t e^{a_i (t - \tau)} (\pm b_i) d\tau \\
0 &= e^{a_i t} x_i + b_i \int_0^t e^{-a_i \tau} d\tau \\
e^{a_i t} x_i &= \pm b_i \int_0^t e^{-a_i \tau} d\tau \\
e^{a_i t} x_i &= \pm b_i \frac{1}{a_i} (e^{-a_i t} - e^{-a_i t_0}) \\
u^*(t) &= +1 \\
e^{a_i t_0} x_i &= \frac{b_i}{a_i} (e^{-a_i t} - e^{-a_i t_0})
\end{align*}
\]

\[
\frac{1^0 t \to \infty}{2^0 t \to t_0} \Rightarrow -\frac{|b_i|}{a_i} < x_i^0 \leq 0
\]

For \( u^*(t) = -1 \):

\[
\begin{align*}
e^{a_i t_0} x_i &= -\frac{b_i}{a_i} (e^{-a_i t} - e^{-a_i t_0}) \\
1^0 t \to \infty &\to 0 \\
2^0 t \to t_0 &\Rightarrow 0 \leq x_i^0 < \frac{|b_i|}{a_i}
\end{align*}
\]

For \( u^*(t) = +1 \) and \( u^*(t) = -1 \):

\[
-\frac{|b_i|}{a_i} < x_i < \frac{|b_i|}{a_i}
\]

For these initial states we can reach the origin.

If \( |x_i^0| > \frac{|b_i|}{a_i} \) an optimum contour doesn't exist.