6.3 Pontryagin's Minimum Principle and State Inequality Constraints

\[ J(u) - J(u^*) = \Delta J \geq 0 \]

\[ \Delta J(u^*, \delta u) = \delta J(u^*, \delta u) + \text{h.o.t.} \]

\( \text{linear in } \delta u \)

The necessary condition for \( u^* \) to be an extremal control is that the variation \( \delta J(u^*, \delta u) = 0 \) for all admissible controls.

If \( |u(t)| \leq 1 \), \( t \in [t_0, t_f] \)
then \( \delta u \) is arbitrary only if the extremal control is strictly within the boundary for all time on the interval \( [t_0, t_f] \).
If the extremal control lies on a boundary during at least one subinterval \( [t_1, t_2] \subset [t_0, t_f] \) then

\[ t \in [t_1, t_2] \]
\[ \Rightarrow \delta J(u^*, \delta u) \geq 0 \]
Conclusion: \[ \begin{cases} \delta J(u^*, \delta u) \geq 0 & \text{on the boundary} \\ \delta J(u^*, \delta u) = 0 & \text{inside the boundary} \end{cases} \]

(Eqs. 5.1-9 and 5.1-13 from Sec 5.1)

\[
\Delta J(u^*, \delta u) = \left[ \frac{\partial^2 J}{\partial x^2}(x^*(t_f), t_f) - p^*(t_f) \right]^T \delta x + \\
+ \left[ H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial H}{\partial u} (x^*(t_f), t_f) \right] \delta u \\
+ \left[ \frac{\partial H}{\partial p} (x^*(t_f), u^*(t_f), p^*(t_f), t_f) - x^*(t_f) \right]^T \delta p \right] dt + r.o.t
\]

- \[ x^* = \frac{\partial H}{\partial p} = 0 \text{ satisfied} \]

- \[ p^* = -\frac{\partial H}{\partial x} \text{ can be chosen arbitrary} \]

- boundary conditions assumed to be satisfied

\[
\Rightarrow \Delta J(u^*, \delta u) = \int_0^{t_f} \left[ \frac{\partial H}{\partial u} (x^*(t_f), u^*(t_f), p^*(t_f), t_f) \right] \delta u \, dt + r.o.t
\]

\[
\left[ \frac{\partial H}{\partial u} (x^*, u^*, p^*, t_f) \right]^T \delta u = H(x^*, u^* + \delta u, p^*, t_f) - H(x^*, u^*, p^*, t_f)
\]

\[
\Rightarrow \Delta J(u^*, \delta u) = \int_0^{t_f} \left[ H(x^*, u^* + \delta u, p^*, t_f) - H(x^*, u^*, p^*, t_f) \right] dt + r.o.t
\]

\[
\Delta J(u^*, \delta u) \geq 0
\]

\[
\Rightarrow H(x^*, u^* + \delta u, p^*, t_f) - H(x^*, u^*, p^*, t_f) \geq 0
\]
or, for $u^*$ to be a minimizing control it is necessary that

$$H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t)$$

Pontryagin's minimum principle

$\Rightarrow$

Optimal control must minimize the Hamiltonian

**Summary:**

$$\dot{x} = \alpha(x, u, t)$$

$$J(u) = \ell(x(t_f), t_f) + \int_0^H \ell(x(t), u(t), t) dt$$

$$H(x, u, p, t) = \ell(x, u, t) + p^T \alpha(x, u, t)$$

**Necessary Conditions for the Optimum for Constrained Control**

$$\dot{x} = \frac{\partial H}{\partial p}$$  \hspace{1cm} (4)

$$\dot{p} = -\frac{\partial H}{\partial x}$$  \hspace{1cm} (2)

$$H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t)$$  \hspace{1cm} (3)

Terminal conditions:

$$\left(\frac{\partial \ell}{\partial x}(x(t_f), t_f) - p^*(t_f)\right)^T \delta x + \left[H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial H}{\partial u} (x^*(t_f), t_f)\right] \delta u = 0$$

If control is not bounded (3) $\Rightarrow$

$$\frac{\partial H}{\partial u}(x^*, p^*, u^*, t) = 0$$

$$\frac{\partial^2 H}{\partial u^2} > 0 \Rightarrow$$ successful conditions
Additional Necessary Conditions:

1. If the final time is fixed and the Hamiltonian does not depend explicitly on time, then the Hamiltonian must be constant when evaluated on an extremal trajectory, that is

\[ H(x^*, p^*, u^*) = c_1 \quad \forall t \in [t_0, t_f] \]

2. If the final time is free, and the Hamiltonian does not explicitly depend on time, then the Hamiltonian must be identically zero when evaluated on an extremal trajectory, that is

\[ H(x^*, p^*, u^*) = 0 \quad \forall t \in [t_0, t_f] \]

Proof:

Problem 5.5

2. Show that

\[ \frac{\partial H}{\partial t} = 0 \quad \Rightarrow \quad H(t) = \text{const} \quad \forall t \in [t_0, t_f] \]

Since

\[ H(x(t_f), u(t_f), p(t_f)) = \frac{\partial L}{\partial \dot{x}} = 0 \quad \Rightarrow \quad H(t) = 0 \]

1. \( p(t) = \frac{\partial L}{\partial x}(x(t), u(t)) \Rightarrow \text{const} \) or similarly \( x(t), p(t), u(t) = \text{const} \)

\[ H = g + p^T \alpha \]

\[ H(t_f) = g(t_f) + p(t_f) \alpha(t_f) = \text{const} \]
State Variable Inequality Constraints

\[ f(x, t) \geq 0 \quad \forall x, \text{ for exist } \quad \text{dem } f = 0 \]

\[ x_{n+1} = (a_1)^2 s(-s_1) + (a_2)^2 s(-s_2) + \ldots + (a_e)^2 s(-s_e) = 0, \quad \text{Heaviside step function} \]

\[ x_{n+1} = 0 \quad \forall t \]

\[ x_{n+1} = 0 \quad \text{when all constraints are satisfied} \]

\[ \int_{t_0}^{t} x_{n+1}(t) \, dt = x_{n+1}(t) - x_{n+1}(t_0) \]

Assume \( x_{n+1}(t_0) = 0 \) and \( x_{n+1}(T_f) = 0 \)

\[ \Rightarrow x_{n+1}(t) = 0 \quad \forall t \in [t_0, T_f] \quad (\text{as all constraints are satisfied}) \]

\[ \begin{cases} 
    x = a(x, u) \\
    J = \frac{1}{2} \int_{t_0}^{T_f} q(x, u, t) \, dt + r(x(T_f), T_f) \\
    \int_{t}^{T_f} f(x, t) \, dt = 0 
\end{cases} \]

Form the Hamiltonian:

\[ H(x, u, p, t) = q(x, u, t) + p_1 a_1 + \ldots + p_n a_n \\
\quad + \sum_{i=1}^{n} \left( s_i^2 s(-s_i) + \ldots + (a_e)^2 s(-s_e) \right) \]
Necessary conditions:

\[ x'_1 = a_1 \]
\[ x'_n = a_n \]
\[ x'_{n+1} = a_{n+1} \]

\[ x_1(t_0) = 0 \]
\[ x_{n+1}(t_0) = 0 \]

\[ p_1 = -\frac{\partial H}{\partial x_1} \]

\[ p_n = -\frac{\partial H}{\partial x_{n+1}} = 0 \quad \text{no } x_{n+1} \text{ in } H \]

\[ H(x^*_1, u^*, p^*_1, t) \leq H^*(x^*_1, u, p^*_1, t) \]